Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 83, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE SOLUTIONS FOR A NONLINEAR n-TH ORDER m-POINT BOUNDARY-VALUE PROBLEM

JIEHUA ZHANG, YANPING GUO, YUDE JI

Abstract. Using the Leggett-Williams fixed point theorem in cones, we prove the existence of at least three positive solutions to the nonlinear n-th order m-point boundary-value problem

$$\Delta^n u(k) + a(k)f(k,u) = 0, \quad k \in \{0, N\},$$

$$u(0) = 0,\ \Delta u(0) = 0, \dots, \Delta^{n-2}u(0) = 0, \quad u(N+n) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i).$$

1. Introduction

Multi-point boundary value problems arise in a variety of areas of applied mathematics and physics. The solvability of two-point difference and multi-point differential boundary value problems has been studied extensively in the literature in recent years; see [1, 2, 3, 4, 5, 6, 8, 9, 10, 12] and their references. Guo [8] used Leggett-Williams fixed point theorem to obtain the existence of at least three positive solutions for the second-order m-point boundary value problem

$$u''(t) + f(t,u) = 0, \quad 0 \le t \le 1,$$

$$u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = 0,$$

where $k_i > 0$ $(i = 1, 2, \dots, m-2)$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$, $0 < \sum_{i=1}^{m-2} k_i\xi_i < 1$ are given, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

Recently, Eloe and Ahmad [7] discussed the existence of at least one positive solution for the nonlinear n-th order three-point boundary value problem

$$u^{(n)}(t) + a(t)f(u) = 0, \quad t \in (0,1),$$

$$u(0) = 0,\ u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u(1) = \alpha u(\eta),$$

2000 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Boundary value problem; positive solution; fixed point theorem; Green's function.

Submitted March 12, 2010. Published June 24, 2011.
Supported by grants: 10971045 the Natural Science Foundation of China, and A2009000664 from the Natural Science Foundation of Hebei Province.

where $n \geq 2, 0 < \eta < 1, 0 < \alpha\eta^{n-1} < 1, f(t) \in C([0,1],[0,\infty))$ is either superlinear or sublinear. The method they used is the Krasnoselskii's fixed point theorem in cones.

Motivated by the results [7, 11], in this paper, we investigate the existence of positive solutions for the following nonlinear n-th order m-point boundary value problem

$$\Delta^n u(k) + a(k) f(k,u) = 0, \quad k \in \{0, N\}, \tag{1.1}$$

$$u(0) = 0, \quad \Delta u(0) = 0, \dots, \Delta^{n-2} u(0) = 0, \quad u(N+n) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.2}$$

where $n \geq 2$, $\alpha_i \geq 0$ for $i = 1, 2, \dots, m-3$, and $\alpha_{m-2} > 0$, ξ_i is an integer, satisfying $n = \xi_0 \leq \xi_1 < \xi_2 < \dots < \xi_{m-2} < \xi_{m-1} = N + n$,

$$0 < \sum_{i=1}^{m-2} \alpha_i (\sum_{j=1}^{n-1} \prod_{l=1}^{j} (\xi_i - n + l) + 1) < \sum_{j=1}^{n-1} \prod_{l=1}^{j} (N + l) + 1.$$

We denote $\{i, j\} = \{k \in \mathbb{N} : i \leq k \leq j\}$ and assume that:

- (A1) $f : \{0, N\} \times [0, \infty) \to [0, \infty)$ is continuous;
- (A2) $a(k) \geq 0$, for $k \in \{0, N\}$ and there exists $k_0 \in \{\xi_{m-2}, N\}$ such that $a(k_0) > 0$.

This article is organized as follows. In Section 2, we present some preliminaries that will be used to prove our main results. In Section 3, using the Leggett-Williams fixed point theorem, we show that (1.1)–(1.2) has at least three positive solutions.

2. Preliminaries

In this section, we present some notation and lemmas, which are fundamental in the proof of our main results.

Let E be a Banach space over $\mathbb{R}$. A nonempty convex closed set $K \subset E$ is said to be a cone provided that

- (i) $au \in K$ for all $u \in K$ and all $a \geq 0$;
- (ii) $u, -u \in K$ implies $u = 0$.

