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STRUCTURE OF GROUND STATE SOLUTIONS FOR SINGULAR ELLIPTIC EQUATIONS WITH A QUADRATIC GRADIENT TERM

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Abstract. We establish results on existence, non-existence, and asymptotic behavior of ground state solutions for the singular nonlinear elliptic problem

$$-\Delta u = g(u)|\nabla u|^2 + \lambda\psi(x)f(u) \quad \text{in } \mathbb{R}^N,$$
$$u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0,$$

where $\lambda \in \mathbb{R}$ is a parameter, $\psi \geq 0$, not identically zero, is a locally Hölder continuous function; $g : (0,\infty) \to \mathbb{R}$ and $f : (0,\infty) \to (0,\infty)$ are continuous functions, (possibly) singular in 0; that is, $f(s) \to \infty$ and either $g(s) \to \infty$ or $g(s) \to -\infty$ as $s \to 0$. The main purpose of this article is to complement the main theorem in Porru and Vitolo [15], for the case $\Omega = \mathbb{R}^N$. No monotonicity condition is imposed on f or g.

1. Introduction

In this article, we establish results concerning non-existence, existence and asymptotic behavior of positive ground state solutions; that is, entire positive classical solutions (in $C^2(\mathbb{R}^N)$) vanishing at infinity, for the singular nonlinear elliptic problem

$$\begin{gathered}-\Delta u = g(u)|\nabla u|^2 + \lambda\psi(x)f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0,\end{gathered} \tag{1.1}$$

where $g : (0,\infty) \to \mathbb{R}$ and $f : (0,\infty) \to (0,\infty)$ are continuous functions, possibly, singular in 0 in the sense, for example, that either $g(s) \to \infty$ or $g(s) \to -\infty$ and $f(s) \to \infty$ as $s \to 0$; $\psi : \mathbb{R}^N \to [0,\infty)$, $\psi \neq 0$ is a locally Hölder continuous function and $\lambda \in \mathbb{R}$ is a real parameter.

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The search for classical solutions to (1.1) with $\lambda = 1$ and $g = 0$; that is, for the problem

$$\begin{gathered} -\Delta u = \psi(x) f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.2}$$

where ψ and f are as above with f singular at 0, has received much attention in recent years; see [3, 6, 7, 8, 11, 12, 19, 21, 22] and references therein. For more general nonlinearities, we refer the reader to Mohammed [13], and for nonlinearities including singular terms in the origin and super-linear terms at infinity to Santos [16].

For further studies on (1.1), the reader is referred to [20] and the references therein. However, [20] does not include the nonlinearity in the coefficient of the gradient term. For the version of (1.1) on bounded domains,

$$\begin{gathered} -\Delta u = \lambda g(u)|\nabla u|^2 + \psi(x) f(u) \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \quad u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a regular bounded domain, λ is a real parameter, $\psi : \Omega \to [0, \infty)$ and f, g are appropriate functions, see for example [1, 2, 4, 14, 15] and their references.

Problems such as (1.3) were studied in [1, 9, 14] with $f(s) = 1,\ s > 0$. In [2] and [14], (1.3) was considered with general terms f but in all cases g is non-singular in 0, that is, g is continuously extendable to 0. In [15], (1.3) was studied with $\psi(x) = 1$, in Ω. Under some conditions on f and g the authors showed existence and, in particular cases, asymptotic behavior of solutions to (1.3). In most cases, monotonicity conditions are imposed upon f or g.

To establish our main results regarding problem (1.1), we shall denote by

$$G(s) = \int_1^s g(t)dt, \quad s > 0,$$

a primitive of g. We define

$$f_{go} = \liminf_{s\to 0} \frac{e^{G(s)} f(s)}{\int_0^s e^{G(t)}dt}, \quad f_{g\infty} = \limsup_{s\to\infty} \frac{e^{G(s)} f(s)}{\int_0^s e^{G(t)}dt},$$
$$\underline{f}_{go} = \liminf_{s\to 0} \frac{e^{G(s)} f(s)}{[\int_s^1 e^{G(t)}dt]^q}, \quad \overline{f}_{go} = \limsup_{s\to 0} \frac{e^{G(s)} f(s)}{[\int_s^1 e^{G(t)}dt]^p}$$

with $1 < q \le p < \infty$.

We will say that ψ satisfies the condition (ψ_∞) if the problem

$$\begin{gathered} -\Delta u = \psi(x) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0 \end{gathered} \tag{1.4}$$

has a unique solution $w_\psi \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^N)$, for some $\alpha \in (0, 1)$. Also we will say that ψ satisfies the condition $(\psi_\infty)'$ if

$$0 < \liminf_{|x|\to\infty} \frac{\psi(x)}{|x|^\gamma} \le \limsup_{|x|\to\infty} \frac{\psi(x)}{|x|^\gamma} < \infty, \tag{1.5}$$

where $\psi > 0$, and γ is a negative constant such that $\gamma < -2p$ with p given in $\overline{f}_{go}$.

Remark 1.1. Concerning the hypothesis (ψ_∞), we have: (1) If

$$\int_0^\infty \Big[s^{1-N}\int_0^s t^{N-1}\hat{\psi}(t)dt\Big]ds < \infty, \tag{1.6}$$

where $\hat{\psi}(r) = \max_{|x|=r}\psi(x),\ r > 0$, then (ψ_∞) holds. In this case,

$$w_\psi(x) \le \int_{|x|}^\infty \Big[s^{1-N}\int_0^s t^{N-1}\hat{\psi}(t)dt\Big]ds := \hat{w}_\psi(|x|),\ x \in \mathbb{R}^N, \tag{1.7}$$

because $\hat{w}_\psi(|\cdot|)$ is an upper solution of (1.4). (see details in Santos [17]).

