Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 82, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MAXIMUM PRINCIPLE AND EXISTENCE RESULTS FOR NONLINEAR COOPERATIVE SYSTEMS ON A BOUNDED DOMAIN

LIAMIDI LEADI, ABOUBACAR MARCOS

Abstract. In this work we give necessary and sufficient conditions for having a maximum principle for cooperative elliptic systems involving p-Laplacian operator on a bounded domain. This principle is then used to yield solvability for the considered cooperative elliptic systems by an approximation method.

1. Introduction

This article studies the general nonlinear cooperative elliptic system

$$\begin{gathered} -\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)h(u,v) + f \quad \text{in } \Omega \\ -\Delta_q v = dn(x)|v|^{q-2}v + cn_1(x)k(u,v) + g \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is an bounded domain of class $C^{2,\nu}$ of $\mathbb{R}^N$ ($N \geq 1$). Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, is the p-Laplacian operator. The parameters a, b, c, d are nonnegative real numbers. The functions $h, k : \mathbb{R}^2 \to \mathbb{R}$ are continuous and have like the weight functions m, m_1, n, n_1, some properties which will be specified later.

Our aim is to construct a Maximum Principle with inverse positivity assumptions which means that if f, g are nonnegative functions then any solution (u, v) of (1.1) obey $u \geq 0$; $v \geq 0$ on Ω.

Many works have been devoted to the study of linear and nonlinear elliptic cooperative systems either on a bounded domain or an unbounded domain of $\mathbb{R}^N$ (cf. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15]). Most of those works deal with Maximum Principle for a certain class of functions h an k. In this work, we deal with a more general class of functions h, k. For specific interest for our purposes is the work in [4] where a study of problems such as (1.1) was carried out in the particular case where the weights $m = m_1 = n = n_1 = 1$; $h(s,t) = |s|^\alpha |t|^\beta t$ and $k(s,t) = |s|^\alpha s|t|^\beta$, α and β are some nonnegative real parameters. Clearly, our work extends the work

2000 *Mathematics Subject Classification.* 35B50, 35J57, 35J60.
Key words and phrases. Maximum Principle; elliptic systems; p-Laplacian operator; sub-super solutions; approximation method.

Submitted November 11, 2010. Published June 24, 2011.

in [4] first by considering a problem with weights and next by dealing with a more general class of functions h, k. For instance our result can apply for the case

$$h(s,t) = \begin{cases} |\sin s|^{\alpha}|\arctan t|^{\beta} t & \text{for } t \geq 0,\ s \in \mathbb{R} \\ |s|^{\alpha}|t|^{\beta} t & \text{for } t \leq 0,\ s \in \mathbb{R}. \end{cases}$$

$$k(s,t) = \begin{cases} |\sin s|^{\alpha} s|\arctan t|^{\beta} & \text{for } s \geq 0,\ t \in \mathbb{R} \\ |s|^{\alpha} s|t|^{\beta} & \text{for } s \leq 0,\ t \in \mathbb{R}. \end{cases}$$

which is not taking into account in [4].

The remainder of this article is organized as follows: in the preliminary Section 2, we specify the required assumptions on the data of our problem and we collect some known results relative to the principal positive eigenvalue of the p-Laplacian operator. In Section 3, the Maximum Principle for (1.1) is given and is shown to be proven full enough to yield existence of solution for (1.1) in Section 4.

2. Preliminaries

Throughout this work we assume that:

(B1) $\alpha, \beta \geq 0$; $p, q > 1$ and $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$;

(B2) $b, c \geq 0$, $f \in L^{p'}(\Omega)$, $g \in L^{q'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$;

(B3) m, m_1, n, n_1 are smooth weights such that $m, n \in L^{\infty}(\Omega)$ and $0 < m_1$, $n_1 \leq m^{(\alpha+1)/p} n^{(\beta+1)/q}$.

(B4) The functions h and k satisfy the sign conditions: $th(s,t) \geq 0$, $sk(s,t) \geq 0$ for $(s,t) \in \mathbb{R}^2$ and there exits $\Gamma > 0$ such that

$$h(s,-t) \leq -h(s,t) \quad \text{for } t \geq 0,\ s \in \mathbb{R}$$
$$h(s,t) = \Gamma^{\alpha+\beta+2-p}|s|^{\alpha}|t|^{\beta} t \quad \text{for } t \leq 0,\ s \in \mathbb{R}$$

and

$$k(-s,t) \leq -k(s,t) \quad \text{for } s \geq 0, t \in \mathbb{R}$$
$$k(s,t) = \Gamma^{\alpha+\beta+2-q}|s|^{\alpha} s|t|^{\beta} \quad \text{for } s \leq 0, t \in \mathbb{R}$$

Here and henceforth the Lebesgue norm in $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$ and the usual norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|$. The positive and negative part of a function u are defined respectively as $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. Equalities (and inequalities) between two functions must be understood a.e. in Ω.

Let us recall some results on eigenvalue problems with weight (cf [1, 2]) useful in the sequel for this work. Given $g \in L^{\infty}(\Omega)$, it was known that the eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

admits, an unique positive first eigenvalue $\lambda_1(g,p)$ with a nonnegative eigenfunction. Moreover, this eigenvalue is isolated, simple and as a consequence of its variational characterization one has

$$\lambda_1(g,p)\int_{\Omega} g(x)|u|^p \leq \int_{\Omega} |\nabla u|^p \quad \forall u \in W_0^{1,p}(\Omega).$$

Now we denote by Φ (respectively Ψ) the positive eigenfunction associated with $\lambda_1(m,p)$ (respectively $\lambda_1(n,q)$) normalized by $\int_{\Omega} m(x)|\Phi|^p = 1$ (resp $\int_{\Omega} n(x)|\Psi|^q =$

1). The functions ϕ and ψ belong to $C^{1,\alpha}(\bar{\Omega})$ (see [16, 17]) and by the weak maximum principle (see [18])

$$\frac{\partial \Phi}{\partial \nu} < 0 \quad \text{and} \quad \frac{\partial \Psi}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

where ν is the unit exterior normal. Finally, let us define

$$\Theta := \frac{\inf_{\Omega} k_1(x)}{\sup_{\Omega} k_2(x)},$$

where

$$k_1(x) := [\frac{n_1(x)}{n(x)}]^{(\beta+1)/q}[\frac{\Phi(x)^p}{\Psi(x)^q}]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}, \quad k_2(x) := [\frac{m(x)}{m_1(x)}]^{(\alpha+1)/p}[\frac{\Phi(x)^p}{\Psi(x)^q}]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}.$$

3. A Maximum Principle for system (1.1)

We say that a Maximum Principle holds for system (1.1) if $f \geq 0$ and $g \geq 0$ implies $u \geq 0$ and $v \geq 0$.