A map α is said to be a nonnegative continuous concave functional on K provided that $\alpha : K \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$. Similarly, we say a map β is a nonnegative continuous convex functional on K provided that $\beta : K \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Let α be a nonnegative continuous concave functional on K. Then, for nonnegative real numbers $0 < b < d$ and c, we define the convex sets

$$P_c = \{x \in K | \|x\| < c\},$$
$$P(\alpha, b, d) = \{x \in K | b \leq \alpha(x), \|x\| \leq d\}.$$

Theorem 2.1 (Leggett-Williams fixed point theorem). *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \le \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \le c$ such that*

- (C1) $\{x \in P(\alpha, b, d) | \alpha(x) > b\} \neq \emptyset$, *and* $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, d)$,
- (C2) $\|Ax\| < a$ *for* $\|x\| \le a$, *and*
- (C3) $\alpha(Ax) > b$ *for* $x \in P(\alpha, b, c)$, *with* $\|Ax\| > d$.

Then A has at least three fixed point x_1, x_2 and x_3 such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$ with $\alpha(x_3) < b$.

Lemma 2.2 ([12]). *Assume that u satisfies the difference inequality $\Delta^n u(k) \le 0$, $k \in \{0, N\}$, and the homogeneous boundary conditions, $u(0) = \cdots = u(n-2) = 0$, $u(N+n) = 0$. Then, $u(k) \ge 0$, $k \in \{0, N+n\}$.*

For a finite or infinite sequence $u(0), u(1), \dots$, the value $k = 0$ is a node for the sequence if $u(0) = 0$, and a value $k > 0$ is a node for u if $u(k) = 0$ or $u(k-1)u(k) < 0$. The following lemma, obtained in [12], is a discrete analogue of Rolle's Theorem.

Lemma 2.3. *Suppose that the finite sequence $u(0), \dots, u(j)$ has N_j nodes and the sequence $\Delta u(0), \dots, \Delta u(j-1)$ has M_j nodes. Then, $M_j \ge N_j - 1$.*

Theorem 2.4. *Assume $n \le \xi_1 < \xi_2 < \cdots < \xi_{m-2} < N + n$,*

$$0 < \sum_{i=1}^{m-2} \alpha_i \Big(\sum_{j=1}^{n-1} \prod_{l=1}^{j} (\xi_i - n + l) + 1 \Big) < \sum_{j=1}^{n-1} \prod_{l=1}^{j} (N + l) + 1,$$

and $y(k) \ge 0$, $k \in \{0, N\}$. Then, the difference equation

$$\Delta^n u(k) + y(k) = 0, \quad k \in \{0, N\}, \tag{2.1}$$

coupled with the boundary conditions (1.2), has a unique solution

$$u(k) = \begin{cases} 0, & \text{for } k \in \{0, n-2\}, \\ \frac{\delta}{M(n-1)!}, & \text{for } k = n-1, \\ -\frac{1}{(n-1)!} \sum_{s=0}^{k-n} y(s) \prod_{j=1}^{n-1} (k - n + j - s) & \\ +\frac{\delta}{M(n-1)!} \sigma, & \text{for } k \in \{n, N+n\}, \end{cases} \tag{2.2}$$

where

$$M = \Big(\sum_{j=1}^{n-1} \prod_{l=1}^{j} (N + l) + 1 \Big) - \sum_{i=1}^{m-2} \alpha_i \Big(\sum_{j=1}^{n-1} \prod_{l=1}^{j} (\xi_i - n + l) + 1 \Big),$$

$$\delta = \sum_{s=0}^{N} y(s) \prod_{j=1}^{n-1} (N + j - s) - \sum_{i=1}^{m-2} \alpha_i \sum_{s=0}^{\xi_i - n} y(s) \prod_{j=1}^{n-1} (\xi_i - n + j - s),$$

$$\sigma = \sum_{j=1}^{n-1} \prod_{l=1}^{j} (k - n + l) + 1.$$

Proof. Let $\Delta^{n-1} u(0) = A$, since $u(0) = 0$, $\Delta u(0) = 0, \dots, \Delta^{n-2} u(0) = 0$, it follows that $\Delta^{n-z} u(z-1) = A$, for $z \in \{1, n-1\}$, $u(0) = \cdots = u(n-2) = 0$, $u(n-1) = A$.

