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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A CLASS OF LINEAR NON-AUTONOMOUS NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study a class of linear non-autonomous neutral delay differential equations, and establish a criterion for the asymptotic behavior of their solutions, by using the corresponding characteristic equation.

1. Introduction

Let $\mathbb{C}$ be the complex numbers with norm $|\cdot|$. For $r \geq 0$, let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{C})$ be the space of continuous functions taking $[-r, 0]$ into $\mathbb{C}$ with norm defined by $\|\varphi\| = \max_{-r\leq\theta\leq 0} |\varphi|$. A functional differential equation of neutral type, or shortly a neutral equation, is a system of the form

$$\frac{d}{dt} M x_t = L(t) x_t, \quad t \geqslant t_0 \in \mathbb{R}, \tag{1.1}$$

where $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$, $M : \mathcal{C} \to \mathbb{C}$ is continuous, linear and atomic at zero, (see [5, page 255] for the concept of atomic at zero),

$$M\varphi = \varphi(0) - \int_{-r}^{0} \varphi(\theta)\, d\mu(\theta), \tag{1.2}$$

where $\operatorname{Var}_{[s,0]} \mu \to 0$, as $s \to 0$.

For (1.1), $L(t)$ denote a family of bounded linear functionals on $\mathcal{C}$. By the Riesz representation theorem, for each $t \geqslant t_0$, there exists a complex valued function of bounded variation $\eta(t, \cdot)$ on $[-r, 0]$, normalized so that $\eta(t, 0) = 0$ and $\eta(t, \cdot)$ is continuous from the left in $(-r, 0)$ such that

$$L(t)\varphi = \int_{-r}^{0} \varphi(\theta)\, d_\theta \eta(t, \theta). \tag{1.3}$$

For any $\varphi \in \mathcal{C}$, $\sigma \in [t_0, \infty)$, a function $x = x(\sigma, \varphi)$ defined on $[\sigma - r, \sigma + A)$ is said to be a solution of (1.1) on $(\sigma, \sigma + A)$ with initial φ at σ if x is continuous on $[\sigma - r, \sigma + A)$, $x_\sigma = \varphi$, Mx_t is continuously differentiable on $(\sigma, \sigma + A)$, and relation (1.1) is satisfied on $(\sigma, \sigma + A)$. For more information on this type of equations, see [5].

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The initial-value problem (IVP) is stated as

$$\begin{aligned} \frac{d}{dt}Mx_t &= L(t)x_t \quad t \geqslant \sigma, \\ x_\sigma &= \varphi. \end{aligned} \tag{1.4}$$

For $\mu = 0$ in (1.2), $M\varphi = \varphi(0)$ and equation (1.1) becomes the retarded functional differential equation

$$x'(t) = L(t)x_t. \tag{1.5}$$

Consider the *characteristic equation* associated with (1.5),

$$\lambda(t) = \int_0^r \exp\Big(-\int_{t-\theta}^t \lambda(s)ds\Big)\, d_\theta\eta(t,\theta) \tag{1.6}$$

which is obtained by looking for solutions to (1.5) of the form

$$x(t) = \exp\Big[\int_0^t \lambda(s)\,ds\Big]. \tag{1.7}$$

The solutions of (1.6) are continuous functions $\lambda(\cdot)$ defined in $[t_0 - r, \infty)$ which satisfy (1.5).

Cuevas and Frasson [1] studied the asymptotic behavior of solutions of (1.5) with initial condition $x_\sigma = \varphi$, and obtained the following result.

Theorem 1.1. *Assume that $\lambda(t)$ is a solution of (1.6) such that*

$$\limsup_{t\to\infty} \int_0^r \theta|e^{-\int_{t-\theta}^t \lambda(s)ds}|d_\theta|\eta|(t,\theta) < 1.$$

Then for each solution x of (1.5), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t \lambda(s)ds}$$

exists, and

$$\lim_{t\to\infty} \Big[x(t)e^{-\int_{t_0}^t \lambda(s)ds}\Big]' = 0.$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^t \lambda(s)ds} = \lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^t \lambda(s)ds},$$

if $\lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^t \lambda(s)ds}$ exists.

Motivated by the work in [1], we provide a generalization of [1], and consider the asymptotic behavior of solutions to (1.4). The method for the proving our main result is similar to the one in [1, 2]. In Section 2, we state the main results. In Section 3, some examples will be shown as applications of the main results of this paper.