By a solution (u, v) of (1.1), we mean a weak solution; i.e., $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\begin{gathered}
\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla w = \int_\Omega [am(x)|u|^{p-2}uw + bm_1(x)h(u,v)w + fw] \\
\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla z = \int_\Omega [dn(x)|v|^{q-2}vz + cn_1(x)k(u,v)z + gz]
\end{gathered} \tag{3.1}$$

for all $(w, z) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Note that by assumptions (B1)–(B4), the integrals in (3.1) are well-defined. We are now ready to state the validity of the Maximum Principle for (1.1).

Theorem 3.1. *Assume (B1)–(B4). Then the Maximum Principle holds for* (1.1) *if*

(C1) $\lambda_1(m, p) > a$,
(C2) $\lambda_1(n, q) > d$,
(C3) $(\lambda_1(m, p) - a)^{(\alpha+1)/p}(\lambda_1(n, q) - d)^{(\beta+1)/q} > b^{(\alpha+1)/p}c^{(\beta+1)/q}$.

Conversely if the Maximum Principle holds, then conditions (C1)–(C4) *are satisfied, where*

(C4) $(\lambda_1(m, p) - a)^{(\alpha+1)/p}(\lambda_1(n, q) - d)^{(\beta+1)/q} > \Theta b^{(\alpha+1)/p}c^{(\beta+1)/q}$

Proof. The proof is partly adapted from [4, 12]

The condition is necessary. Assume that the Maximum Principle holds for system (1.1). If $\lambda_1(m, p) \leq a$ then the functions $f := (a - \lambda_1(m, p))m(x)\Phi^{p-1}$ and $g := 0$ are nonnegative, however $(-\Phi, 0)$ satisfies (1.1), which contradicts the Maximum Principle.

Similarly, if $\lambda_1(n, q) \leq d$ then $f := 0$ and $g := (d - \lambda_1(n, q))n(x)\Psi^{q-1}$ are nonnegative functions and $(0, -\Psi)$ satisfies (1.1), which is a contradiction with the Maximum Principle.

Now, assume that $\lambda_1(m, p) > a$, $\lambda_1(n, q) > d$ and that (C4) does not hold; that is,

(C4') $(\lambda_1(m, p) - a)^{(\alpha+1)/p}(\lambda_1(n, q) - d)^{(\beta+1)/q} < \Theta b^{(\alpha+1)/p}c^{(\beta+1)/q}$

Set

$$A=\Big(\frac{\lambda_1(m,p)-a}{b}\Big)^{(\alpha+1)/p},\quad B=\Big(\frac{\lambda_1(n,q)-d}{c}\Big)^{(\beta+1)/q},$$

then (C4') becomes $AB\le\Theta$ which implies

$$A\Theta_2\le\frac{\Theta_1}{B},\quad \text{where } \Theta_1=\inf_{\Omega}k_1(x),\quad \Theta_2=\sup_{\Omega}k_2(x). \tag{3.2}$$

Hence there exists $\xi>0$ such that

$$A\Theta_2\le\xi\le\frac{\Theta_1}{B}.$$

Let c_1,c_2 be two positive real numbers such that

$$\xi=\Big(\frac{c_2^q\Gamma^q}{c_1^p\Gamma^p}\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}.$$

Using (3.2), (B1) and the above expression of ξ, we have

$$[\lambda_1(m,p)-a]m(x)[c_1\Phi(x)]^{p-1}\le\Gamma^{\alpha+\beta+2-p}bm_1(x)[c_1\Phi(x)]^{\alpha}[c_2\Psi(x)]^{\beta+1}$$

a.e, for $x\in\Omega$ and

$$[\lambda_1(n,q)-d]n(x)[c_2\Psi(x)]^{q-1}\le\Gamma^{\alpha+\beta+2-q}cn_1(x)[c_1\Phi(x)]^{\alpha+1}[c_2\Psi(x)]^{\beta}$$

a.e, for $x\in\Omega$. Furthermore, using the inequalities in (B4), we obtain

$$-[\lambda_1(m,p)-a]m(x)[c_1\Phi(x)]^{p-1}-bm_1(x)h(-c_1\Phi,-c_2\Psi)\ge0\quad\text{a.e, for}\quad x\in\Omega$$

and

$$-[\lambda_1(n,q)-d]n(x)[c_2\Psi(x)]^{q-1}-cn_1(x)k(-c_1\Phi,-c_2\Psi)\ge0\quad\text{a.e, for } x\in\Omega.$$

Hence

$$0\le-[\lambda_1(m,p)-a]m(x)[c_1\Phi(x)]^{p-1}-bm_1(x)h(-c_1\Phi,-c_2\Psi)=f,$$
$$0\le-[\lambda_1(n,q)-d]n(x)[c_2\Psi(x)]^{q-1}-cn_1(x)k(-c_1\Phi,-c_2\Psi)=g$$

are nonnegative functions and $(-c_1\Phi,-c_2\Psi)$ is a solution of (1.1). This is a contradiction with the Maximum Principle.

The condition is sufficient. Assume that the conditions (C1)–(C3) are satisfied. So for $f\ge0$, $g\ge0$, suppose that there exists a solution (u,v) of system (1.1). Multipling the first equation in (1.1) by u^- and the second one by v^- and integrating over Ω we have

$$\int_\Omega|\nabla u^-|^p=a\int_\Omega m(x)|u^-|^p-b\int_\Omega m_1(x)h(u,v)u^--\int_\Omega fu^-$$
$$\int_\Omega|\nabla v^-|^q=d\int_\Omega n(x)|v^-|^q-c\int_\Omega n_1(x)k(u,v)v^--\int_\Omega gv^-.$$

Then, using the sign conditions in (B4) we obtain

$$\int_\Omega|\nabla u^-|^p\le a\int_\Omega m(x)|u^-|^p-b\int_\Omega m_1(x)h(u,-v^-)u^-$$
$$\int_\Omega|\nabla v^-|^q\le d\int_\Omega n(x)|v^-|^q-c\int_\Omega n_1(x)k(-u^-,v)v^-.$$