Summing (2.1) from 0 to $k-1$, one gets $\Delta^{n-1}u(k) = -\sum_{s=0}^{k-1} y(s) + A$. Again summing the equality above, from 1 to $k-1$, it follows that

$$\Delta^{n-2}u(k) = -\sum_{s_1=0}^{k-2}\sum_{s=0}^{s_1} y(s) + (k-1)A + A.$$

Repeat the summing in this way in proper order, we get

$$u(k) = -\sum_{s_{n-1}=0}^{k-n} \cdots \sum_{s=0}^{s_1} y(s) + A\sigma.$$

It can be expressed that

$$\begin{aligned}\sum_{s_1=0}^{k-2}\sum_{s=0}^{s_1} y(s) &= \sum_{s=0}^{0} y(s) + \sum_{s=0}^{1} y(s) + \cdots + \sum_{s=0}^{s_2} y(s)\\ &= (s_2+1)y(0) + s_2 y(1.1) + \cdots + y(s_2)\\ &= \sum_{s=0}^{s_2}(s_2+1-s)y(s),\end{aligned}$$

by repeating this process coupled with the mathematical induction, we have

$$\sum_{s_{n-1}=0}^{k-n} \cdots \sum_{s=0}^{s_1} y(s) = \frac{1}{(n-1)!}\sum_{s=0}^{k-n} y(s)\prod_{j=1}^{n-1}(k-n+j-s).$$

From $u(N+n) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i)$, we have $A = \delta/(M(n-1)!)$. Hence, (2.2) is the unique solution. □

Theorem 2.5. *Assume that* $n \le \xi_1 < \xi_2 < \cdots < \xi_{m-2} < N+n$ *and that* $0 < \sum_{i=1}^{m-2}\alpha_i(\sum_{j=1}^{n-1}\prod_{l=1}^{j}(\xi_i - n + l) + 1) < \sum_{j=1}^{n-1}\prod_{l=1}^{j}(N+l) + 1$. *Then, the Green's function for the boundary value problem*

$$-\Delta^n u(k) = 0, \quad k \in \{0, N\},$$

$$u(0) = 0, \quad \Delta u(0) = 0, \dots, \Delta^{n-2}u(0) = 0, \quad u(N+n) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i),$$

is given by

$$G(k,s) = \begin{cases} 0, & \text{for } k \in \{0, n-2\},\\ \frac{h(\xi_{r-1},\xi_r;s)}{(n-1)!}, & \text{for } k = n-1,\\ \frac{-\prod_{j=1}^{n-1}(k-n+j-s) + h(\xi_{r-1},\xi_r;s)\sigma}{(n-1)!}, & \text{for } 0 \le s \le k-n \le N,\\ \frac{h(\xi_{r-1},\xi_r;s)\sigma}{(n-1)!}, & \text{for } 0 < k-n+1 \le s \le N,\end{cases}$$

where

$$h(\xi_{r-1},\xi_r;s) = \begin{cases} \frac{\prod_{j=1}^{n-1}(N+j-s) - \sum_{i=1}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i - n + j - s)}{M},\\ \quad \text{for } 0 \le s \le \xi_1 - n,\\ \frac{\prod_{j=1}^{n-1}(N+j-s) - \sum_{i=r}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i - n + j - s)}{M},\\ \quad \text{for } s \in \{\xi_{r-1}-n+1, \xi_r - n\}, r \in \{2, m-1\}.\end{cases}$$

Proof. Make the assumption that $\sum_{i=m_1}^{m_2} f(i) = 0$ for $m_2 < m_1$. For $n \le k \le \xi_1$, the unique solution of (2.1) (1.2) can be expressed as

$$\begin{aligned} u(k) = \frac{1}{M(n-1)!}\{&\sum_{s=0}^{k-n}[-M\prod_{j=1}^{n-1}(k-n+j-s) \\ &+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=1}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s) \\ &+\sum_{s=k-n+1}^{\xi_1-n}\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=1}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma y(s) \\ &+\sum_{r=2}^{m-1}\sum_{s=\xi_{r-1}-n+1}^{\xi_r-n}\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=r}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma y(s)\} \end{aligned}$$