(2) If we assume $N \ge 3$ and

$$\textstyle\int_1^\infty r\hat{\psi}(r)dr < \infty,$$

then (1.6) will be true (see details in Goncalves and Santos [7]).

To state our next theorem, we consider the problem

$$\begin{gathered} -\Delta u = \lambda\psi(x)u \quad \text{in } \Omega,\\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.8}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain and ψ is a non-negative and suitable function. We know that the first eigenvalue $\lambda_1(\psi,\Omega)$ of (1.8) is positive and non-increasing in the sense that $\lambda_1(\psi,\Omega_2) \le \lambda_1(\psi,\Omega_1)$ if $\Omega_1 \subseteq \Omega_2$. So there exists

$$\lambda_1(\psi) = \lim_{k\to\infty}\lambda_1(\psi, B_k(0)) \in [0,\infty), \tag{1.9}$$

where $B_k(0)$ is the ball centered in the origin of $\mathbb{R}^N$ with radius k. For more details concerning the principal eigenvalue $\lambda_1(\psi)$, we refer to Santos [17].

Our main results read as follows:

Theorem 1.2. *Assume that* $\int_0^1 e^{G(t)}dt < \infty$, (ψ_∞) *and* $f_{go} \in (0,\infty]$ *hold. Then* (1.1) *admits a solution* $u = u_\lambda \in C^2(\mathbb{R}^N)$ *if* $\lambda_1(\psi)/f_{g0} < \lambda < \lambda^\star$ *for some* $\lambda^\star > 0$.

Remark 1.3. The $\lambda^\star > 0$ and the solution u, given by Theorem 1.2 depend on the behavior of g and f at infinity. More specifically, denoting by

$$F(s) = \int_0^s e^{G(t)}dt, s \ge 0, \quad F_\infty = \lim_{s\to\infty} F(s) = \int_0^\infty e^{G(t)}dt, \tag{1.10}$$

we have

- (i) If $F_\infty = \infty$ and
  - (1) $0 \le f_{g\infty} < \infty$, then $\lambda^\star \ge \frac{1}{\|w_\psi\|_\infty f_{g\infty}}$,
  - (2) $f_{g\infty} = \infty$, then $\lambda^\star$ is a positive constant.
- (ii) If $F_\infty < \infty$, then
  - (1) $\lambda^\star = \frac{1}{\|w_\psi\|_\infty}\frac{1}{F_\infty}\int_0^{F_\infty}\Big(s^{-1}\int_0^s\Big[\sup_{r>F^{-1}(t)}\frac{e^{G(r)}f(r)}{F(r)}\Big]^{-1}dt\Big)ds \in (0,\infty]$,
  - (2) $\|u\|_\infty \le F_\infty$.

As an example that satisfies all the assumptions of Theorem 1.2, we have

$$\begin{gathered} -\Delta u = -\frac{\mu}{u}|\nabla u|^2 + \lambda\psi(x)f(u) \quad \text{in } \mathbb{R}^N,\\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{1.11}$$

if $-\infty < \mu < 1$, $\lim_{s\to 0} f(s)/s > 0$ and ψ satisfies (ψ_∞). Furthermore, if we have $\lim_{s\to\infty} f(s)/s = 0$, then $\lambda^\star = \infty$.

In the next result and Theorem 1.8, we assume that f is a C^1-function and $N \geq 3$.

Theorem 1.4. *Assume that $\int_0^1 e^{G(t)}dt = \infty$, $(\psi_\infty)'$, $\underline{f}_{go} \in (0,\infty]$ and $\overline{f}_{go} \in [0,\infty)$ hold. Then there exists $\lambda^\star > 0$ such that for all $\lambda \in (0,\lambda^\star)$ the problem (1.1) has a solution.*

Remark 1.5. Again here $\lambda^\star > 0$ depends on the behavior of f and g at infinity. That is, if

$$\limsup_{s\to\infty} \frac{e^{G(s)}f(s)}{[\int_s^{s+1} e^{G(t)}dt]^p} < \infty, \tag{1.12}$$

where $p > 1$ is defined in $\overline{f}_{go}$, then, for some positive constant c,

$$\lambda^\star \geq c \inf_{s>0} \Big[\int_s^{s+1} e^{G(t)}dt\Big]^{1-p}.$$

Consider the example

$$\begin{gathered} -\Delta u = -\frac{\mu}{u}|\nabla u|^2 + \lambda\psi(x)f(u) \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0. \end{gathered} \tag{1.13}$$

All hypotheses of Theorem 1.4 are satisfied if ψ satisfies $(\psi_\infty)'$, $\mu \geq 1$ and f satisfies

$$\lim_{s\to 0} \frac{f(s)}{s(\ln 1/s)^p} > 0, \quad \text{if } \mu = 1, \quad \text{and} \quad \lim_{s\to 0} \frac{f(s)}{s^{\mu(1-p)+p}} > 0, \quad \text{if } \mu > 1,$$

where $p = q$ is given in $(\psi_\infty)'$. Besides this, $\lambda^\star = \infty$, if

$$\lim_{s\to \infty} \frac{f(s)}{s(\ln 1/s)^p} < \infty, \quad \text{if } \mu = 1, \quad \text{and} \quad \lim_{s\to \infty} \frac{f(s)}{s^{\mu(1-p)}} < \infty, \quad \text{if } \mu > 1.$$

For the non-existence, we have the following result.