Recalling the conditions in (B4), we derive that

$$h(u,-v^-)u^-=-\Gamma^{\alpha+\beta+2-p}(u^-)^{\alpha+1}(v^-)^{\beta+1},$$

$$k(-u,v)v^- = -\Gamma^{\alpha+\beta+2-q}(u^-)^{\alpha+1}(v^-)^{\beta+1}$$

and hence

$$\begin{gathered}
\int_\Omega |\nabla u^-|^p \le a\int_\Omega m|u^-|^p + b\Gamma^{\alpha+\beta+2-p}\int_\Omega m_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}\\
\int_\Omega |\nabla v^-|^q \le d\int_\Omega n|v^-|^q + c\Gamma^{\alpha+\beta+2-q}\int_\Omega n_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}.
\end{gathered}$$

Combining the variational characterization of $\lambda_1(m,p)$ and $\lambda_1(n,q)$ with the Hölder inequality and assumption (B3), we have

$$\begin{aligned}
&(\lambda_1(m,p)-a)\int_\Omega m(x)|u^-|^p\\
&\le b\Gamma^{\alpha+\beta+2-p}\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\alpha+1)/q}\Big(\int_\Omega (n(x)|v^-|^q)\Big)^{(\beta+1)/p},
\end{aligned}$$

$$\begin{aligned}
&(\lambda_1(n,q)-d)\int_\Omega n(x)|v^-|^q\\
&\le c\Gamma^{\alpha+\beta+2-q}\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\alpha+1)/q}\Big(\int_\Omega (n(x)|v^-|^q)\Big)^{(\beta+1)/p},
\end{aligned}$$

which implies

$$\begin{gathered}
\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\alpha+1)/p}\Big[(\lambda_1(m,p)-a)\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\beta+1)/q}\\
-b\Gamma^{\alpha+\beta+2-p}\Big(\int_\Omega n(x)|v^-|^q\Big)^{(\beta+1)/q}\Big]\le 0,\\
\Big(\int_\Omega n(x)|v^-|^q\Big)^{(\beta+1)/q}\Big[(\lambda_1(n,q)-d)\Big(\int_\Omega n(x)|v^-|^q\Big)^{(\alpha+1)/p}\\
-c\Gamma^{\alpha+\beta+2-q}\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\alpha+1)/p}\Big]\le 0.
\end{gathered}\tag{3.3}$$

Let us show that $u^- = v^- = 0$.

• If $\int_\Omega m(x)|u^-|^p = 0$ or $\int_\Omega n(x)|v^-|^q = 0$ then, using the fact that $m > 0$, $n > 0$, and (3.3), we obtain $u^- = v^- = 0$, which implies that the Maximum Principle holds.

• If, $\int_\Omega m(x)|u^-|^p \ne 0$ and $\int_\Omega n(x)|v^-|^p \ne 0$, then we have

$$\begin{gathered}
(\lambda_1(m,p)-a)\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\beta+1)/q} \le b\Gamma^{\alpha+\beta+2-p}\Big(\int_\Omega n(x)|v^-|^q\Big)^{(\beta+1)/q}\\
(\lambda_1(n,q)-d)\Big(\int_\Omega n(x)|v^-|^q\Big)^{(\alpha+1)/p} \le c\Gamma^{\alpha+\beta+2-q}\Big(\int_\Omega m(x)|u^-|^p\Big)^{(\alpha+1)/p},
\end{gathered}$$

which implies

$$\begin{aligned}
&\Big(\lambda_1(m,p)-a\Big)^{(\alpha+1)/p}\Big(\int_\Omega m(x)|u^-|^p\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\\
&\le b^{\frac{\alpha+1}{p}}\Gamma^{(\alpha+\beta+2-p)\frac{\alpha+1}{p}}\Big(\int_\Omega n(x)|v^-|^q\Big)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}},
\end{aligned}$$

$$
\begin{aligned}
&(\lambda_1(n,q)-d)^{(\beta+1)/q}\Big(\int_\Omega n(x)|v^-|^q\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}\\
&\le c^{(\beta+1)/q}\Gamma^{(\alpha+\beta+2-q)\frac{\beta+1}{q}}\Big(\int_\Omega m(x)|u^-|^p\Big)^{\frac{\beta+1}{q}\frac{\alpha+1}{p}}.
\end{aligned}
$$

Multiplying the two inequalities above and using the fact that

$$
\begin{aligned}
&(\alpha+\beta+2-p)\frac{\alpha+1}{p}+(\alpha+\beta+2-q)\frac{\beta+1}{q}\\
&=(\alpha+\beta+2)(\frac{\alpha+1}{p}+\frac{\beta+1}{q})-(\alpha+1)-(\beta+1)=0
\end{aligned}
\tag{3.4}
$$

one has

$$
\begin{aligned}
&(\lambda_1(m,p)-a)^{(\alpha+1)/p}(\lambda_1(n,q)-d)^{(\beta+1)/q}\\
&\times\Big[\Big(\int_\Omega m(x)|u^-|^p\Big)\Big(\int_\Omega n(x)|v^-|^q\Big)\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\\
&\le b^{\frac{\alpha+1}{p}}c^{(\beta+1)/q}\Big[\Big(\int_\Omega m(x)|u^-|^p\Big)\Big(\int_\Omega n|v^-|^q\Big)\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}
\end{aligned}
$$

and then

$$
\begin{aligned}
&\big[(\lambda_1(m,p)-a)^{(\alpha+1)/p}(\lambda_1(n,q)-d)^{(\beta+1)/q}-b^{\frac{\alpha+1}{p}}c^{(\beta+1)/q}\big]\\
&\times\Big[\Big(\int_\Omega m(x)|u^-|^p\Big)\Big(\int_\Omega n(x)|v^-|^q\Big)\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}\le 0
\end{aligned}
$$

Since (C1)–(C3) are satisfied, the inequality above is not possible. Consequently $u^-=v^-=0$ and the Maximum Principle holds. $\square$

When $p=q$ and $m=n$, the number θ is equal to 1 and as a consequence of Theorem 3.1, we have the following result.

Corollary 3.2. . *Consider the cooperative system* (1.1) *with* $p=q>1$ *and* $m=n$. *Then the Maximum Principle holds if and only if* (C1)–(C3) *are satisfied.*

Remark 3.3. Our result is reduced to the one in [4] when $h(s,t)=|s|^\alpha|t|^\beta t$, $k(s,t)=|s|^\alpha s|t|^\beta$ and $m=n=1$. When $p=q$ and $\alpha=\beta=p-2$, we obtain the result in [10].

4. Existence of Solutions

We prove in this section that, under some conditions, system (1.1) admits at least one solution.