If $\xi_{t-1}+1 \le k \le \xi_t$, $2 \le t \le m-2$, the unique solution of (2.1) (1.2) can be expressed as

$$\begin{aligned} u(k) = \frac{1}{M(n-1)!}\{&\sum_{s=0}^{\xi_1-n}[-M\prod_{j=1}^{n-1}(k-n+j-s) \\ &+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=1}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s) \\ &+\sum_{r=2}^{t-1}\sum_{s=\xi_{r-1}-n+1}^{\xi_r-n}[-M\prod_{j=1}^{n-1}(k-n+j-s) \\ &+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=r}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s) \\ &+\sum_{s=\xi_{t-1}-n+1}^{k-n}[-M\prod_{j=1}^{n-1}(k-n+j-s) \\ &+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=t}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s) \\ &+\sum_{s=k-n+1}^{\xi_t-n}\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=t}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma y(s) \\ &+\sum_{r=t+1}^{m-1}\sum_{s=\xi_{r-1}-n+1}^{\xi_r-n}\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=r}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma y(s)\}. \end{aligned}$$

For $\xi_{m-2}+1 \le k \le N+n$, the unique solution of (2.1) (1.2) can be expressed as

$$\begin{aligned} u(k) = \frac{1}{M(n-1)!}\{&\sum_{s=0}^{\xi_1-n}[-M\prod_{j=1}^{n-1}(k-n+j-s) \\ &+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=1}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=2}^{m-2}\sum_{s=\xi_{r-1}-n+1}^{\xi_r-n}[-M\prod_{j=1}^{n-1}(k-n+j-s)\\
&+\big(\prod_{j=1}^{n-1}(N+j-s)-\sum_{i=r}^{m-2}\alpha_i\prod_{j=1}^{n-1}(\xi_i-n+j-s)\big)\sigma]y(s)\\
&+\sum_{s=\xi_{m-2}-n+1}^{k-n}\big(-M\prod_{j=1}^{n-1}(k-n+j-s)+\sigma\prod_{j=1}^{n-1}(N+j-s)\big)y(s)\\
&+\sum_{s=k-n+1}^{N}\big(\prod_{j=1}^{n-1}(N+j-s)\big)\sigma y(s).
\end{aligned}
$$

Therefore, the unique solution of (2.1) (1.2) is $u(k)=\sum_{s=0}^{N}G(k,s)y(s)$. By the method which Eloe has recently used to obtain the sign of Green's function and related inequalities in [6], it can be verified directly that $G(k,s)\geq 0$ on $\{0,N+n\}\times\{0,N\}$. So, $u(k)\geq 0$, $k\in\{0,N+n\}$. The proof is complete. $\square$

Theorem 2.6. *Assume that* $n\leq\xi_1<\xi_2<\dots<\xi_{m-2}<N+n$, *and that* $0<\sum_{i=1}^{m-2}\alpha_i(\sum_{j=1}^{n-1}\prod_{l=1}^{j}(\xi_i-n+l)+1)<\sum_{j=1}^{n-1}\prod_{l=1}^{j}(N+l)+1$. *If u satisfies* $\Delta^n u(k)\leq 0$, $k\in\{0,N\}$, *with the nonlocal conditions* (1.2), *then*

$$
\min_{k\in\{\xi_{m-2},N+n\}}u(k)\geq\gamma\|u\|, \tag{2.3}
$$

where

$$
\begin{aligned}
\gamma=\min\Big\{&\frac{\alpha_{m-2}(N+n-\xi_{m-2})}{N+n-\alpha_{m-2}\xi_{m-2}},\ \frac{\alpha_{m-2}\prod_{i=0}^{n-2}(\xi_{m-2}-i)}{\prod_{i=0}^{n-2}(N+n-i)},\ \frac{\alpha_1\prod_{i=0}^{n-2}(\xi_1-i)}{\prod_{i=0}^{n-2}(N+n-i)},\\
&\frac{\prod_{i=0}^{n-2}(\xi_{m-2}-i)}{\prod_{i=0}^{n-2}(N+n-i)}\Big\}.
\end{aligned}
$$

Proof. We will show the details in the case that u satisfies the strict difference inequality $\Delta^n u(k)<0$, $k\in\{0,N\}$. Once (2.3) is obtained for functions satisfying the strict inequality, one assumes that u satisfies the difference inequality and sets

$$
\begin{aligned}
u(\epsilon,k)&=u(k)+\epsilon\big(\prod_{j=0}^{n-2}(k-j)\big)\\
&\quad\times\Big(\frac{(N+n)\prod_{j=0}^{n-2}(N+n-j)-\sum_{i=1}^{m-2}\alpha_i\xi_i\prod_{j=0}^{n-2}(\xi_i-j)}{\prod_{j=0}^{n-2}(N+n-j)-\sum_{i=1}^{m-2}\alpha_i\prod_{j=0}^{n-2}(\xi_i-j)}-k\Big).
\end{aligned}
$$

Then for each $\epsilon>0$, $u(\epsilon,k)$ satisfies the strict difference inequality and the nonlocal conditions (1.2). Thus, (2.3) holds for each $\epsilon>0$ and by limiting, it holds for $\epsilon=0$.

