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ASYMPTOTIC BEHAVIOR OF GROUND STATE SOLUTIONS FOR SUBLINEAR AND SINGULAR NONLINEAR DIRICHLET PROBLEMS

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Abstract. In this article, we are concerned with the asymptotic behavior of the classical solution to the semilinear boundary-value problem

$$\Delta u + a(x)u^{\sigma} = 0$$

in $\mathbb{R}^n$, $u > 0$, $\lim_{|x|\to\infty} u(x) = 0$, where $\sigma < 1$. The special feature is to consider the function a in $C^{\alpha}_{\rm loc}(\mathbb{R}^n)$, $0 < \alpha < 1$, such that there exists $c > 0$ satisfying

$$\frac{1}{c}\frac{L(|x|+1)}{(1+|x|)^{\lambda}} \le a(x) \le c\frac{L(|x|+1)}{(1+|x|)^{\lambda}},$$

where $L(t) := \exp\big(\int_1^t \frac{z(s)}{s}ds\big)$, with $z \in C([1,\infty))$ such that $\lim_{t\to\infty} z(t) = 0$. The comparable asymptotic rate of $a(x)$ determines the asymptotic behavior of the solution.

1. Introduction

In this article, we are interested in estimates for positive solutions of the semilinear problem

$$\begin{gathered} -\Delta u = a(x)g(u), \quad x \in \mathbb{R}^n,\ n \ge 3 \\ u > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{gathered} \tag{1.1}$$

The existence of such solutions and their asymptotic behavior have been extensively studied by many authors when (1.1) has a smooth bounded domain Ω with zero boundary Dirichlet condition. We refer the reader to [1, 3, 6, 8, 9, 14] and the references therein.

In recent years, the study of ground state solutions of problem (1.1) received a lot of interest and numerous existence results have been established (see for instance [2, 4, 5, 7, 10, 12] and the references therein).

More specifically, Lair and Shaker [7] established the existence and the uniqueness of positive classical solution, where g is a positive nonincreasing and continuously differentiable function on $(0,\infty)$ and a is a nontrivial nonnegative function

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in $C^{\alpha}_{\rm loc}(\mathbb{R}^n)$, satisfying

$$\int_0^\infty t \max_{|x|=t} a(x)dt < \infty. \tag{1.2}$$

Moreover, they showed that this condition on a is nearly optimal.

Furthermore, Brezis and Kamin [2] proved the existence of a unique positive solution to the problem

$$\begin{gathered}
-\Delta u = a(x)u^{\sigma}, \quad x \in \mathbb{R}^n,\ n \geq 3\\
u > 0,\\
\liminf_{|x|\to\infty} u(x) = 0,
\end{gathered}$$

where $0 < \sigma < 1$ and a is a nonnegative measurable function potentially bounded, that is the function $x \mapsto \int_{\mathbb{R}^n} \frac{a(y)}{|x-y|^{n-2}}dy$ is in $L^\infty(\mathbb{R}^n)$.

Throughout this article, we denote $\mathcal{K}$ the set of all functions L defined on $[1,\infty)$, by

$$L(t) := c\exp\Big(\int_1^t \frac{z(s)}{s}ds\Big),$$

where c is a positive constant and $z \in C([1,\infty))$ such that $\lim_{t\to\infty} z(t) = 0$.

Remark 1.1. It is obvious that $L \in \mathcal{K}$ if and only if L is a positive function in $C^1([1,\infty))$ such that $\lim_{t\to\infty} \frac{tL'(t)}{L(t)} = 0$.

Example 1.2. Let $m \in \mathbb{N}^*$, $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$ and ω be a positive real number sufficiently large such that the function

$$L(t) = \prod_{k=1}^m (\log_k(wt))^{-\lambda_k}$$

is defined and positive on $[1,\infty)$, where $\log_k x = \log \circ \log \circ \dots \circ \log x$ (k times).Then $L \in \mathcal{K}$.

In this paper, we give precise asymptotic behavior of the solution to the problem

$$\begin{gathered}
-\Delta u = a(x)u^{\sigma}, \quad x \in \mathbb{R}^n,\ n \geq 3,\\
u > 0 \quad \text{in } \mathbb{R}^n,\\
\lim_{|x|\to\infty} u(x) = 0,
\end{gathered} \tag{1.3}$$

where $\sigma < 1$ and a satisfies the hypothesis

(H1) a is a nonnegative function in $C^{\alpha}_{\rm loc}(\mathbb{R}^n)$, $0 < \alpha < 1$, satisfying

$$a(x) \approx \frac{L(1+|x|)}{(1+|x|)^{\lambda}},$$

where $\lambda \geq 2$ and $L \in \mathcal{K}$ such that $\int_1^\infty t^{1-\lambda}L(t)dt < \infty$.