Theorem 4.1. *Assume* (B1), (B2), (C1), (C2), (C3) *are satisfied. Then for* $f\in L^{p'}(\Omega)$ *and* $g\in L^{q'}(\Omega)$, *system* (1.1) *admits at least one solution in* $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

The proof will be given in several steps. It borrows some ideas from [4, 12], and requires the Lemmas state below.

We choose $r>0$ such that $a+r>0$ and $d+r>0$. Hence (1.1) reads as follows:

$$
\begin{gathered}
-\Delta_p u+rm(x)|u|^{p-2}u=(a+r)m(x)|u|^{p-2}u+bn_1(x)h(u,v)+f\quad\text{in }\Omega\\
-\Delta_q v+rn(x)|v|^{p-2}v=cn_1k(u,v)+(d+r)n(x)|v|^{p-2}v+g\quad\text{in }\Omega\\
u=v=0\quad\text{on }\partial\Omega
\end{gathered}
\tag{4.1}
$$

Following [3] and [4], for $0 < \epsilon < 1$, we introduce the system

$$\begin{gathered} -\Delta_p u_\epsilon + rm(x)|u_\epsilon|^{p-2}u_\epsilon = \hat{h}(x, u_\epsilon, v_\epsilon) + f \quad \text{in } \Omega \\ -\Delta_q v_\epsilon + rn(x)|v_\epsilon|^{q-2}v_\epsilon = \hat{k}(x, u_\epsilon, v_\epsilon) + g \quad \text{in } \Omega \\ u_\epsilon = v_\epsilon = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.2}$$

where

$$\begin{gathered} \hat{h}(x, s, t) = (a + r)m(x)|s|^{p-2}s(1 + \epsilon^{\frac{1}{p}}|s|^{p-1})^{-1} + bm_1(x)h(s, t)(1 + \epsilon|h(s, t)|)^{-1}, \\ \hat{k}(x, s, t) = (d + r)n(x)|t|^{p-2}t(1 + \epsilon^{1/q}|t|^{q-1})^{-1} + cn_1k(s, t)(1 + \epsilon|k(s, t)|)^{-1} \end{gathered}$$

Lemma 4.2. *System* (4.2) *has a solution in* $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$

Proof. Let $\epsilon > 0$ be fixed

- Construction of sub-solution and super-solution for system

$$\begin{gathered} -\Delta_p u + rm(x)|u|^{p-2}u = \hat{h}(x, u, v) + f \quad \text{in } \Omega \\ -\Delta_q v + rn(x)|v|^{p-2}v = \hat{k}(x, u, v) + g \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{4.3}$$

From (B3), the functions $\hat{h}$ and $\hat{k}$ are bounded; that is, there exists a positive constant M such that

$$|\hat{h}(x, u, v)| < M, \quad |\hat{k}(x, u, v)| < M \quad \forall (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$$

Let $u^0 \in W_0^{1,p}(\Omega)$ (respectively $v^0 \in W_0^{1,q}(\Omega)$) be a solution of

$$\begin{gathered} -\Delta_p u^0 + rm(x)|u^0|^{p-2}u^0 = M + f \\ (\text{resp. } -\Delta_p v^0 + rn(x)|v^0|^{q-2}v^0 = M + g) \end{gathered}$$

and $u_0 \in W_0^{1,p}(\Omega)$ (resp $v_0 \in W_0^{1,q}(\Omega)$) be a solution of equation

$$\Delta_p u_0 + rm(x)|u_0|^{p-2}u_0 = -M + f(\text{resp} - \Delta_p v_0 + rn(x)|v^0|^{q-2}v_0 = -M + g)$$

The existence of u_0, u^0, v_0, v^0 is proved in [14]. Moreover we have

$$\begin{gathered} -\Delta_p u_0 + rm(x)|u_0|^{p-2}u_0 - \hat{h}(x, u_0, v) - f \le 0 \quad \forall v \in [v_0, v^0], \\ -\Delta_p u^0 + rm(x)|u^0|^{p-2}u^0 - \hat{h}(x, u^0, v) - f \ge 0 \quad \forall v \in [v_0, v^0], \\ -\Delta_q v_0 + rn(x)|v_0|^{q-2}v_0 - \hat{k}(x, u, v_0) - g \le 0 \quad \forall u \in [u_0, u^0], \\ -\Delta_q v^0 + rn(x)|v^0|^{q-2}v^0 - \hat{k}(x, u, v^0) - g \ge 0 \quad \forall u \in [u_0, u^0] \end{gathered}$$

So (u_0, u^0) and (v_0, v^0) are sub-super solutions of (4.3).

- Let $K = [u_0, u^0] \times [v_0, v^0]$ and let $T : (u, v) \mapsto (w, z)$ the operator such that

$$\begin{gathered} -\Delta_p w + rm(x)|w|^{p-2}w = \hat{h}(x, u, v) + f \quad \text{in } \Omega \\ -\Delta_q z + rn(x)|z|^{q-2}z = \hat{k}(x, u, v) + g \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.4}$$

- Let us prove that $T(K) \subset K$. If $(u, v) \in K$, then

$$-(\Delta_p w - \Delta_p \xi^0) + rm(x)(|w|^{p-2}w - |\xi^0|^{p-2}\xi^0) = [\hat{h}(x, u, v) - M]) \tag{4.5}$$

Taking $(w - \xi^0)^+$ as test function in (4.5), we have

$$\int_\Omega (|\nabla w|^{p-2}\nabla w - |\nabla \xi^0|^{p-2}\nabla \xi^0)\nabla (w - \xi^0)^+$$

$$
\begin{aligned}
&+ r\int_\Omega m(x)(|w|^{p-2}w - |\xi^0|^{p-2}\xi^0)(w-\xi^0)^+ \\
&= \int_\Omega [(h(x,u,v) - M)](w-\xi^0)^+ \leq 0.
\end{aligned}
$$

Since the weight m is positive, by the monotonicity of the function $s \mapsto |s|^{p-2}s$ and that of the p-Laplacian, we deduce that the last integral equal zero and then $(w-\xi^0)^+ = 0$; that is, $w \leq \xi^0$. Similarly we obtain $\xi_0 \leq w$ by taking $(w-\xi_0)^-$ as test function in (4.5). So we have $\xi_0 \leq w \leq \xi^0$ and $\eta_0 \leq z \leq \eta^0$ and the step is complete.

• To show that T is completely continuous we need the following Lemma.