Under the assumption $\Delta^n u(k)<0$, $k\in\{0,N\}$, we have to distinguish two cases.

Case (i): $0<\sum_{i=1}^{m-2}\alpha_i<1$. Suppose $u(\xi_r)=\max_{i\in\{1,m-2\}}u(\xi_i)$, then $u(N+n)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i)\leq\sum_{i=1}^{m-2}\alpha_i u(\xi_r)<u(\xi_r)$. It follows by repeated applications of Lemma 2.3 that for each $j\in\{1,n-1\}$, $\Delta^j u$ has precisely one node, $k_j\in\{n-1-j,N+n-j\}$ and $k_{j+1}<k_j$, $j\in\{1,n-2\}$. Assume that $\|u\|=u(\overline{k})$, if Δu vanishes and $\|u\|$ is attained at more than one point, choose $\overline{k}$ to be the largest value producing $\|u\|$, then that node occurs at $k_1=\overline{k}-1$. Otherwise, $k_1=\overline{k}$. Moreover, with the strict difference inequality $\Delta^n u(k)<0$, $k\in\{0,N\}$, we know that u is increasing on $\{n-2,\overline{k}\}$ and decreasing, concave down on $\{\overline{k},N+n\}$. And, if $k\neq k_j$, $k\in\{n-1-j,N+n-j\}$, $\Delta^j u$ does not have a node at k. So, it is easy to see that $\min_{k\in\{\xi_{m-2},N+n\}}u(k)=u(N+n)$.

First assume that $\overline{k}\leq\xi_{m-2}<N+n$. Since $u(N+n)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i)\geq\alpha_{m-2}u(\xi_{m-2})$, and by the decreasing, negative concavity nature of u, we have

$$\begin{aligned} u(\overline{k}) &\leq u(N+n)+\frac{u(N+n)-u(\xi_{m-2})}{N+n-\xi_{m-2}}\big(\overline{k}-(N+n)\big)\\ &\leq u(N+n)+\big(\frac{1}{\alpha_{m-2}}u(N+n)-u(N+n)\big)\frac{N+n}{N+n-\xi_{m-2}}\\ &=\frac{N+n-\alpha_{m-2}\xi_{m-2}}{\alpha_{m-2}(N+n-\xi_{m-2})}u(N+n); \end{aligned}$$

i.e.,

$$\min_{k\in\{\xi_{m-2},N+n\}}u(k)\geq\frac{\alpha_{m-2}(N+n-\xi_{m-2})}{N+n-\alpha_{m-2}\xi_{m-2}}\|u\|.$$

Second, if $\xi_{m-2}<\overline{k}<N+n$, let

$$h(k)=u(k)-\frac{\|u\|\prod_{i=0}^{n-2}(k-i)}{\prod_{i=0}^{n-2}(\overline{k}-i)},\quad k\in\{0,\overline{k}\}.$$

We can prove directly that $\Delta^n h(k)<0$, $k\in\{0,\overline{k}-n\}$, $h(0)=\dots=h(n-2)=0$, $h(\overline{k})=0$. Apply Lemma 2.2, it follows that $h(k)\geq 0$; i.e.,

$$u(k)\geq\frac{\|u\|\prod_{i=0}^{n-2}(k-i)}{\prod_{i=0}^{n-2}(\overline{k}-i)},\quad k\in\{0,\overline{k}\}.$$

So, in particular,

$$u(\xi_{m-2})\geq\frac{\|u\|\prod_{i=0}^{n-2}(\xi_{m-2}-i)}{\prod_{i=0}^{n-2}(\overline{k}-i)}>\frac{\|u\|\prod_{i=0}^{n-2}(\xi_{m-2}-i)}{\prod_{i=0}^{n-2}(N+n-i)},\tag{2.4}$$

which implies

$$u(N+n)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i)\geq\alpha_{m-2}u(\xi_{m-2})\geq\frac{\alpha_{m-2}\prod_{i=0}^{n-2}(\xi_{m-2}-i)}{\prod_{i=0}^{n-2}(N+n-i)}\|u\|.$$

Case (ii): $\sum_{i=1}^{m-2}\alpha_i\geq 1$. Again, using the argument given in the first case, we obtain the similar nature of u.

