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SINGULAR EQUILIBRIUM SOLUTIONS FOR A REPLICATOR DYNAMICS MODEL

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Abstract. We evaluate explicitly certain classes of singular equilibrium solutions for a specific one-dimensional replicator dynamics equation. These solutions are linear combinations of Dirac delta functions. Equilibrium solutions are important in the study of equilibrium selection in non-cooperative games.

1. Introduction–the replicator dynamics model

The replicator dynamics models are popular models in evolutionary game theory. They have significant applications in economics, population biology and other areas of science.

Let $A = (a_{ij})$ be an $m \times m$ matrix (the *payoff matrix*). Then a typical replicator dynamics equation is

$$u_i'(t) = \Big[\sum_{j=1}^{m} a_{ij} u_j(t) - \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij} u_i(t) u_j(t)\Big] u_i(t), \quad i = 1, \dots, m.$$

The set $S = \{1, \dots, m\}$ is the *strategy space*. The term in the square brackets is a measure of the success of strategy i and it is assumed to be the difference of the payoff of the players playing strategy i from the average payoff of the population. It is then assumed that the logarithmic derivative of $u_i(t)$, where u_i is the percentage of the population playing i, is equal to this success measure; i.e., that agents update their strategies proportionally to the success of the strategy i.

The vector

$$u(t) = (u_1(t), \dots, u_m(t))^T,$$

is a probability distribution on S, hence

$$u_j(t) \geq 0, \quad j = 1, \dots, m; \quad \sum_{j=1}^{m} u_j(t) = 1.$$

If these conditions on $u(t)$ are satisfied for $t = 0$, then it is easy to see that they are satisfied for all $t \geq 0$.

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The replicator dynamics equation can be written in a compact form

$$u_t = (Au)u - (u, Au)u = [Au - (u, Au)]u, \tag{1.1}$$

where $(Au)u$ denotes the vector whose j-th component is the product of the j-th components of the vectors (Au) and u.

This model was introduced by Taylor and Jonker [10] and Maynard Smith [9]. See also Imhof [2], where a stochastic version of the model is discussed.

Infinite-dimensional versions of this evolutionary strategy model have been proposed, e.g., Bomze [1] and Oechssler and Riedel [6, 7], in connection to certain economic applications. However, the abstract form of the proposed equations does not provide any insight on the form of solutions.

In order to make some progress in this direction, in earlier papers Papanicolaou et al. [3, 8, 4], had focused on the case where S is a "continuum" (i.e. a region of $\mathbb{R}^d$, $d \geq 1$) and A a differential operator or an integral operator. In this short note we are only interested in equilibrium solutions to (1.1) in a special infinite-dimensional case. Smooth (or even continuous) equilibrium solutions do not exist in the considered case, but there are infinitely many singular solutions. Inspired by an example of Krugman [5] we take S to be the interval $[a, b]$, with $a < b$, of the real line, and choose A to be the (compact) self-adjoint, integral (hence nonlocal) operator given by

$$(Av)(x) = \frac{1}{2r} \int_a^b e^{-r|x-\xi|} v(\xi) d\xi, \tag{1.2}$$

where r is a strictly positive constant.

Let us first determine the inverse operator of A. To do that we proceed as follows: By letting

$$(Av)(x) = w(x), \quad w(x) = \frac{1}{2r} \int_a^b e^{-r|x-\xi|} v(\xi) d\xi, \tag{1.3}$$

it is straightforward to show that

$$-w''(x) + r^2 w(x) = v(x), \tag{1.4}$$

$$w'(a) = rw(a), \quad w'(b) = -rw(b). \tag{1.5}$$

For the rest of this article, without loss of generality, we assume that $a = 0$.