Lemma 4.3. *If $(u_n, v_n) \to (u, v)$ in $L^p(\Omega) \times L^q(\Omega)$ as $n \to \infty$, then*

(1) *$X_n = m(x)\frac{|u_n|^{p-2}u_n}{1+|\epsilon^{1/p}u_n|^{p-1}}$ converges to $X = m(x)\frac{|u|^{p-2}u}{1+|\epsilon^{1/p}u|^{p-1}}$ in $L^{p'}(\Omega)$ as $n \to \infty$.*

(2) *$Y_n = m_1(x)\frac{h(u_n,v_n)}{1+\epsilon|h(u_n,v_n)|}$ converges to $Y = m_1(x)\frac{h(u,v)}{1+\epsilon|h(u,v)|}$ in $L^{q'}(\Omega)$ as $n \to \infty$.*

Proof. Since $u_n \to u$ in $L^p(\Omega)$, there exists a subsequence still denoted (u_n) such that

$$
\begin{gathered}
u_n(x) \to u(x) \quad \text{a.e. on } \Omega, \\
|u_n(x)| \leq \eta(x) \quad \text{a.e. on } \Omega \text{ with } \eta \in L^p(\Omega)
\end{gathered} \tag{4.6}
$$

Let

$$
X_n = m(x)\frac{|u_n|^{p-2}u_n}{1+|\epsilon^{1/p}u_n|^{p-1}}\,.
$$

Then

$$
\begin{gathered}
X_n(x) \to X(x) = m(x)\frac{|u(x)|^{p-2}u(x)}{1+|\epsilon^{1/p}u(x)|^{p-1}} \quad \text{a.e. on } \Omega, \\
|X_n| \leq \|m\|_\infty |u_n|^{p-1} \leq \|m\|_\infty |\eta|^{p-1}
\end{gathered}
$$

in $L^{p'}(\Omega)$. Thus, from Lebesgue's dominated convergence theorem one has

$$
X_n \to X = m(x)\frac{|u|^{p-2}u}{1+|\epsilon^{1/p}u|^{p-1}} \quad \text{in } L^{p'}(\Omega) \quad \text{as } n \to \infty
$$

So (1) is proved.

Moreover, since $v_n \to v$ in $L^q(\Omega)$, there exists a subsequence still denoted (v_n) such that

$$
\begin{gathered}
v_n(x) \to v(x) \quad \text{a.e on}\Omega, \\
|v_n(x)| \leq \zeta(x) \quad \text{a.e on } \Omega \text{ with}\zeta \in L^q(\Omega)
\end{gathered} \tag{4.7}
$$

Using (B4), one has

$$
|Y_n| \leq \|m_1\|_\infty |h(u_n, v_n)| \leq \Gamma^{\alpha+\beta+2-p}\|m_1\|_\infty |\eta|^\alpha |\zeta|^{\beta+1}
$$

in $L^{p'}(\Omega)$, since $\frac{\alpha}{p} + \frac{\beta+1}{q} = \frac{1}{p'}$. Let

$$
Y_n = m_1(x)\frac{h(u_n,v_n)}{1+\epsilon|h(u_n,v_n)|}
$$

Then

$$
Y_n(x) \to Y(x) = m_1(x)\frac{h(u(x),v(x))}{1+\epsilon|h(u(x),v(x))|} \quad \text{a.e in } \Omega
$$

So, we can apply the Lebesgue's dominated convergence theorem and then we obtain

$$Y_n(x) \to Y(x) = m_1(x)\frac{h(u(x),v(x))}{1+\epsilon|h(u(x),v(x))|} \quad \text{in } L^{p'}(\Omega).$$

as $n \to \infty$. □

Remark 4.4. We can similarly prove that, as $n \to \infty$,

$$n(x)|v_n|^{q-2}v_n(1+|\epsilon^{1/q}v_n|^{q-1})^{-1} \to n(x)|v|^{q-2}v(1+|\epsilon^{1/q}v|^{q-1})^{-1} \quad \text{in} L^{q'}(\Omega),$$
$$n_1(x)k(u_n,v_n)(1+\epsilon|k(u_n,v_n)|)^{-1} \to n_1(x)k(u,v)(1+\epsilon|k(u,v)|)^{-1} \quad \text{in } L^{q'}(\Omega)$$

• To complete the continuity of T. Let us consider a sequence (u_n,v_n) such that $(u_n,v_n) \to (u,v)$ in $L^p(\Omega)\times L^q(\Omega)$ as $n\to\infty$. We will prove that $(w_n,z_n) = T(u_n,v_n) \to (w,z) = T(u,v)$. Note that $(w_n,z_n) = T(u_n,v_n)$ if only if

$$\begin{aligned}
&(-\Delta_p w_n + rm(x)|w_n|^{p-2}w_n) - (-\Delta_p w + rm(x)|w|^{p-2}w)\\
&= \hat{h}(x,u_n,v_n) - \hat{h}(x,u,v)\\
&= (a+r)[m(x)\frac{|u_n|^{p-2}u_n}{1+|\epsilon^{1/p}u_n|^{p-1}} - m(x)\frac{|u|^{p-2}u}{1+|\epsilon^{1/p}u|^{p-1}}]\\
&\quad + bm_1(x)[\frac{h(u_n,v_n)}{1+\epsilon|h(u_n,v_n|} - \frac{h(u,v)}{1+\epsilon|h(u,v)|}]\\
&= (a+r)(X_n - X) + b(Y_n - Y)
\end{aligned} \tag{4.8}$$

Multiplying by $(w_n - w)$ and integrating over Ω one has

$$\begin{aligned}
&\int_\Omega (|\nabla w_n|^{p-2}\nabla w_n - |\nabla w|^{p-2}\nabla w)\nabla(w_n - w)\\
&+ r\int_\Omega m(x)(|w_n|^{p-2}w_n - |w|^{p-2}w).(w_n - w)\\
&= (a+r)\int_\Omega (X_n - X)(w_n - w) + b\int_\Omega (Y_n - Y)(w_n - w)\\
&\le (a+r)\Big(\int_\Omega |X_n - X|^{p'}\Big)^{1/p'}\Big(\int_\Omega |w_n - w|^p)^{1/p}\\
&\quad + b\Big(\int_\Omega |Y_n - Y|^{p'}\Big)^{1/p'}\Big(\int_\Omega |w_n - w|^p\Big)^{1/p}
\end{aligned}$$

Combining Lemma 4.3 and the inequality

$$\|x-y\|^p \le c[(\|x\|^{p-2}x - \|y\|^{p-2}y)(x-y)]^{s/2}[\|x\|^p + \|y\|^p]^{1-s/2}, \tag{4.9}$$

where $x,y\in\mathbb{R}^{\mathbb{N}}$, $c = c(p) > 0$ and $s = 2$ if $p \ge 2$, $s = p$ if $1 < p < 2$ (cf. e.g. [13]), we can conclude that $w_n \to w$ in $W_0^{1,p}(\Omega)$ when $n\to\infty$.