2. Main results

For equation (1.1), the characteristic equation is

$$\lambda(t) = \int_{-r}^0 d\mu(\theta)\lambda(t+\theta)\exp\Big(-\int_{t+\theta}^t \lambda(s)ds\Big) + \int_{-r}^0 d_\theta\eta(t,\theta)\exp\Big(-\int_{t+\theta}^t \lambda(s)ds\Big), \tag{2.1}$$

which is obtained by looking for solutions of (1.1) of the form (1.7) and the solutions of (2.1) are continuous functions defined in $[\sigma - r, \infty)$ satisfying (2.1). For autonomous neutral functional differential equations (NFDEs), the constant solutions of (2.1) are the roots of the so called characteristic equation, for detailed discussion of this type, refer to [3, 4, 5].

Theorem 2.1. *Assume that $\lambda(t)$ is a solution of (2.1) such that*

$$\limsup_{t\to\infty} \chi_{\lambda,t} < 1, \tag{2.2}$$

where

$$\begin{aligned} \chi_{\lambda,t} = &\int_{-r}^{0} |e^{-\int_{t+\theta}^{t} \lambda(s)\,ds}|\, d|\mu|(\theta) \\ &+ \int_{-r}^{0} (-\theta)|e^{-\int_{t+\theta}^{t} \lambda(s)\,ds}|\Big(|\lambda(t+\theta)|\, d|\mu|(\theta) + d_\theta|\eta|(t,\theta)\Big). \end{aligned}$$

Then for each solution x of (1.4), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t} \lambda(s)\,ds} \tag{2.3}$$

exists, and

$$\lim_{t\to\infty} \Big[x(t)e^{-\int_{t_0}^{t} \lambda(s)\,ds}\Big]' = 0. \tag{2.4}$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^{t} \lambda(s)\,ds} = \lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^{t} \lambda(s)\,ds} \tag{2.5}$$

if the limit in the right-hand side exists.

Proof. From (2.2), there exists $t_1 \ge t_0$, such that

$$\sup_{t\ge t_1} \chi_{\lambda,t} < 1.$$

Hence without loss of generality, we assume that $t_0 = 0$ and define

$$\Gamma_\lambda := \sup_{t\ge 0} \chi_{\lambda,t} < 1.$$

For solutions x of (1.4), we set

$$y(t) = x(t)e^{-\int_{0}^{t} \lambda(s)\,ds}, \quad t \geqslant -r.$$

Then (1.4) becomes

$$\begin{aligned} &y'(t) + \lambda(t)y(t) - \int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ &= \int_{-r}^{0} y(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds}\Big(\lambda(t+\theta)\, d\mu(\theta) + d_\theta\eta(t,\theta)\Big) \end{aligned} \tag{2.6}$$

and the initial condition is equivalent to

$$y(t) = \varphi(t)e^{-\int_{0}^{t} \lambda(s)\,ds}, \quad -r \le t \le 0. \tag{2.7}$$

Combining (2.7) with (2.1), for $t \ge -r$, we have

$$\begin{aligned} y'(t) = &\int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ &- \int_{-r}^{0} e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \int_{-r}^{0} y'(s)\,ds\Big(\lambda(t+\theta)\, d\mu(\theta) + d_\theta\eta(t,\theta)\Big). \end{aligned} \tag{2.8}$$

From the definition of the solutions to (1.4), we know that $y'(t)$ is continuous, Let

$$M_{\varphi,\lambda_0} = \max\{|\varphi'(t)e^{-\int_0^t \lambda(s)\,ds} - \lambda(t)\varphi(t)e^{-\int_0^t \lambda(s)\,ds}| : -r \le t \le 0\}.$$

We shall show that M_φ is also a bound of y' on the whole interval $[-r,\infty)$; i.e.,

$$|y'(t)| \le M_{\varphi,\lambda_0}, \quad t \ge -r. \tag{2.9}$$

For this purpose, let us consider an arbitrary number $\varepsilon > 0$. Then

$$|y'(t)| < M_{\varphi,\lambda_0} + \varepsilon \quad \text{for } t \ge -r. \tag{2.10}$$

Indeed, in the opposite case, we suppose there exists a point $t^* > 0$ such that

$$\begin{gathered} |y'(t)| < M_{\varphi,\lambda_0} + \varepsilon \quad \text{for } -r \le t < t^*, \\ |y(t^*)| = M(\lambda_0,\mu_0;\phi) + \varepsilon. \end{gathered} \tag{2.11}$$