Formulas (1.3)–(1.5) tell us that A is the inverse of the self-adjoint (local) differential operator L, defined as

$$L := -\frac{d^2}{dx^2} + r^2, \tag{1.6}$$

and whose domain consists of sufficiently smooth functions $v = v(x)$ satisfying the boundary conditions

$$w'(0) = rw(0), \quad w'(b) = -rw(b). \tag{1.7}$$

Now let $(\cdot, \cdot)$ be the standard inner product for the Hilbert space $L_2(0, b)$, namely

$$(f, g) = \int_0^b f(x)\overline{g(x)} dx.$$

A simple integration by parts yields

$$\begin{aligned}(Lw,w) &= \int_0^b [-w''(x)\overline{w(x)} + r^2 w(x)\overline{w(x)}]dx \\ &= \int_0^b [|w'(x)|^2 + r^2|w(x)|^2]dx - w'(b)\overline{w(b)} + w'(0)\overline{w(0)}.\end{aligned}$$

The boundary conditions (1.7) imply that

$$(Lw,w) = \int_0^b [|w'(x)|^2 + r^2|w(x)|^2]dx + r[|w(b)|^2 + |w(0)|^2] \ge 0;$$

i.e., L is positive. Therefore, A, the operator in (1.2), which is L^{-1}, is also a positive operator and the corresponding replicator dynamics equation (1.1) has the form

$$u_t = [Au - (u, Au)]u, \quad t > 0,\ x \in [0, b], \tag{1.8}$$

with

$$\int_0^b u(0,x)dx = 1, \quad u(0,x) \ge 0, \quad x \in [0,b]. \tag{1.9}$$

Integration of both sides of (1.8) with respect to x gives

$$\frac{\partial}{\partial t}\int_0^b u(t,x)dx = (u, Au)\big[1 - \int_0^b u(t,x)dx\big]. \tag{1.10}$$

It follows from (1.10) that the set of probability measures on $S = [0, b]$ is invariant under the flow (1.8) and this is, of course, a desirable feature of the model. This "conservation of probability" is essential for the applicability of (1.8)–(1.9) in the context of evolutionary dynamics modelling.

2. Singular equilibrium solutions

The *equilibrium solutions* to (1.8)–(1.9) are the solutions u which are independent of t. Equivalently, u is an equilibrium solution if $u(t, x) = v(x)$, where $v(x)$ satisfies

$$[Av - (v, Av)]v = 0, \tag{2.1}$$

with

$$\int_0^b v(x)dx = 1, \quad v(x) \ge 0, \quad x \in [0,b]. \tag{2.2}$$

Suppose $v(x)$ is an equilibrium solution such that $v(x) > 0$ on an open interval (c_1, c_2) with $c_1 < c_2$. Then, (2.1) implies

$$(Av)(x) = \gamma, \quad x \in (c_1, c_2), \tag{2.3}$$

where$\gamma := (v, Av)$. We now apply A^{-1} , namely L of (1.6), to both sides of (2.1). Since L is a local operator and we are interested in its effect only on the interval $x \in (c_1, c_2)$, we do not need to know the value of $(Av)(x)$ when $x \notin (c_1, c_2)$:

$$v(x) = r^2\gamma, \quad \text{for } x \in (c_1, c_2);$$

i.e., $v(x)$ must be constant on (c_1, c_2). In particular, if $v(x)$ is continuous on $[0, b]$, thus we must have

$$v(x) = \frac{1}{b}, \quad x \in [0,b]. \tag{2.4}$$

However, with $v(x)$ given in (2.4), the quantity $(Av)(x)$ is not constant, and this contradicts (2.3). Therefore, there are no continuous equilibrium solutions.