Similarly we show that $z_n \to z$ in $W_0^{1,q}(\Omega)$ as $n \to \infty$ and then, the continuity of T is proved

• Compactness of the operator T. Suppose (u_n,v_n) a bounded sequence in K and let $(w_n,z_n) = T(u_n,v_n)$. Multiplying the first equality in the definition of T by w_n and integrating by parts on Ω, we notice the boundness of w_n in $W_0^{1,p}(\Omega)$ and then we use the compact imbedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ to conclude.

The same argument is valid with (z_n) in $L^q(\Omega)$. Thus T is completely continuous. Since the set K is convex, bounded and closed in $L^p(\Omega)\times L^q(\Omega)$, the Schauder's fixed point theorem, yields existence of a fixed point for T and accordingly the existence of solution of system (4.2). So Lemma 4.2 is proved. □

Proof of Theorem 4.1. The proof will be given in three steps.

Step 1. Let us first prove that (u_ϵ, v_ϵ) is bounded in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. Indeed assume that $\|u_\epsilon\| \to \infty$ or $\|v_\epsilon\| \to \infty$ as $\epsilon \to 0$. Let

$$t_\epsilon = \max\{\|u_\epsilon\|; \|v_\epsilon\|\}, \quad w_\epsilon = \frac{u_\epsilon}{t_\epsilon^{1/p}}, \quad z_\epsilon = \frac{v_\epsilon}{t_\epsilon^{1/q}}$$

We have $\|w_\epsilon\| \le 1$ and $\|z_\epsilon\| \le 1$ with either $\|w_\epsilon\| = 1$ or $\|z_\epsilon\| = 1$. Dividing the first equation in (4.2) by $(t_\epsilon)^{\frac{1}{p'}}$ we obtain

$$\begin{aligned}
&-\Delta_p w_\epsilon + rm(x)|w_\epsilon|^{p-2}w_\epsilon\\
&= (a+r)m(x)|w_\epsilon|^{p-2}w_\epsilon(1+|\epsilon^{\frac{1}{p}}u_\epsilon|^{p-1})^{-1}\\
&\quad + t_\epsilon^{-1/p'}bm_1(x)h({t_\epsilon}^{1/p}w_\epsilon, t_\epsilon^{\frac{1}{q}}z_\epsilon)(1+\epsilon|h(u_\epsilon,v_\epsilon)|)^{-1} + t_\epsilon^{-1/p'}f.
\end{aligned}$$

Similarly dividing the second equation in (4.2) by $(t_\epsilon)^{1/q'}$ we obtain

$$\begin{aligned}
&-\Delta_q z_\epsilon + rn(x)|z_\epsilon|^{q-2}w_\epsilon\\
&= (d+r)n(x)|w_\epsilon|^{\alpha}w_\epsilon(1+|\epsilon^{\frac{1}{p}}u_\epsilon|^{\alpha+1})^{-1}\\
&\quad + t_\epsilon^{-1/q'}cn_1(x)k({t_\epsilon}^{1/p}w_\epsilon, t_\epsilon^{\frac{1}{q}}z_\epsilon)(1+\epsilon|k(u_\epsilon,v_\epsilon)|)^{-1} + t_\epsilon^{-1/q'}g
\end{aligned}$$

Testing the first equation in the system above by w_ϵ and using (B4), we obtain

$$\begin{aligned}
\int_\Omega |\nabla w_\epsilon|^p &\le a\int_\Omega m(x)|w_\epsilon|^p + b\Gamma^{\alpha+\beta+2-p}\int_\Omega m(x)^{\frac{\alpha+1}{p}}|w_\epsilon|^{\alpha+1}n(x)^{(\beta+1)/q}|z_\epsilon|^{\beta+1}\\
&\quad + (t_\epsilon)^{\frac{-1}{p'}}\int_\Omega |f||w_\epsilon|.
\end{aligned}$$

which, by the Hölder inequality, implies

$$\begin{aligned}
\int_\Omega |\nabla w_\epsilon|^p &\le a\int_\Omega m|w_\epsilon|^p + b\Gamma^{\alpha+\beta+2-p}\Big(\int_\Omega m|w_\epsilon|^p\Big)^{(\alpha+1)/p}\Big(\int_\Omega n|w_\epsilon|^q\Big)^{(\beta+1)/q}\\
&\quad + (t_\epsilon)^{-1/p'}\|f\|_{(p^*)'}\|z_\epsilon\|_{p^*}
\end{aligned}$$

Using the variational characterization of $\lambda_1(m,p)$ and the imbedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$. one has

$$\begin{aligned}
\|w_\epsilon\|^p &\le \frac{a}{\lambda_1(m,p)}\|w_\epsilon\|^p + b\Gamma^{\alpha+\beta+2-p}\frac{\|w_\epsilon\|^{\alpha+1}}{[\lambda_1(m,p)]^{(\alpha+1)/p}}\frac{\|z_\epsilon\|^{\beta+1}}{[\lambda_1(n,q)]^{(\beta+1)/q}}\\
&\quad + c(p,\Omega)(t_\epsilon)^{\frac{-1}{p'}}\|f\|_{(p^*)'}\|z_\epsilon\|,
\end{aligned}$$

where $c(p,\Omega)$ is the imbedding constant. So, one gets

$$\begin{aligned}
&(\lambda_1(m,p)-a)\frac{(\|w_\epsilon\|^p)^{(\beta+1)/q}}{\lambda_1(m,p)}\\
&\le \frac{b\Gamma^{\alpha+\beta+2-p}(\|z_\epsilon\|^q)^{(\beta+1)/q}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}}\lambda_1(n,q)^{(\beta+1)/q}} + (t_\epsilon)^{-1/p'}\Big(\int_\Omega |f|^{p'}\Big)^{1/p'}\Big(\int_\Omega |\nabla w_\epsilon|^p\Big)^{-\alpha/p},
\end{aligned} \tag{4.10}$$

and accordingly

$$
\begin{aligned}
&(\lambda_1(m,p)-a)^{(\alpha+1)/p}\frac{(\limsup\|w_\epsilon\|^p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}}}\\
&\le b^{\frac{\alpha+1}{p}}\frac{\Gamma^{(\alpha+\beta+2-p)(\frac{\alpha+1}{p})}(\limsup\|z_\epsilon|^q)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{(\frac{\alpha+1}{p})^2}\lambda_1(n,q)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}.
\end{aligned}
\tag{4.11}
$$