Here and throughout the paper, for two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, for $x \in S$ means that there exists $c > 0$ such that $\frac{1}{c}f(x) \leq g(x) \leq cf(x)$, for all $x \in S$.

Remark 1.3. (i) Note that we need to verify the condition $\int_1^\infty t^{1-\lambda}L(t)dt < \infty$ in hypothesis (H1), only for $\lambda = 2$ (see Remark 2.2).

(ii) It is obvious to see that if a satisfies hypothesis $(H1)$, then a is potentially bounded and a verifies (1.2). This implies from [7] and [2], that problem (1.3) has a unique classical positive solution in $C^{2,\alpha}(\mathbb{R}^n)$. Thus it becomes interesting to know the asymptotic behavior of such solution, as $t \to \infty$.

Our main result is the following.

Theorem 1.4. *Assume* (H1). *Then the solution u of problem* (1.3) *satisfies*

$$u(x) \approx \theta_\lambda(x), \tag{1.4}$$

where $x \in \mathbb{R}^n$, and θ_λ is defined on $\mathbb{R}^n$ by

$$\theta_\lambda(x) := \begin{cases} \big(\int_{|x|+1}^{\infty} \frac{L(t)}{t} dt\big)^{1/(1-\sigma)}, & \text{for } \lambda = 2, \\ \frac{(L(1+|x|))^{1/(1-\sigma)}}{(1+|x|)^{(\lambda-2)/(1-\sigma)}}, & \text{for } 2 < \lambda < n-(n-2)\sigma, \\ \frac{1}{(1+|x|)^{n-2}} \big(\int_1^{|x|+2} \frac{L(t)}{t} dt\big)^{1/(1-\sigma)}, & \text{for } \lambda = n-(n-2)\sigma, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{for } \lambda > n-(n-2)\sigma. \end{cases} \tag{1.5}$$

To obtain estimates (1.5), we shall adopt a sub-supersolution method. For the reader's convenience, we recall the definition.

A positive function $v \in C^{2,\alpha}(\mathbb{R}^n)$ is called a subsolution of problem (1.3) if

$$\begin{gathered} -\Delta v \le a(x) v^\sigma \quad x \in \mathbb{R}^n, \\ \lim_{|x|\to\infty} v(x) = 0. \end{gathered} \tag{1.6}$$

If the above inequality is reversed, v is called a supersolution of problem (1.3).

The outline of this article is as follows. In Section 2, we state some already known results on functions in $\mathcal{K}$, useful for our study and we give estimates on some potential functions. The proof of Theorem 1.4 is given in Section 3. The last section is reserved to some applications.

We close this section by giving the following notation. For a nonnegative measurable function a in $\mathbb{R}^n$, we denote by Va the potential of a defined on $\mathbb{R}^n$ by

$$Va(x) = \int_{\mathbb{R}^n} G(x,y) a(y) dy,$$

where $G(x,y) = \frac{c_n}{|x-y|^{n-2}}$ is the Green function of the Laplacian Δ in $\mathbb{R}^n$ ($n \ge 3$), and $c_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$. We point out that for any nonnegative function f in $C^{\alpha}_{\rm loc}(\mathbb{R}^n)$ ($0 < \alpha < 1$) such that $Vf \in L^\infty(\mathbb{R}^n)$, we have $Vf \in C^{2,\alpha}_{\rm loc}(\mathbb{R}^n)$ and satisfies $-\Delta(Vf) = f$ in $\mathbb{R}^n$; see [11, Theorem 6.3].

2. Key estimates

2.1. **Technical lemmas.** In what follows, we collect some fundamental properties of functions belonging to the class $\mathcal{K}$. First, we need the following elementary result.

Lemma 2.1 (Karamata's Theorem)**.** *Assume that $g \in C^1([\beta,\infty),(0,\infty))$ and that $\lim_{t\to\infty} tg'(t)/g(t) = \gamma$. Then we have the following properties:*

(i) *If $\gamma < -1$, then $\int_\beta^\infty g(s)ds$ converges and*

$$\int_t^\infty g(s)ds \sim -\frac{tg(t)}{\gamma+1}, \quad \text{as } t \to \infty.$$

(ii) *If* $\gamma > -1$*, then* $\int_\beta^\infty g(s)ds$ *diverges and*

$$\int_\beta^t g(s)ds \sim \frac{tg(t)}{\gamma+1} \quad \text{as } t \to \infty.$$

Remark 2.2. Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$. Applying Lemma 2.1 to $g(t) = t^\gamma L(t)$, we obtain that

- If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s)ds$ diverges and $\int_t^\infty s^\gamma L(s)ds \sim -\frac{t^{1+\gamma}L(t)}{\gamma+1}$, as $t \to \infty$;
- If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)ds$ converges and $\int_1^t s^\gamma L(s)ds \sim \frac{t^{1+\gamma}L(t)}{\gamma+1}$ as $t \to \infty$.