Motivated by the above observations we try to find equilibrium solutions to (1.8)–(1.9), namely solutions to (2.1)–(2.2), in the class of singular probability measures. More precisely, we look for solutions of the form

$$v(x) = \sum_{j=1}^{n} \alpha_j \delta_{c_j}(x) = \sum_{j=1}^{n} \alpha_j \delta(x - c_j), \tag{2.5}$$

where $\delta_c(x) := \delta(x-c)$ with δ denoting the Dirac delta function, while the constants $c_1 < \dots < c_n$ lie in $(0, b)$, and the positive constants α_j satisfy

$$\sum_{j=1}^{n} \alpha_j = 1. \tag{2.6}$$

Obviously any such $v(x)$ satisfies (2.2) and we only need to check (2.1); i.e., we need to find the $v(x)$'s for which

$$(Av)(x) = (v, Av), \quad x = c_1, \dots, c_n. \tag{2.7}$$

To satisfy (2.7) we apply A given in (1.2) on (2.5). Using

$$(A\delta_c)(x) = \frac{1}{2r} \int_0^b e^{-r|x-\xi|} \delta(\xi - c) d\xi = \frac{e^{-r|x-c|}}{2r},$$

we obtain

$$(Av)(x) = \frac{1}{2r} \sum_{j=1}^{n} \alpha_j e^{-r|x-c_j|}. \tag{2.8}$$

For the right side of (2.7) we observe that

$$(v, Av) = \frac{1}{2r} \int_0^b \Big[\sum_{j=1}^{n} \alpha_j \delta(x - c_j) \Big] \Big[\sum_{k=1}^{n} \alpha_k e^{-r|x-c_k|} \Big] dx = \frac{1}{2r} \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \alpha_k e^{-r c_{jk}},$$

where

$$c_{jk} := |c_j - c_k|. \tag{2.9}$$

Let us view $c_1, \dots, c_n$ as given and treat $\alpha_1, \dots, \alpha_n$ as unknowns. Then, we claim that all we need to do is to find (positive) $\alpha_1, \dots, \alpha_n$ and λ satisfying (2.6) and

$$\sum_{j=1}^{n} \alpha_j e^{-r c_{jk}} = \lambda, \quad k = 1, \dots, n. \tag{2.10}$$

To justify this claim, we notice that, in view of (2.8), (2.10) can be written in the form

$$(Av)(x) = \frac{\lambda}{2r}, \quad x = c_1, \dots, c_n. \tag{2.11}$$

Then, (2.6) and (2.11) imply

$$(v, Av) = \int_0^b v(x) \frac{\lambda}{2r} dx = \frac{\lambda}{2r} = (Av)(x), \quad x = c_1, \dots, c_n. \tag{2.12}$$

Since the support of v is in $\{c_1, \dots, c_n\}$, we infer from (2.12) that $v(x)$ of (2.5) is an equilibrium solution. Notice that the special case with $n = 1$ yields the equilibrium solution

$$v(x) = \delta(x - c_1), \tag{2.13}$$

where c_1 is any point in the interval $(0, b)$.

From now on we concentrate on the system (2.6) and (2.10) with $n \geq 2$. If we set

$$b_1 := e^{-rc_{12}}, \quad b_2 := e^{-rc_{23}}, \quad \dots, \quad b_{n-1} := e^{-rc_{n-1,n}}, \tag{2.14}$$

then (2.10) takes the form

$$\begin{gathered}
\alpha_1 + b_1\alpha_2 + (b_1b_2)\alpha_3 + (b_1b_2b_3)\alpha_4 + \dots + (b_1 \dots b_{n-1})\alpha_n = \lambda \\
b_1\alpha_1 + \alpha_2 + b_2\alpha_3 + (b_2b_3)\alpha_4 + \dots + (b_2 \dots b_{n-1})\alpha_n = \lambda \\
(b_1b_2)\alpha_1 + b_2\alpha_2 + \alpha_3 + b_3\alpha_4 + \dots + (b_3 \dots b_{n-1})\alpha_n = \lambda \\
\dots \\
(b_1 \dots b_{n-1})\alpha_1 + (b_2 \dots b_{n-1})\alpha_2 + (b_3 \dots b_{n-1})\alpha_3 + \dots + \alpha_n = \lambda
\end{gathered} \tag{2.15}$$