In a similar way, we obtain

$$
\begin{aligned}
&(\lambda_1(n,q)-d)^{(\beta+1)/q}\frac{(\limsup\|z_\epsilon\|^q)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(n,q)^{(\beta+1)/q}}\\
&\le c^{(\beta+1)/q}\frac{\Gamma^{(\alpha+\beta+2-q)(\frac{\beta+1}{q})}(\limsup\|w_\epsilon\|^p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(n,q)^{(\frac{\beta+1}{q})^2}\lambda_1(m,p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}.
\end{aligned}
\tag{4.12}
$$

Multiplying term by term the expressions in (4.11) and (4.12), and using (3.4), we obtain

$$
\begin{aligned}
&[(\lambda_1(m,p)-a)^{\frac{\alpha+1}{p}}(\lambda_1(n,q)-d)^{(\beta+1)/q}-b^{\frac{\alpha+1}{p}}c^{(\beta+1)/q}]\\
&\times\frac{(\limsup\|w_\epsilon\|^p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}(\limsup\|z_\epsilon\|^p)^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}}\lambda_1(n,q)^{(\beta+1)/q}}\le 0.
\end{aligned}
$$

Since conditions (C1)–(C3) hold, one has

$$\limsup\|w_\epsilon\|^p=\limsup\|z_\epsilon\|^p=0.$$

This yields a contradiction since $\|w_\epsilon\|=1$ or $\|z_\epsilon\|=1$, and consequently (u_ϵ,v_ϵ) is bounded in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Step 2. $(\epsilon^{1/p}u_\epsilon;\epsilon^{1/q}v_\epsilon)$ converges strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ when ϵ approaches 0. It is obvious due to the boundness of (u_ϵ,v_ϵ) in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Step 3. Let us prove that (u_ϵ,v_ϵ) converges strongly in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ when ϵ approaches 0. Since (u_ϵ,v_ϵ) is bounded in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ we can extract a subsequence still denoted (u_ϵ,v_ϵ) which converges weakly to (u_0,v_0) in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ and strongly in $L^p(\Omega)\times L^q(\Omega)$ when $\epsilon\to 0$.

As $u_\epsilon\to u_0$ in $L^p(\Omega)$, $v_\epsilon\to v_0$ in $L^q(\Omega)$ when $\epsilon\to 0$ then there exists a function $\eta\in L^p(\Omega)$, $\zeta\in L^q(\Omega)$ such that

$$
\begin{gathered}
u_\epsilon(x)\to u_0(x)\quad\text{a.e. as }\epsilon\to 0\text{ and }|u_\epsilon|\le\eta\text{ in }L^p(\Omega).\\
v_\epsilon(x)\to v_0(x)\quad\text{a.e. as }\epsilon\to 0\text{ and }|v_\epsilon|\le\zeta\text{ in }L^q(\Omega).
\end{gathered}
$$

Hence we have

$$
\begin{gathered}
||u_\epsilon|^{p-2}u_\epsilon(x)(1+|\epsilon^{\frac{1}{p}}u_\epsilon|^{p-1})^{-1}|\le|u_\epsilon|^{p-1}\le\eta^{p-1}\quad\text{in }L^{p'}(\Omega),\\
||v_\epsilon|^{p-2}v_\epsilon(1+|\epsilon^{1/q}v_\epsilon|^{q-1})^{-1}|\le|v_\epsilon|^{q-1}\le\zeta^{q-1}\quad\text{in }L^{q'}(\Omega).
\end{gathered}
$$

Since $(\epsilon^{1/p}u_\epsilon)\to 0$, $(\epsilon^{1/q}v_\epsilon)\to 0$ a.e. when $\epsilon\to 0$, one can deduce that

$$
\begin{gathered}
|u_\epsilon(x)|^{p-2}u_\epsilon(x)(1+|\epsilon^{\frac{1}{p}}u_\epsilon(x)|^{p-1})^{-1}\to|u_0(x)|^{p-2}u_0(x),\\
|v_\epsilon(x)|^{q-2}u_\epsilon(x)(1+|\epsilon^{1/q}v_\epsilon(x)|^{q-1})^{-1}\to|v_0(x)|^{q-2}v_0(x),
\end{gathered}
$$

a.e in Ω as $\epsilon\to 0$.

Applying the dominated convergence theorem we obtain

$$|u_\epsilon|^{p-2}u_\epsilon(1+|\epsilon^{1/p}u_\epsilon|^{p-1})^{-1}\to|u_0|^{p-2}u_0,$$

$$|v_\epsilon|^{q-2}v_\epsilon(1+|\epsilon^{1/q}v_\epsilon|^{q-1})^{-1} \to |v_0|^{q-2}v_0$$

in $L^{p'}(\Omega)$ as $\epsilon \to 0$. Similarly we have

$$\begin{aligned}
\frac{|h(u_\epsilon,v_\epsilon)|}{1+\epsilon|h(u_\epsilon,v_\epsilon)|} &\le \Gamma^{\alpha+\beta+2-p}|\eta|^{\alpha}|\zeta|^{\beta+1} \quad \text{in } L^{p'}(\Omega) \text{ since } \frac{\alpha}{p}+\frac{\beta+1}{q}=\frac{1}{p'},\\
\frac{|k(u_\epsilon,v_\epsilon)|}{1+\epsilon|k(u_\epsilon,v_\epsilon)|} &\le \Gamma^{\alpha+\beta+2-q}|\eta|^{\alpha+1}|\zeta|^{\beta} \quad \text{in } L^{q'}(\Omega) \text{ since } \frac{\alpha+1}{p}+\frac{\beta}{q}=\frac{1}{q'},
\end{aligned}$$

and

$$\begin{aligned}
\frac{h(u_\epsilon(x),v_\epsilon(x))}{1+\epsilon|h(u_\epsilon(x),v_\epsilon(x))|} &\to h(u_0(x),v_0(x)) \quad \text{a.e. as } \epsilon\to 0,\\
\frac{k(u_\epsilon(x),v_\epsilon(x))}{1+\epsilon|k(u_\epsilon(x),v_\epsilon(x))|} &\to k(u_0(x),v_0(x)) \quad \text{a.e. as } \epsilon\to 0.
\end{aligned}$$

Again using the dominated converge theorem we have

$$\begin{aligned}
\frac{h(u_\epsilon,v_\epsilon)}{1+\epsilon|h(u_\epsilon,v_\epsilon)|} &\to h(u_0,v_0) \quad \text{in } L^{p'}(\Omega) \text{ as } \epsilon\to 0,\\
\frac{k(u_\epsilon,v_\epsilon)}{1+\epsilon|k(u_\epsilon,v_\epsilon)|} &\to k(u_0,v_0) \quad \text{in } L^{q'}(\Omega) \text{ as } \epsilon\to 0.
\end{aligned}$$

Now, we conclude the strong convergence of (u_ϵ,v_ϵ) in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ by applying (4.9).