Then combining (2.8) and (2.11), we obtain

$$\begin{aligned} &M(\lambda_0,\mu_0;\phi) + \varepsilon \\ &= y'(t^*) \\ &\le \Big| \int_{-r}^0 y'(t^*+\theta)e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}\, d\mu(\theta)\Big| \\ &\quad + \Big| \int_{-r}^0 e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds} \int_{-r}^0 y'(s)\,ds\Big(\lambda(t^*+\theta)\,d\mu(\theta) + d_\theta\eta(t^*,\theta)\Big)\Big| \\ &\le (M_{\varphi,\lambda_0} + \varepsilon)\Big\{ \int_{-r}^0 |e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\, d|\mu|(\theta) \\ &\quad + \int_{-r}^0 (-\theta)|e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\Big(|\lambda(t^*+\theta)|\, d|\mu|(\theta) + d_\theta|\eta|(t^*,\theta)\Big)\Big\} \\ &= (M_{\varphi,\lambda_0} + \varepsilon)\Gamma_\lambda \\ &< (M_{\varphi,\lambda_0} + \varepsilon), \end{aligned} \tag{2.12}$$

which is a contradiction, so (2.10) holds. Since (2.10) holds for every $\varepsilon > 0$, it follows that $|y'(t)| \le M_{\varphi,\lambda_0}$, for all $t \ge -r$. By using (2.8) and (2.9), for $t \ge 0$ we have

$$\begin{aligned} |y'(t)| &\le \Big| \int_{-r}^0 y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)ds}\, d\mu(\theta)\Big| \\ &\quad + \Big| \int_{-r}^0 e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \int_{-r}^0 y'(s)\,ds\Big(\lambda(t+\theta)\,d\mu(\theta) + d_\theta\eta(t,\theta)\Big)\Big| \\ &\le M_{\varphi,\lambda_0}\Big\{ \int_{-r}^0 |e^{-\int_{t+\theta}^{t} \lambda(s)\,ds}|\, d|\mu|(\theta) \\ &\quad + \int_{-r}^0 (-\theta)|e^{-\int_{t+\theta}^{t} \lambda(s)\,ds}|\Big(|\lambda(t+\theta)|\, d|\mu|(\theta) + d_\theta|\eta|(t,\theta)\Big)\Big\} \\ &= M_{\varphi,\lambda_0}\Gamma_\lambda, \end{aligned} \tag{2.13}$$

which means, for $t \ge 0$,

$$|y'(t)| \le M_{\varphi,\lambda_0}\Gamma_{\lambda_0}.$$

One can show by induction, that $y'(t)$ satisfies

$$|y'(t)| \le M_{\varphi,\lambda_0}(\Gamma_\lambda)^n \quad \text{for } t \ge nr - r, \quad (n = 0,1,2,3,\dots). \tag{2.14}$$

Since $0 \le \chi_{\lambda,t} < 1$, it follows that $y'(t)$ tends to zero as $t \to \infty$. So we proved (2.4). In the following, we will show (2.3) holds.

To prove that $\lim_{t\to\infty} y(t)$ exists, we consider (2.14). For an arbitrary $t \ge 0$, we set $n = [t/r] + 1$ (the greatest integer less than or equal to $t/r + 1$), then from $n = [t/r] + 1 \le t/r + 1 \le [t/r] + 2 = n + 1$, we have $t/r \le n$. From (2.14),

$$|y'(t)| \le M_{\varphi,\lambda_0}(\Gamma_\lambda)^n \le M_{\varphi,\lambda_0}(\Gamma_\lambda)^{t/r} \quad \text{for } t \ge nr - r. \tag{2.15}$$

Now we use the Cauchy convergence criterion, for $t > T \ge 0$, from (2.15), we have

$$\begin{aligned} |y(t) - y(T)| &\le \int_T^t |y'(s)|\,ds \le \int_T^t M_{\varphi,\lambda_0}(\Gamma_\lambda)^{s/r}\,ds \\ &= M_{\varphi,\lambda_0}\frac{r}{\ln\Gamma_\lambda}\Big[(\Gamma_\lambda)^{s/r}\Big]_{s=T}^{s=t} \\ &= M_{\varphi,\lambda_0}\frac{r}{\ln\Gamma_\lambda}\Big[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r}\Big]. \end{aligned} \tag{2.16}$$

Let $T \to \infty$, we have $t \to \infty$, and by (2.16), we have

$$M_{\varphi,\lambda}\frac{r}{\ln\Gamma_\lambda}\Big[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r}\Big] \to 0;$$

and $\lim_{T\to\infty}|y(t) - y(T)| = 0$. The Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} y(t)$. We obtain (2.5) by a straight forward application of (2.4). $\square$

Remark 2.2. Under the conditions of Theorem 2.1, a solution of (1.4) can not grow faster than the exponential function; i.e., there exists a constant $M > 0$, such that

$$|x(t)| \le Me^{\int_0^t \lambda(s)\,ds}, \quad \text{for } t \ge 0. \tag{2.17}$$

From (2.17), it is not difficult to show that:

- Every solution of (1.4) is bounded if and only if $\limsup_{t\to\infty}\int_0^t \lambda(s)\,ds < \infty$;
- Every solution of (1.4) tends to zero if and only if $\limsup_{t\to\infty}\int_0^t \lambda(s)\,ds \to -\infty$.