Firstly, suppose $u(\xi_{m-2})>u(N+n)$, then $\min_{k\in\{\xi_{m-2},N+n\}}u(k)=u(N+n)$, which implies $\xi_1<\overline{k}<N+n$. In fact, if $n-2<\overline{k}\leq\xi_1$, then $u(\xi_1)\geq u(\xi_2)\geq\dots\geq u(\xi_{m-2})>u(N+n)$, and

$$u(N+n)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i)>\sum_{i=1}^{m-2}\alpha_i u(N+n)\geq u(N+n).$$

Which is a contradiction. Thus (2.4) is readily modified to obtain

$$u(\xi_1)\geq\frac{\|u\|\prod_{i=0}^{n-2}(\xi_1-i)}{\prod_{i=0}^{n-2}(N+n-i)},$$

which implies

$$u(N+n)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i)\geq \alpha_1 u(\xi_1)\geq \frac{\alpha_1\prod_{i=0}^{n-2}(\xi_1-i)}{\prod_{i=0}^{n-2}(N+n-i)}\|u\|.$$

Secondly, if $u(\xi_{m-2})\leq u(N+n)$, then $\min_{k\in\{\xi_{m-2},N+n\}}u(k)=u(\xi_{m-2})$; thus, $\xi_{m-2}\leq \overline{k}\leq N+n$. Hence, we have (2.4). The proof is complete. □

3. Main results

In this section, we will impose suitable growth conditions on f, which enable us to apply Theorem 2.1 to obtain three positive solutions for (1.1)) (1.2).

Let $E=\{u:\{0,N+n\}\to\mathbb{R}\}$, and choose the cone $K\subset E$,

$$K=\big\{u\in E:u(k)\geq 0,\ k\in\{0,N+n\},\ \text{and}\ \min_{k\in\{\xi_{m-2},N+n\}}u(k)\geq\gamma\|u\|\big\}.$$

Define an operator A by

$$Au(k)=\sum_{s=0}^{N}G(k,s)a(s)f\big(s,u(s)\big).$$

Obviously, u is a solution of (1.1) (1.2) if and only if u is a fixed point of operator A.

Finally, we define the nonnegative continuous concave functional α on K by

$$\alpha(u)=\min_{k\in\{\xi_{m-2},N+n\}}u(k).$$

Note that, for each $u\in K$, $\alpha(u)\leq\|u\|$.

For of convenience, we denote

$$\lambda_1=\max_{k\in\{0,N+n\}}\sum_{s=0}^{N}G(k,s)a(s),\quad \lambda_2=\min_{k\in\{\xi_{m-2},N+n\}}\sum_{s=\xi_{m-2}}^{N}G(k,s)a(s).$$

Then $0<\lambda_2<\lambda_1$. To present our main result, we assume there exist constants $0<a<b<\min\{\gamma,\frac{\lambda_2}{\lambda_1}\}c$ such that

- (H1) $f(k,u)\leq c/\lambda_1$, for $(k,u)\in\{0,N+n\}\times[0,c]$;
- (H2) $f(k,u)<a/\lambda_1$, for $(k,u)\in\{0,N+n\}\times[0,a]$;
- (H3) $f(k,u)>b/\lambda_2$, for $(k,u)\in\{\xi_{m-2},N+n\}\times[b,b/\gamma]$.

Theorem 3.1. *Under assumptions* (H1)–(H3), *the boundary value problem* (1.1) (1.2) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\|<a,\quad b<\min_{k\in\{\xi_{m-2},N+n\}}u_2(k),\quad \|u_3\|>a,\quad \min_{k\in\{\xi_{m-2},N+n\}}u_3(k)<b. \tag{3.1}$$

Proof. First, We note that $A:\overline{P_c}\to\overline{P_c}$ is completely continuous. If $u\in\overline{P_c}$, then $\|u\|\leq c$, and by condition (H1), we have

$$\|Au\|=\max_{k\in\{0,N+n\}}\sum_{s=0}^{N}G(k,s)a(s)f\big(s,u(s)\big)\leq\frac{c}{\lambda_1}\max_{k\in\{0,N+n\}}\sum_{s=0}^{N}G(k,s)a(s)=c.$$

Hence, $A:\overline{P_c}\to\overline{P_c}$. Standard applications of Arzela-Ascoli theorem imply that A is completely continuous. In an analogous argument, the condition (H2) implies the condition (C2) of Theorem 2.1.