Theorem 1.6. *Assume that $g : (0,\infty) \to \mathbb{R}$, $f : (0,\infty) \to [0,\infty)$, $\psi : \mathbb{R}^N \to [0,\infty)$ are continuous functions and $\lambda \leq 0$. Then (1.1) has no solution.*

Concerning the asymptotic behavior, we have the following result.

Theorem 1.7. *Assume that (1.6) holds and $N \geq 3$, then the solution given by Theorem 1.2 (which we shall denote as $u = u_\lambda$) satisfies*

$$F^{-1}(c|x|^{2-N}) \leq u(x) \leq F^{-1}(d|x|^{2-N}), \quad |x| \geq 1,$$

for some positive constants c and d with F defined in (1.10). In particular, if $g = 0$, then

$$c|x|^{2-N} \leq u(x) \leq d|x|^{2-N}, \quad |x| \geq 1.$$

For example the solution of (1.11), given by Theorem 1.2, satisfies

$$c|x|^{4-2N} \leq u(x) \leq d|x|^{4-2N}, \quad |x| \geq 1,$$

if in addition we assume $\lim_{s\to 0} f(s)/s < \infty$.

Theorem 1.8. *The solution given by Theorem 1.4 (which we shall denote as $u = u_\lambda$) satisfies*

$$F_0^{-1}(c|x|^{\frac{\gamma+2}{1-q}}) \leq u(x) \leq F_0^{-1}(d|x|^{\frac{\gamma+2}{1-p}}), \quad |x| \geq R,$$

for some positive constants c, d and R with

$$F_0(s) = \int_s^1 e^{G(t)}dt, \quad 0 < s < 1.$$

For example the solution of (1.13) with $\mu > 1$ satisfies

$$\frac{1}{c + |x|^{\frac{\gamma+2}{(1-p)(\mu-1)}}} \le u(x) \le \frac{1}{d + |x|^{\frac{\gamma+2}{(1-p)(\mu-1)}}}, \quad |x| \ge R,$$

for some constants $c, d, R > 0$.

Remark 1.9. Examples of $\psi : \mathbb{R}^N \to (0, \infty)$ satisfying (ψ_∞) with $\nu > 2$ are as follows:

$$\psi(x) = \frac{1}{1 + |x|^\nu}, \quad \psi(x) = \frac{1}{2 + \sin(|x|^2) + |x|^\nu}$$

while

$$\psi(x) = \frac{1}{1 + |x|^{2p+1}}, \quad x \in \mathbb{R}^N$$

satisfies $(\psi_\infty)'$, where $p > 2$.

The proof of Theorem 1.2 is based on the suitable diffeomorphisms and in Santos's arguments which showed existence of at least one entire positive solution for the problem (1.2) in the presence of singular and super linear terms at infinity without imposing any monotonicity condition in $f(s)$ or $f(s)/s$ (for more details see [16]).

Proof of Theorem 1.2. Consider the function defined in (1.10); that is, $F : [0, \infty) \to [0, \infty)$ with

$$F(s) = \int_0^s e^{G(t)}dt, \quad s \ge 0 \quad \text{and} \quad F_\infty = \lim_{s\to\infty} \int_0^s e^{G(t)}dt.$$

Thus, F is increasing, $F(0) = 0$. Now, we will consider two separate cases.

Case 1: $F_\infty = \infty$. In this case, $F(s) \to \infty$ as $s \to \infty$. Now, let the continuous function $h(s) = F'(F^{-1}(s))f(F^{-1}(s))$, $s > 0$ and for each $\tau, \lambda > 0$ given, consider the continuous function $\widetilde{H}_\lambda : (0, \infty) \times (0, \infty) \to (0, \infty)$ defined by

$$\widetilde{H}_\lambda(\tau, s) = \begin{cases} \lambda s \sup_{s\le t\le\tau} \frac{h(t)}{t}, & s \le \tau, \\ \lambda s \frac{h(\tau)}{\tau}, & s \ge \tau. \end{cases}$$

So, it is easy to check that

(i) $\widetilde{H}_\lambda(\tau, s) \ge \lambda h(s)$, $0 < s \le \tau$,
(ii) $\widetilde{H}_\lambda(\tau, s)/s$ is non-increasing in $s > 0$,
(iii) $\lim_{s\to 0^+} \widetilde{H}_\lambda(\tau, s)/s = \lambda \sup_{0<t\le\tau} h(t)/t$,
(iv) $\lim_{s\to\infty} \widetilde{H}_\lambda(\tau, s)/s = \lambda h(\tau)/\tau$.

By (iii), the function $\hat{H}_\lambda : (0, \infty) \times (0, \infty) \to (0, \infty)$, given by

$$\hat{H}_\lambda(\tau, s) = \frac{s^2}{\int_0^s \frac{t}{\widetilde{H}_\lambda(\tau,t)}dt}$$

is a well-defined and continuous function. Using (ii), we have

$$\hat{H}_\lambda(\tau, s) \ge \widetilde{H}_\lambda(\tau, s), \quad \forall\ \tau, s > 0.$$

Besides this, $\hat{H}_\lambda(\tau,\cdot) \in C^1(0,\infty)$, for each $\tau > 0$. Using (i)-(iv), it follows that for each $\lambda \geq 0$, $\hat{H}_\lambda$ satisfies the following.