or, in matrix notation

$$B\alpha = \lambda \mathbf{1}, \tag{2.16}$$

where B is the $n \times n$ symmetric matrix

$$B := \begin{bmatrix} 1 & b_1 & (b_1b_2) & \dots & (b_1 \dots b_{n-1}) \\ b_1 & 1 & b_2 & \dots & (b_2 \dots b_{n-1}) \\ (b_1b_2) & b_2 & 1 & \dots & (b_3 \dots b_{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (b_1 \dots b_{n-1}) & (b_2 \dots b_{n-1}) & (b_3 \dots b_{n-1}) & \dots & 1 \end{bmatrix}, \tag{2.17}$$

while the n-vectors α and $\mathbf{1}$ are given by

$$\alpha := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \mathbf{1} := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.18}$$

From (2.16), we have

$$\alpha_j = \frac{\det B_j}{\det B}\lambda, \quad j = 1, \dots, n, \tag{2.19}$$

where B_j is the matrix obtained from B by replacing its j-th column by $\mathbf{1}$.

We will now evaluate the determinants $\det B$ and $\det B_j$, for $j = 1, \dots, n$. To make our computations more transparent we introduce Δ and Δ_j defined as

$$\Delta(b_1, \dots, b_{n-1}) := \det B, \quad \Delta_j(b_1, \dots, b_{n-1}) := \det B_j. \tag{2.20}$$

Multiply the second row of B by b_1 and subtract the resulting row from the first we obtain

$$\Delta(b_1, \dots, b_{n-1}) = (1 - b_1^2) \det \begin{bmatrix} 1 & b_2 & \dots & (b_2 \dots b_{n-1}) \\ b_2 & 1 & \dots & (b_3 \dots b_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ (b_2 \dots b_{n-1}) & (b_3 \dots b_{n-1}) & \dots & 1 \end{bmatrix}.$$

Hence

$$\Delta(b_1, \dots, b_{n-1}) = (1 - b_1^2)\Delta(b_2, \dots, b_{n-1}),$$

which yields

$$\Delta(b_1, \dots, b_{n-1}) = \prod_{j=1}^{n-1} (1 - b_j^2. \tag{2.21}$$

In a similar way, one computes $\det B_1$: Multiply the second row of B_1 by b_1 and subtract resulting row from the first we obtain

$$\Delta_1(b_1,\dots,b_{n-1}) = (1-b_1)\prod_{j=2}^{n-1}(1-b_j^2). \tag{2.22}$$

To compute $\det B_2$ we, again, multiply the second row of B_2 by b_1 and subtract resulting row from the first. This yields

$$\Delta_2(b_1,\dots,b_{n-1}) = (1-b_1^2)\Delta_1(b_2,\dots,b_{n-1}) - (1-b_1)b_1\Delta(b_2,\dots,b_{n-1}).$$

Thus, (2.21) and (2.22) give

$$\Delta_2(b_1,\dots,b_{n-1}) = (1-b_1^2)(1-b_2)\prod_{j=3}^{n-1}(1-b_j^2) - (1-b_1)b_1\prod_{j=2}^{n-1}(1-b_j^2),$$

or

$$\Delta_2(b_1,\dots,b_{n-1}) = (1-b_1b_2)(1-b_1)(1-b_2)\prod_{j=3}^{n-1}(1-b_j^2). \tag{2.23}$$

The computation of $\det B_3$ is simpler: Multiply the second row of B_3 by b_1 and subtract resulting row from the first we obtain

$$\begin{aligned}
&\Delta_3(b_1,\dots,b_{n-1})\\
&= (1-b_1^2)\Delta_2(b_2,\dots,b_{n-1})\\
&\quad + (1-b_1)\det\begin{bmatrix} b_1 & 1 & \dots & (b_2\dots b_{n-1})\\ (b_1b_2) & b_2 & \dots & (b_3\dots b_{n-1})\\ \vdots & \vdots & \ddots & \vdots\\ (b_1\dots b_{n-1}) & (b_2\dots b_{n-1}) & \dots & 1\end{bmatrix}.
\end{aligned} \tag{2.24}$$