Remark 2.3. If the characteristic equation (2.1) has a constant solution $\lambda(t) = \lambda_0$, then from Theorem 2.1, $\lim_{t\to\infty} x(t)e^{-\lambda_0 t}$ exists.

3. Examples

Example 3.1. Consider the linear differential equation with distributed delay

$$x'(t) - \frac{1}{2}x'(t-1) = \int_{-1}^0 \frac{x(t+\theta)}{2(t+\theta)}\,d\theta, \quad t > 1. \tag{3.1}$$

This equation can be written in the form (1.1) by setting $\mu(\theta) = -1/2$ for $\theta \le -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = \ln t + \frac{1}{2}\ln(t+\theta)$ for $t > 1$ and $\theta \in [-1, 0]$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, $|\mu| = \mu$, $|\eta| = \eta$.

The characteristic equation associated with (3.1) is

$$\lambda(t) = \frac{\lambda(t-1)}{2}\exp\Big[-\int_{t-1}^t \lambda(s)\,ds\Big] + \int_{-1}^0 \frac{1}{2(t+\theta)}\exp\Big[-\int_{t+\theta}^t \lambda(s)\,ds\Big]\,d\theta, \tag{3.2}$$

which has a solution

$$\lambda(t) = 1/t. \tag{3.3}$$

For this $\lambda(t)$ and for $t > 1$, using the expression of $\chi_{\lambda,t}$, we have

$$\frac{1}{2}\big(1-\frac{1}{t}\big)+\frac{1}{4t}+\int_{-1}^{0}\frac{-\theta}{2(t+\theta)}\exp\Big[-\int_{t+\theta}^{t}\frac{ds}{s}\Big]\,d\theta=\frac{1}{2}<1 \quad \text{as } t\to\infty.$$

Hence the hypothesis (2.2) of Theorem 2.1 is fulfilled. So we obtain that

$$\lim_{t\to\infty}\frac{x(t)}{t} \text{ exists}, \quad \lim_{t\to\infty}\Big[\frac{x(t)}{t}\Big]'=0 \quad \text{and} \quad \lim_{t\to\infty}\frac{x'(t)}{t}=0. \tag{3.4}$$

Example 3.2. Consider the equation with variable delay

$$x'(t)-\frac{2}{3}x'(t-1)=\frac{x(t-\tau(t))}{3(t+c-\tau(t))}, \quad t\geqslant t_0. \tag{3.5}$$

where $c\in\mathbb{R}$ and $\tau:[0,\infty)\to[-1,0]$ is a continuous function such that $t+c-\tau(t)>0$ for $t\geqslant t_0$. Equation (3.5) can be written in the form (1.1) by letting $\mu(\theta)=-2/3$ for $\theta\le -1$, $\mu(\theta)=0$ for $\theta>-1$, $\eta(t,\theta)=0$ for $\theta<\tau(t)$, $\eta(t,\theta)=(t+c-\tau(t))/3$ for $\theta\geqslant\tau(t)$. Since both $\theta\mapsto\eta(t,\theta)$ and $\theta\mapsto\mu(\theta)$ are increasing functions, we have that $|\mu|=\mu, |\eta|=\eta$.

The characteristic equation associated with (3.5) is

$$\lambda(t)=\frac{2\lambda(t-1)}{3}\exp\Big[-\int_{t-1}^{t}\lambda(s)ds\Big]+\frac{1}{3(t+c-\tau(t))}\exp\Big[-\int_{t-\tau(t)}^{t}\lambda(s)ds\Big] \tag{3.6}$$

and we have that a solution of (3.6) is

$$\lambda(t)=\frac{1}{t+c}. \tag{3.7}$$

For (3.7), the left hand side of (2.2) reads as

$$\begin{aligned}&\limsup_{t\to\infty}\Big[\frac{2}{3}\Big(1-\frac{1}{t+c}\Big)+\frac{1}{6(t+c)}+\int_{-1}^{0}(-\theta)|e^{-\int_{t-\theta}^{t}\lambda(s)ds}|d_\theta|\eta|(t,\theta)\Big]\\&=\limsup_{t\to\infty}\Big[\frac{2}{3}-\frac{\tau(t)}{3(t+c)}\Big]=\frac{2}{3}<1.\end{aligned}$$

and hence hypothesis (2.2) of Theorem 2.1 is fulfilled and therefore, for all solutions $x(t)$ of (3.5), we have that

$$\lim_{t\to\infty}\frac{x(t)}{t+c} \text{ exists, and } \lim_{t\to\infty}\Big[\frac{x(t)}{t+c}\Big]'=0. \tag{3.8}$$

Manipulating further the limits in (3.5), we are able to establish that $x(t)=O(t)$ and $x'(t)=o(t)$ as $t\to\infty$.

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