Lemma 1.10. *If $\int_0^1 e^{G(t)} dt < \infty$, then, for each $\tau > 0$,*

- (i) *$\hat{H}_\lambda(\tau,s)/s$ is non-increasing for $s > 0$,*
- (ii) *$\lim_{s\to 0} \hat{H}_\lambda(\tau,s)/s = \lambda \sup_{0<t\leq\tau} h(t)/t$,*
- (iii) *$\lim_{s\to\infty} \hat{H}_\lambda(\tau,s)/s = \lambda h(\tau)/\tau$.*

Now, we define the continuous function

$$H_\lambda(\tau) = \frac{1}{\|w_\psi\|_\infty \tau} \int_0^\tau \frac{t}{\hat{H}_\lambda(\tau,t)} dt, \quad \tau > 0,$$

where w_ψ is given by the hypothesis (ψ_∞). Hence,

$$H_\lambda(\tau) = \frac{1}{\lambda} H_1(\tau), \quad \tau,\lambda > 0. \tag{1.14}$$

Let

$$\lambda^\star = \sup_{\tau\geq 1} H_1(\tau) > 0.$$

Since

$$\liminf_{\tau\to\infty} H_1(\tau) = \frac{1}{\|w_\psi\|_\infty f_{g\infty}}$$

it follows that

$$\frac{1}{\|w_\psi\|_\infty f_{g\infty}} \leq \lambda^\star \leq \infty.$$

This proves Remark 1.3 part (i)(1). So, from (1.14), for each $0 < \lambda < \lambda^\star$, we can take a $\tau_\infty = \tau_\lambda \geq 1$ such that $H_\lambda(\tau_\infty) > 1$. That is,

$$\frac{1}{\tau_\infty} \int_0^{\tau_\infty} \frac{t}{\hat{H}_\lambda(\tau_\infty,t)} dt > \|w_\psi\|_\infty. \tag{1.15}$$

Now, defining the C^2- increasing function

$$\hat{h}_\lambda(s) = \frac{1}{\tau_\infty} \int_0^s \frac{t}{\hat{H}_\lambda(\tau_\infty,t)} dt, \quad s \geq 0$$

and defining $v(x) = \hat{h}_\lambda^{-1}(w_\psi(x))$, $x \in \mathbb{R}^N$, we obtain, using (1.15),

$$v(x) = \hat{h}_\lambda^{-1}(w_\psi(x)) \leq \hat{h}_\lambda^{-1}(\|w_\psi\|_\infty) < \hat{h}_\lambda^{-1}(\hat{h}_\lambda(\tau_\infty)) = \tau_\infty, \quad x \in \mathbb{R}^N$$

and after some calculations, we obtain that $v \in C^2(\mathbb{R}^N)$, $v(x) \to 0$ as $|x| \to \infty$ and that it satisfies

$$\begin{gathered} -\Delta v \geq \psi(x)\hat{H}_\lambda(\tau_\infty,v) \geq \lambda\psi(x)h(v) \quad \text{in } \mathbb{R}^N, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} v(x) = 0. \end{gathered}$$

On the other hand, given $\lambda_1(\psi)/f_{g0} < \lambda < \lambda^\star$ (we point out that $\lambda_1(\psi)/f_{g0} = 0$ if either $f_{g0} = \infty$ or $\lambda_1(\psi) = 0$) we can take from (1.9) a $k_\lambda > 1$ such that

$$\frac{\lambda_1(\psi)}{f_{g0}} \leq \frac{\lambda_1(\psi,k)}{f_{g0}} < \lambda < \lambda^\star, \quad \text{for all } k \geq k_\lambda.$$

As a consequence of this, there exists a $s_0 = s_{0,\lambda,k} \in (0,1)$ such that

$$\lambda h(s) \geq \lambda_1(\psi,k)s, \quad \text{for all } 0 < s < s_0.$$

Now, defining $v_k = \varepsilon_{\lambda,k}\psi_k$, where ψ_k is the positive first eigenfunction of (1.8) with $\Omega = B_k(0)$ and $\varepsilon_{\lambda,k} > 0$ satisfies

$$\varepsilon_{\lambda,k} \max\{\psi_k(x) : x \in \overline{B_k(0)}\} \le s_0,$$

it follows that v_k satisfies

$$\begin{gathered} -\Delta v_k \le \lambda\psi(x)h(v_k) \quad \text{in } B_k(0), \\ v > 0 \quad \text{in } B_k(0), \quad v(x) = 0 \quad \text{on } \partial B_k(0). \end{gathered}$$

Following the arguments of either Mohammed [13] or Santos [16], we have a $v \in C^2(\mathbb{R}^N)$ satisfying

$$\begin{gathered} -\Delta v = \lambda\psi(x)h(v) \quad \text{in } \mathbb{R}^N, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} v(x) = 0. \end{gathered}$$

Let

$$u(x) = u_\lambda(x) = F^{-1}(v(x)), \ x \in \mathbb{R}^N.$$

Such that

$$0 < u \in C^2(\mathbb{R}^N), \quad \lim_{|x|\to\infty} u(x) = F^{-1}\Big(\lim_{|x|\to\infty} v(x)\Big) = F^{-1}(0) = 0$$

and

$$-\Delta u = g(u)|\nabla u|^2 + \lambda\psi(x)f(u), \quad x \in \mathbb{R}^N.$$

Hence, u is a solution of (1.1).