We now show that condition (C1) of Theorem 2.1 is satisfied. Obviously,

$$\{u \in P(\alpha, b, \frac{b}{\gamma}) : \alpha(u) > b\} \neq \emptyset.$$

If $u \in P(\alpha, b, \frac{b}{\gamma})$, then $b \le u(k) \le \frac{b}{\gamma}$, for $k \in \{\xi_{m-2}, N+n\}$. By condition (H3), we obtain

$$\begin{aligned}
\alpha(Au) &= \min_{k\in\{\xi_{m-2},N+n\}} \sum_{s=0}^{N} G(k,s)a(s)f\big(s,u(s)\big) \\
&\ge \min_{k\in\{\xi_{m-2},N+n\}} \sum_{s=\xi_{m-2}}^{N} G(k,s)a(s)f\big(s,u(s)\big) \\
&> \frac{b}{\lambda_2} \min_{k\in\{\xi_{m-2},N+n\}} \sum_{s=\xi_{m-2}}^{N} G(k,s)a(s) = b.
\end{aligned}$$

Therefore, condition (C1) of Theorem 2.1 is satisfied.

Finally, we show that condition (C3) of Theorem 2.1 also holds. If $u \in P(\alpha, b, c)$ and $\|Au\| > \frac{b}{\gamma}$, then

$$\alpha(Au) = \min_{k\in\{\xi_{m-2},N+n\}} Au(k) \ge \gamma\|Au\| > b.$$

So, condition (C3) of Theorem 2.1 is satisfied.

Applying Theorem 2.1, we know that the boundary value problem (1.1) (1.2) has at least three positive solutions u_1, u_2 and u_3 satisfying (3.1). The proof is complete. □

References

[1] P. W. Eloe; *A Generalization of Concavity for Finite Differences*, Comput. Math. Appl. 36 (1998), 109-113.

[2] P. W. Eloe, J. Henderson; *Inequalities based on a generalization of concavity*, Proc. Amer. Math. Soc. 125 (1997), 2103-2108.

[3] P. W. Eloe, J. Henderson; *Singular nonlinear (n-1,1) conjugate boundary value problems*, Georgian Math. Journal, 4 (1997), 501-512.

[4] P.W. Eloe, J. Henderson; *Positive solutions for higher order ordinary differential equations*, Electron. J. Differential Equations 1995, no. 3 (1995), 1-8.

[5] P. W. Eloe, J. Henderson, P. J. Y. Wong; *Positive solutions for two point boundary value problems*, Dynam. Systems Appl. 2 (1996), 135-144.

[6] P. W. Eloe; *Maximum principles for a family of nonlocal boundary value problems*, Adv. Difference Equations 3 (2004), 201-210.

[7] P. W. Eloe, B. Ahmad; *Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions*, Appl. Math. Lett. 18 (2005), 521-527.

[8] Y. Guo, W. Shan, W. Ge; *Positive solutions for second-order m-point boundary value problems*, J. Comput. Appl. Math. 151 (2003), 415-424.

[9] P. Hartman; *Difference equations: Disconjugacy, Principal solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc. 246 (1978), 1-30.

[10] V. A. Il'in, E. I. Moiseev; *Nonlocal boundary value problem of the first kind for a Sturm Liouville operator in its differential and finite difference aspects*, Differ. Equ. 23 (1987), 803-810.

[11] X. Liu, J. Qiu, Y. Guo; *Three positive solutions for second-order m-point boundary value problems*, Appl. Math. Comput. 156 (2004), 733-742.

[12] R. Ma; *Existence theorems for a second order m-point boundary value problems*, J. Math. Anal. Appl. 211 (1997), 545-555.

Jiehua Zhang
College of Sunshine, Fuzhou University, Fuzhou 350015, China
E-mail address: jiehuahappy@163.com

Yanping Guo
College of Sciences, Hebei University of Science and Technology, Shijiazhuang 050018, China
E-mail address: guoyanping65@sohu.com

Yude Ji
College of Sciences, Hebei University of Science and Technology, Shijiazhuang 050018, China
E-mail address: jiyude-1980@163.com