The determinant of the matrix appearing in the second term on the right side of (2.24) is zero because its first column is b_1 times the second column. Hence, (2.24) simplifies to

$$\Delta_3(b_1,\dots,b_{n-1}) = (1-b_1^2)\Delta_2(b_2,\dots,b_{n-1}).$$

Then (2.21) gives

$$\Delta_3(b_1,\dots,b_{n-1}) = (1-b_1^2)(1-b_2b_3)(1-b_2)(1-b_3)\prod_{j=4}^{n-1}(1-b_j^2). \tag{2.25}$$

We can compute $\det B_k$, for $k=4,\dots,n-1$, in a similar way we have computed $\det B_3$. The result is

$$\Delta_k(b_1,\dots,b_{n-1}) = \Big[\prod_{j=1}^{k-2}(1-b_j^2)\Big](1-b_{k-1}b_k)(1-b_{k-1})(1-b_k)\prod_{j=k+1}^{n-1}(1-b_j^2), \tag{2.26}$$

where $k=4,\dots,n-1$ and the empty product that appears in the case $k=n-1$ is taken to be equal to 1.

It remains to compute $\det B_n$. Following the same steps as the ones for computing $\det B_3$, we arrive at the equation

$$\Delta_n(b_1,\dots,b_{n-1}) = (1-b_1^2)\Delta_{n-1}(b_2,\dots,b_{n-1}).$$

Repeating the procedure we get

$$\begin{aligned}\Delta_n(b_1,\dots,b_{n-1}) &= (1-b_1^2)\dots(1-b_{n-2}^2)\Delta_2(b_{n-1})\\ &= (1-b_1^2)\dots(1-b_{n-2}^2)\det\begin{bmatrix}1 & 1\\ b_{n-1} & 1\end{bmatrix},\end{aligned}$$

hence

$$\Delta_n(b_1,\dots,b_{n-1}) = (1-b_{n-1})\prod_{j=1}^{n-2}(1-b_j^2). \tag{2.27}$$

Next, by using (2.21)–(2.27) in (2.19) we obtain

$$\alpha_j = \frac{1-b_{j-1}b_j}{(1+b_{j-1})(1+b_j)}\lambda, \quad j=1,\dots,n, \tag{2.28}$$

where we have set $b_0 = 0$, $b_n = 0$.

Finally, we need to find the value of λ appearing in (2.10). In view of (2.28), (2.6) gives

$$\lambda\sum_{j=1}^{n}\frac{1-b_{j-1}b_j}{(1+b_{j-1})(1+b_j)} = 1. \tag{2.29}$$

After some algebra, (2.29) implies

$$\lambda = \frac{1}{2-n+2\sum_{j=1}^{n-1}(1+b_j)^{-1}}. \tag{2.30}$$

We summarize our results in the following theorem.

Theorem 2.1. *Let A be the operator*

$$(Av)(x) = \frac{1}{2r}\int_0^b e^{-r|x-\xi|}v(\xi)d\xi, \quad x\in[0,b].$$

If $v(x)$ is an equilibrium solution to (1.8)–(1.9) *of the form* (2.5) *with $n \ge 2$, then* (2.28) *holds, where $b_0 = b_n = 0$ and b_j, for $j = 1,\dots,(n-1)$, are given by* (2.14), *while λ is given by* (2.30).

We remark that Theorem 2.1 is true even in the case where S is a semiaxis or the whole real line.

As an example we notice that in the case $n = 2$, (2.28) implies that $\alpha_1 = \alpha_2 = 1/2$, no matter what c_1 and c_2 are.

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