Case 2: $F_\infty < \infty$. The proof of Theorem 1.2 in this case is an adaptation of earlier proof. First, we note that to construct the upper solution, we let the continuous function $h(s) = F'(F^{-1}(s))f(F^{-1}(s))$, $0 < s < F_\infty$ and for each $\lambda > 0$ given, we consider the continuous functions $\widetilde{H}_\lambda, \hat{H}_\lambda$ defined by

$$\widetilde{H}_\lambda(s) = \lambda s \sup_{s\le t\le F_\infty} \frac{h(t)}{t}, \quad 0 < s \le F_\infty$$

and

$$\hat{H}_\lambda(s) = \frac{s^2}{\int_0^s \frac{t}{\widetilde{H}_\lambda(t)}dt}, \quad 0 < s \le F_\infty.$$

Thus, in a similar way to the proof of Lemma 1.10, we have $\hat{H}_\lambda(s) \ge \lambda h(s)$ for $0 < s < F_\infty$ and the following result.

Lemma 1.11. *If $\int_0^1 e^{G(t)}dt < \infty$, then*

(i) *$\hat{H}_\lambda(s)/s$ is non-increasing for $0 < s \le F_\infty$*
(ii) *$\lim_{s\to 0} \hat{H}_\lambda(s)/s = \lambda \sup_{0<t\le F_\infty} h(t)/t$,*
(iii) *$\hat{H}_\lambda(F_\infty) = \lambda F_\infty^2$*
$big/ \int_0^{F_\infty} \big(\sup_{t\le r\le F_\infty} h(r)/r\big)dt$.

Now, we define the continuous function

$$H_\lambda(\tau) = \frac{1}{\|w_\psi\|_\infty \tau} \int_0^\tau \frac{t}{\hat{H}_\lambda(t)}dt, \quad \tau > 0,$$

where w_ψ is given by hypothesis (ψ_∞). Hence,

$$H_\lambda(\tau) = \frac{1}{\lambda} H_1(\tau), \quad \tau,\ \lambda > 0. \tag{1.16}$$

Define

$$\begin{aligned}\lambda^\star &= \lim_{\tau\to F_\infty} H_1(\tau)\\ &= \lim_{\tau\to F_\infty}\frac{1}{\|w_\psi\|_\infty \tau}\int_0^\tau \frac{t}{\hat{H}_1(t)}dt\\ &= \frac{1}{\|w_\psi\|_\infty F_\infty}\int_0^{F_\infty}\frac{t}{\hat{H}_1(t)}dt\\ &= H_1(F_\infty) > 0.\end{aligned}$$

Such that, from (1.16), for each $0 < \lambda < \lambda^\star$, we have

$$H_\lambda(F_\infty) = \frac{1}{\lambda}H_1(F_\infty) = \frac{\lambda^\star}{\lambda} > 1.$$

That is,

$$\frac{1}{F_\infty}\int_0^{F_\infty}\frac{t}{\hat{H}_\lambda(t)}\,dt > \|w_\psi\|_\infty. \tag{1.17}$$

Now, defining the C^2 increasing function

$$\hat{h}_\lambda(s) = \frac{1}{F_\infty}\int_0^s \frac{t}{\hat{H}_\lambda(t)}\,dt, \quad 0 < s \le F_\infty$$

and defining $v(x) = \hat{h}_\lambda^{-1}(w_\psi(x))$, $x \in \mathbb{R}^N$, we obtain, using (1.17),

$$v(x) = \hat{h}_\lambda^{-1}(w_\psi(x)) \le \hat{h}_\lambda^{-1}(\|w_\psi\|_\infty) < \hat{h}_\lambda^{-1}(\hat{h}_\lambda(F_\infty)) = F_\infty,\ x \in \mathbb{R}^N.$$

Now, in a similar way, we construct an upper solution of (1.2) with $f = h$. Secondly, we point out that the lower solution for (1.2) with $f = h$ is constructed the same way as in the proof of Case 1. This proves Theorem 1.2

2. Proof of Theorem 1.4

In this Section, we will deal with the question of existence of a solution for Theorem 1.4. For this, we shall use a modified version of a result by Gonçalves and Roncalli [10] for the existence of an entire blow-up solution which is bounded from below by a positive constant.

We shall consider $k : [0, \infty) \to [0, \infty)$ a C^1-function with $k(0) = 0$ and $k(t) > 0$ for $t > 0$, ψ as before and the problem

$$\begin{gathered}\Delta u = \psi(x)k(u) \quad \text{in } \mathbb{R}^N,\\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = \infty.\end{gathered} \tag{2.1}$$

Lemma 2.1. *Let $\psi \in C^\nu_{\rm loc}(\mathbb{R}^N)$ for some $\nu \in (0,1)$ and $\psi(x) > 0$, $\forall\ x \in \mathbb{R}^N$, $N \ge 3$. Assume that there exist $1 < q \le p < \infty$ such that*

$$\ell_\infty = \liminf_{s\to\infty}\frac{k(s)}{s^q} \in (0,\infty], \tag{2.2}$$

$$S_\infty = \sup_{s>0}\frac{k(s)}{s^p} \in (0,\infty) \tag{2.3}$$

and condition $(\psi_\infty)'$ holds with $\gamma < -2p$. Then (2.1) admits at least one solution $u \in C^2(\mathbb{R}^N)$ such that

$$u(x) \ge a_\psi S_\infty^{\frac{1}{1-p}} > 0 \quad \textit{for all } x \in \mathbb{R}^N, \tag{2.4}$$

for some positive constant a_ψ.

Remark 2.2. The main novelty in Lemma 2.1 is the lower limit of solution u of (2.1) by a positive constant throughout $\mathbb{R}^N$. A similar result was proved in [10] without Claim 2.4.