Finally, using a classical result if nonlinear analysis (cf [14]), we obtain

$$\begin{gathered}
-\Delta_p u_0 + rm(x)|u_0|^{p-2}u_0 = (a+r)m(x)|u_0|^{p-2}u_0 + bm_1(x)h(u_0,v_0)+f \quad \text{in } \Omega\\
-\Delta_q v_0 + rn(x)|v_0|^{q-2}v_0 = (d+r)n(x)|v_0|^{q-2}v_0 + cn_1(x)k(u_0,v_0)+g \quad \text{in } \Omega\\
u_0=v_0=0 \quad \text{on } \partial\Omega
\end{gathered}$$

which can be written again as

$$\begin{gathered}
-\Delta_p u_0 + = am(x)|u_0|^{p-2}u_0 + bm_1(x)h(u_0,v_0)+f \quad \text{in } \Omega\\
-\Delta_q v_0 = dn(x)|v_0|^{q-2}v_0 + cn_1(x)k(u_0,v_0)+g \text{in } \Omega\\
u_0=v_0=0 \quad \text{on } \partial\Omega
\end{gathered}$$

This completes the proof. □

Remark 4.5. One has the same results by interchanging the role of h and k in the second part of the assumption (B4), namely

$$\begin{gathered}
h(s,t)=\Gamma^{\alpha+\beta+2-p}|s|^{\alpha}(t)^{\beta+1} \quad \text{for } t\ge 0,\ s\in\mathbb{R}\\
h(s,-t)\le -h(s,t) \quad \text{for } t\le 0,\ s\in\mathbb{R}
\end{gathered}$$

and

$$\begin{gathered}
k(s,t)=\Gamma^{\alpha+\beta+2-q}(s)^{\alpha+1}|t|^{\beta} \quad \text{for } s\ge 0,\ t\in\mathbb{R},\\
k(-s,t)\le -k(s,t) \quad \text{for } s\le 0,\ t\in\mathbb{R}.
\end{gathered}$$

References

[1] A. Anane; *Simplicité et Isolation de la première valeur propre du p-laplacien avec poids*, Comptes rendus Acad.Sc.Paris, 305, (1987), 725-729.

[2] W. Allegretto and Y. X. Huang; *Eigenvalues of the indefinite weight p-Laplacian in weighted spaces*, Funck.Ekvac., 8 (1995), 233-242.

[3] L. Boccardo, J. Fleckinger and F. de Thélin; *Existence of solutions for some nonlinear cooperative systems*, Diff. and Int. Equations,7 (1994), 689-698.

[4] M. Bouchekif, H. Serag and F. de Thélin; *On maximum principle and existence of solutions for some nonlinear elliptic systems*, Rev. Mat. Apl. 16, pp 1-16.

[5] D. G. de Figueiredo and E. Mitidieri; *Maximum Principles for Linear Elliptic Systems.* Quaterno Matematico 177, Dip. Sc. Mat., Univ Trieste, (1988).

[6] D. G. de Figueiredo and E. Mitidieri; *A Maximum Princilpe for an Elliptic System and Applications to Semilinear Problems* , SIAM J. Math. Anal., 17, (1986), 836-849.

[7] D. G. de Figueiredo and E. Mitidieri; *Maximum Principles for Cooperative Elliptic Systems* , Comptes Rendus Acad.Sc.Paris, 310, (1990), 49-52.

[8] J. Fleckinger, J. Hernandez and F. de Thélin; *Principe du Maximum pour un Système Elliptique non Linéaire*, Comptes Rendus Acad. Sc. Paris, 314, (1992), 665-668.

[9] J. Fleckinger, J. Hernnadez and F. de Thélin; *On maximum principle and existence of postive solutions for some cooperative elliptic systems*, J. Diff. and Int. Equa., vol. 8 (1995), 69-85.

[10] J. Fleckinger and H. Serag; *On maximum principle and existence of solutions for elliptic systems on* $\mathbb{R}^N$, J.Egypt. Math.Soc., vol. 2 (1994), pp 45-51.

[11] J. Fleckinger and H. Serag; *Semilinear cooperative elliptic systems on* $\mathbb{R}^N$, Rend. di Mat., vol. seri VII,15 Roma (1995), 89-108.

[12] L. Leadi and A. Marcos; *Maximum principle and existence results for elliptic system on* $\mathbb{R}^N$, Electron. J. Diff. Equa., Vol. 2010(2010), No. 60, 1-13.

[13] P. Lindqvist; *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc., 109 (1990), 157-166. Addendum in Proc. Amer.Math.Soc., 116 (1992), 583-584.

[14] J. L. Lions; *Quelques methodes de resolutions des problèmes aux limites non linéaires* , Dunod, Paris, 1969.

[15] H. M. Serag and E. A. El-Zahrani; *Maximum principle and existence of positive solutions for nonlinear systems on* $\mathbb{R}^N$, Electron. J. Diff. Equa. (EJDE), Vol. 2005 (2005), No. 85, 1-12.

[16] J. Serrin; *Local behavior of solutions of quasilinear equations*, Acta Math., 111 (1964), 247-302.

[17] P. Tolksdorf; *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equat., 51 (1984), 126-150.

[18] J. L. Vázquez; *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. and Optimization 12 (1984) 191-202.

Liamidi Leadi

Institut de Mathématiques et de Sciences Physiques, Université d'Abomey Calavi, 01 BP: 613 Porto-Novo, Bénin (West Africa)

E-mail address: leadiare@imsp-uac.org, leadiare@yahoo.com

Aboubacar Marcos

Institut de Mathématiques et de Sciences Physiques, Université d'Abomey Calavi, 01 BP: 613 Porto-Novo, Bénin (West Africa)

E-mail address: abmarcos@imsp-uac.org