Proof of Lemma 2.1. We follows similar arguments as those in [10, Theorem 1.1]. In fact, from conditions (2.2) and $(\psi_\infty)'$ there exists a $R_\psi > 0$ such that

$$b_1 = \frac{1}{2}\liminf_{|x|\to\infty}\frac{\psi(x)}{|x|^\gamma} \le \frac{\psi(x)}{|x|^\gamma} \le 2\liminf_{|x|\to\infty}\frac{\psi(x)}{|x|^\gamma} = b_2, \quad \forall |x| \ge R_\psi$$

and

$$k(s) \ge \frac{\ell_\infty}{2}s^q,\ \forall\ s \ge R_\psi. \tag{2.5}$$

Now, defining

$$\alpha = \frac{\gamma+2}{1-p} > 2, \quad \beta = \frac{\gamma+2}{1-q} > 2,$$

$$A_\psi = \max_{[0,R_\psi]}\frac{\left[\frac{t^\alpha}{(N+\alpha-2)\alpha} + 1 + \frac{t^2}{2N}\right]^p}{1+t^{\alpha-2}}, \quad B_\psi = \max_{[R_\psi,\infty)}\frac{\left[\frac{1}{(N+\alpha-2)\alpha} + \frac{1}{t^\alpha} + \frac{t^{2-\alpha}}{2N}\right]^p}{1+t^{-\gamma-\alpha p}},$$

$$C_\psi = \min_{[0,R_\psi]}\frac{\left[\frac{t^\beta}{(N+\beta-2)\beta} + 1 + \delta + \frac{t^2}{2N}\right]^q}{1+t^{\beta-2}}, \quad D_\psi = \min_{[R_\psi,\infty)}\frac{\left[\frac{1}{(N+\beta-2)\beta} + \frac{1+\delta}{t^\beta} + \frac{t^{2-\beta}}{2N}\right]^q}{1+t^{-\gamma-\beta q}},$$

$$\delta = \begin{cases} 0, & \text{if } \alpha = \beta, \\ \frac{[\alpha\beta(N+\beta-\alpha)]^{\frac{\alpha}{\beta-\alpha}}}{(\beta-\alpha)^{\frac{\alpha}{\beta-\alpha}}[\alpha(\alpha+1)(N+\alpha-2)]^{\frac{\beta}{\beta-\alpha}}}, & \text{if } \alpha < \beta, \end{cases}$$

we have

$$\begin{aligned} 0 < \tilde\lambda &= \min\left\{\left(M_\psi S_\infty A_\psi\right)^{\frac{1}{1-p}}, \left(b_2 S_\infty B_\psi\right)^{\frac{1}{1-p}}\right\} \\ &\le \tilde\Lambda = \max\left\{\tilde\lambda, \frac{R_\psi}{1+\delta}, \left(\frac{m_\psi \ell_\infty C_\psi}{2}\right)^{\frac{1}{1-q}},\ \left(\frac{b_1\ell_\infty D_\psi}{2}\right)^{\frac{1}{1-q}}\right\} < \infty, \end{aligned}$$

where

$$M_\psi = \max_{|x|\le R_\psi}\psi(x) \quad \text{and} \quad m_\psi = \min_{|x|\le R_\psi}\psi(x).$$

In the sequel, we use the notation

$$\underline{u}(x) = \tilde\lambda\Big[\frac{|x|^\alpha}{(N+\alpha-2)\alpha} + \frac{|x|^2}{2N} + 1\Big], \quad x \in \mathbb{R}^N,$$

$$\overline{u}(x) = \tilde\Lambda\Big[\frac{|x|^\beta}{(N+\beta-2)\beta} + \frac{|x|^2}{2N} + \delta + 1\Big], \quad x \in \mathbb{R}^N$$

and separately considering the cases $|x| \le R_\psi$ and $|x| \ge R_\psi$. We obtain by direct computations, using (2.5), that

$$\begin{gathered} \underline{u}(x) \le \overline{u}(x), \quad x \in \mathbb{R}^N, \\ \underline{u}(x), \quad \overline{u}(x) \to \infty \quad \text{as } |x| \to \infty \\ \Delta\underline{u}(x) \le \psi(x)k(\underline{u}(x)), \quad \Delta\overline{u}(x) \ge \psi(x)k(\overline{u}(x)). \end{gathered}$$

Now, by applying [10, Theorem 2.1], we have a solution $u \in C^2(R^N)$ of (2.1) with

$$0 < a_\psi(S_\infty)^{\frac{1}{1-p}} \le \tilde\lambda \le \underline{u}(x) \le u(x) \le \overline{u}(x), \quad \forall\ x \in R^N,$$

where

$$a_\psi = \min\left\{(M_\psi A_\psi)^{\frac{1}{1-p}}, (b_2 B_\psi)^{\frac{1}{1-p}}\right\} > 0.$$

This completes the proof. □

Proof of Theorem 1.4. For each $\tau > 0$ given, define $F_\tau : (0,\tau] \to (0,\infty)$ by

$$F_\tau(s) = \int_s^{\tau+1} e^{G(t)} dt.$$

So, F_τ is a decreasing continuous function. From $\int_0^1 e^{G(t)} dt = \infty$, we have

$$\lim_{s\to 0} F_\tau(s) = \infty \quad \text{and} \quad \lim_{s\to\tau} F_\tau(s) = F_\tau(\tau).$$

Now, we consider the C^1-function $k_\tau : [0,\infty) \to [0,\infty)$ defined by

$$k_\tau(s) = \begin{cases} s^p, & 0 \le s < \frac{F_\tau(\tau)}{2}, \\ \xi_\tau(s), & \frac{F_\tau(\tau)}{2} \le s \le F_\tau(\tau), \\ e^{G(F_\tau^{-1}(s))} f(F_\tau^{-1}(s)), & s \ge F_\tau(\tau), \end{cases}$$

for appropriate function ξ_τ, and the τ-problems family

$$\begin{gathered} \Delta v = \lambda\psi(x) k_\tau(v) \quad \text{in } \mathbb{R}^N, \\ v > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} v(x)\infty. \end{gathered} \tag{2.6}$$

We claim that

$$\ell_{\infty,\tau} = \liminf_{s\to\infty} \frac{k_\tau(s)}{s^q} \in (0,\infty].$$

In fact, making $t = F_\tau^{-1}(s)$, $0 < s \le \tau$, we have

$$\begin{aligned} \ell_{\infty,\tau} &= \liminf_{s\to\infty} \frac{k_\tau(s)}{s^q} = \liminf_{t\to 0} \frac{e^{G_\tau(t)} f(t)}{F_\tau(t)^q} \\ &= \liminf_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^1 e^{G(s)} ds + b_2(\tau)\right]^q} = \liminf_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^1 e^{G(s)} ds\right]^q [1 + \frac{b_2(\tau)}{\int_t^1 e^{G(s)} ds}]^q} \\ &= \liminf_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^1 e^{G(s)} ds\right]^q} = \underline{f}_{go}, \end{aligned}$$

where $b_2(\tau)$ denotes a real positive constant. Since $\underline{f}_{go} \in (0,\infty]$, we obtain (2.2) of Lemma 2.1.

Also, since

$$\limsup_{s\to 0} \frac{k_\tau(s)}{s^p} = 1$$

and making $t = F_\tau^{-1}(s),\ 0 < s \le \tau$, we have

$$\begin{aligned} \limsup_{s\to\infty} \frac{k_\tau(s)}{s^p} &= \limsup_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^{\tau+1} e^{G(r)} dr\right]^p} \\ &= \limsup_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^1 e^{G(r)} dr + \int_1^{\tau+1} e^{G(r)} dr\right]^p} \\ &\le \limsup_{t\to 0} \frac{e^{G(t)} f(t)}{\left[\int_t^1 e^{G(r)} dr\right]^p} = \overline{f}_{go}. \end{aligned}$$

By hypothesis the $\overline{f}_{go} \in [0, \infty)$, we have that

$$S_{\infty,\tau} = \sup_{s>0} \frac{k_\tau(s)}{s^p} \in (0, \infty), \tag{2.7}$$

for each $\tau > 0$. Hence, (2.3) of Lemma 2.1 is satisfied. Let

$$\lambda^\star := \sup_{\tau>0} \frac{a_\psi^{p-1}}{F_\tau(\tau)^{p-1} S_{\infty,\tau}} > 0, \tag{2.8}$$

where $a_\psi > 0$ is the constant of Lemma 2.1.

Given $0 < \lambda < \lambda^\star$, pick a $\tau = \tau(\lambda) > 1$ such that

$$F_\tau(\tau) < \frac{1}{\lambda^{\frac{1}{p-1}}} \big[\frac{1}{S_{\infty,\tau}}\big]^{\frac{1}{p-1}} a_\psi. \tag{2.9}$$

and apply Lemma 2.1 to the problem (2.6). That is, there exists a $v = v_\tau = v_{\tau(\lambda)}$ solution of (2.6) satisfying, by (2.4) and (2.9),

$$v_\tau(x) \geq a_\psi [\lambda S_{\infty,\tau}]^{\frac{-1}{p-1}} > F_\tau(\tau), \quad \text{for all } x \in \mathbb{R}^N.$$

Define

$$u_\tau(x) = F_\tau^{-1}(v_\tau(x)), \quad x \in \mathbb{R}^N.$$

Thus, of F_τ^{-1} decreasing, we have

$$u_\tau(x) = F_\tau^{-1}(v_\tau(x)) \leq F_\tau^{-1}(F_\tau(\tau)) = \tau, \ x \in \mathbb{R}^N$$

and from the regularity of F_τ^{-1}, it follows that

$$0 < u_\tau \in C^2(\mathbb{R}^N), \quad \lim_{|x|\to\infty} u_\tau(x) = \lim_{|x|\to\infty} F_\tau^{-1}(v(x)) = 0$$

and

$$-\Delta u_\tau = g(u_\tau)|\nabla u_\tau|^2 + \lambda\psi(x) f(u_\tau), \ x \in \mathbb{R}^N.$$

That is, u_τ is a solution of Problem (1.1). This completes the proof. □

Proof of Remark 1.5. Consider the positive number M defined by

$$M = \sup_{s>0} \frac{e^{G(s)} f(s)}{\big[\int_s^{s+1} e^{G(t)} dt\big]^p},$$

where M is finite by (1.12), and, if necessary redefine, ξ_τ in k_τ such that $0 < \xi_\tau(s) \leq (M+1)s^p$, $\frac{1}{2}F_\tau(\tau) \leq s \leq F_\tau(\tau)$. This is possible because $(M+1)F_\tau(\tau)^p > e^{G(\tau)} f(\tau)$ for each $\tau > 0$ given.

So, it is easy to verify that $S_{\infty,\tau}$, defined in (2.7), satisfies

$$S_{\infty,\tau} \leq M + 1, \quad \text{for all } \tau > 0.$$

Hence, from (2.8), it follows the claim with $c = a_\psi^{p-1}/(M+1) > 0$. □

3. Proofs of main results

Proof of Theorem 1.6. Assume, by contradiction, that (1.1) admits one solution, say $u \in C^2(\mathbb{R}^N)$. Since $u(x) > 0$ for all $x \in \mathbb{R}^N$ and $u(x) \to 0$ as $|x| \to \infty$ it follows that u achieves its maximum $M > 0$ in x_0. That is, $0 < u(x) \le u(x_0) = M$ for all $x \in \mathbb{R}^N$. Set $v : \mathbb{R}^N \to [0, \infty)$ defined by

$$v(x) = \int_{u(x)}^{M} e^{G(t)} dt, \quad x \in \mathbb{R}^N.$$

So, $v \in C^2(\mathbb{R}^N)$, $v \ge 0$, $v \ne 0$ and

$$\Delta v = \lambda \psi(x) f(u) \le 0, \quad x \in \mathbb{R}^N$$

because $\lambda \le 0$. Since

$$v(x_0) = 0 = \min_{x \in \mathbb{R}^N} v(x),$$

it follows, by strong maximum principle, that $v(x) = 0$ for all $x \in \mathbb{R}^N$. This is impossible. This completes the proof. □

Proof of Theorem 1.7. Consider $u = u_\lambda \in C^2(\mathbb{R}^N)$ the solution of (1.1) given by Theorem 1.2. Remembering the proof of Theorem 1.2 (case $F_\infty = \infty$) we have that $u = F(v)$, where $v = v_\lambda \in C^2(\mathbb{R}^N)$ satisfies

$$\frac{1}{\tau_\infty} \int_0^{v(x)} \frac{t}{\hat{H}_\lambda(\tau_\infty, t)}\, dt = w_\psi(x) \le \hat{w}_\psi(|x|), \quad x \in \mathbb{R}^N. \tag{3.1}$$

In this last inequality we used (1.7). Define η by

$$\frac{1}{\tau_\infty} \int_0^{\eta(|x|)} \frac{t}{\hat{H}_\lambda(\tau_\infty, t)}\, dt = \hat{w}_\psi(|x|),\ x \in \mathbb{R}^N.$$

So, $v(x) \le \eta(|x|),\ x \in \mathbb{R}^N$. We claim that

$$\eta(r) \le d r^{2-N}, \quad r \ge 1,$$

where $r = |x|$, $x \in \mathbb{R}^N$, for some positive constant d. To verify this claim, define

$$\phi(r) = \begin{cases} 2\eta(0), & \text{if } 0 \le r \le 1,\\ 2\eta(0) r^{2-N}, & \text{if } r \ge 1. \end{cases}$$

Thus, $\eta(r) \le \phi(r)$, $0 \le r \le 1$. Now, we suppose by contradiction that there exists a $r_0 > 1$ such that

$$\eta(r) \le \phi(r), \quad 0 \le r \le r_0 \quad \text{and} \quad \eta(r_0) = \phi(r_0).$$

Using Díaz and Saa's [5] inequality on $B_{r_0}(0)$ - ball centered in 0 and radius r_0 -, it follows that

$$\begin{aligned} 0 &\le \int_{B_{r_0}(0)} \Big(\frac{-\Delta\phi}{\phi} + \frac{\Delta\eta}{\eta}\Big)(\phi(|x|)^2 - \eta(|x|)^2) dx \\ &\le -\int_{B_{r_0}(0)} \hat{\psi}(|x|) h(\eta(|x|))(\phi(|x|)^2 - \eta(|x|)^2)\, dx\,. \end{aligned}$$

This is impossible, because the last term is negative. This proves the claim.

On the other hand, using classical estimates (see for example Serrin and Zou [18]), we obtain a $c > 0$ constant such that

$$v(x) \ge c|x|^{2-N},\ |x| \ge 1.$$

As a consequence of the last inequality, the prior claim and of F^{-1} being increasing, we have

$$F^{-1}(c|x|^{2-N}) \leq u(x) = F^{-1}(v(x)) \leq F^{-1}(d|x|^{2-N}), \; |x| \geq 1.$$

In a similar manner, we reach this conclusion, if $F_\infty < \infty$ holds. This completes the proof. $\square$

Proof of Theorem 1.8. Consider $u = u_\lambda \in C^2(\mathbb{R}^N)$ the solution of (1.1) given by Theorem 1.4. So, from the demonstration of Theorem 1.4, there exists a $\tau = \tau(\lambda) > 0$ such that u satisfies

$$\underline{u}(x) \leq \int_{u(x)}^{\tau+1} e^{G(t)} dt \leq \overline{u}(x), \quad x \in \mathbb{R}^N, \tag{3.2}$$

where $\underline{u}$ and $\overline{u}$ were defined in the proof of Lemma 2.1. As a consequence of the definition of $\underline{u}$ and $\overline{u}$ there are c, d and R positive constants such that

$$d|x|^\alpha \leq \underline{u}(x) - \int_1^{\tau+1} e^{G(t)} dt, \quad |x| \geq R, \tag{3.3}$$

and

$$\overline{u}(x) - \int_1^{\tau+1} e^{G(t)} dt \leq |x|^\beta c, \quad |x| \geq R. \tag{3.4}$$

Hence from (3.2), (3.3), (3.4) and some calculations, we obtain

$$d|x|^\alpha \leq \int_{u(x)}^{1} e^{G(t)} dt \leq c|x|^\beta, \; |x| \geq R.$$

This completes the proof of Theorem 1.8, remembering that F_0 is decreasing. $\square$

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