Lemma 2.3. *(i) Let* $L_1, L_2 \in \mathcal{K}$*,* $p \in \mathbb{R}$*. Then* $L_1L_2 \in \mathcal{K}$ *and* $L_1^p \in \mathcal{K}$*.*

(ii) Let L *be a function in* $\mathcal{K}$ *then there exists* $m \geq 0$ *such that for every* $\eta > 0$ *and* $t \geq 1$ *,we have*

$$(1+\eta)^{-m}L(t) \leq L(\eta + t) \leq (1+\eta)^m L(t).$$

Proof. Assertion (i) is due to Remark 1.1. Let us prove (ii). Let z be the function in $C([1,\infty))$ such that $\lim_{t\to\infty} z(t) = 0$ and $L(t) = \exp\big(\int_1^t \frac{z(s)}{s}ds\big)$.

Put $m = \sup_{t\in[1,\infty)} |z(t)|$, then for each $\eta > 0$ and $t \geq 1$, we have

$$m \log \frac{t}{t+\eta} \leq \int_t^{t+\eta} \frac{z(s)}{s}ds \leq m \log \frac{t+\eta}{t}.$$

That is,

$$(1+\frac{\eta}{t})^{-m} \leq \exp\Big(\int_t^{t+\eta} \frac{z(s)}{s}ds\Big) \leq (1+\frac{\eta}{t})^m.$$

So (ii) holds. □

Lemma 2.4. *Let* $L \in \mathcal{K}$ *and* $\varepsilon > 0$*, then we have*

$$\lim_{t\to\infty} t^{-\varepsilon}L(t) = 0, \tag{2.1}$$

$$\lim_{t\to\infty} \frac{L(t)}{\int_1^t L(s)/s\,ds} = 0. \tag{2.2}$$

If further $\int_1^\infty L(s)/s\,ds$ *converges, then*

$$\lim_{t\to\infty} \frac{L(t)}{\int_t^\infty L(s)/s\,ds} = 0. \tag{2.3}$$

Proof. Let $L \in \mathcal{K}$ and $\varepsilon > 0$. It is obvious by Remark 1.1 that the function $t \to t^{-\frac{\varepsilon}{2}}L(t)$ is non-increasing in $[\omega,\infty)$, for ω large enough. Then

$$t^{-\frac{\varepsilon}{2}}L(t) \leq \omega^{-\frac{\varepsilon}{2}}L(\omega), \text{ for } t \geq \omega;$$

That is,

$$t^{-\varepsilon}L(t) \leq \frac{L(\omega)}{(\omega t)^{\varepsilon/2}}, \quad \text{for } t \geq \omega.$$

This proves (2.1). For the rest of the proof, we distinguish two cases.

Case 1: $\int_1^\infty \frac{L(s)}{s}ds < \infty$. Since the function $t \to tL(t)$ is nondecreasing in $[\omega,\infty)$, then

$$tL(t)\int_t^\infty \frac{ds}{s^2} \leq \int_t^\infty \frac{L(s)}{s}ds, \quad \text{for } t \geq \omega.$$

Hence

$$0 < L(t) \le \int_t^\infty \frac{L(s)}{s} ds, \quad \text{for } t \ge \omega.$$

Then $\lim_{t\to\infty} L(t) = 0$, which implies (2.2).

Moreover, put $\varphi(t) = L(t)/t$, for $t \ge 1$. Since φ satisfies $\lim_{t\to\infty} t\varphi'(t)/\varphi(t) = -1$, then it follows that

$$\int_t^\infty \varphi(s)ds \sim -\int_t^\infty s\varphi'(s)ds = t\varphi(t) + \int_t^\infty \varphi(s)ds,$$

as $t \to \infty$. This implies that

$$\int_t^\infty \frac{L(s)}{s} ds \sim L(t) + \int_t^\infty \frac{L(s)}{s} ds,$$

as $t \to \infty$. So we deduce (2.3).

Case 2: $\int_1^\infty \frac{L(s)}{s} ds = \infty$. Put $\varphi(t) = L(t)/t$, for $t \ge 1$. Then for ω sufficiently large and $t \ge \omega$, we have

$$\int_\omega^t \varphi(s)ds \sim -t\varphi(t) + \omega\varphi(\omega) + \int_\omega^t \varphi(s)ds,$$

as $t \to \infty$; that is,

$$\int_\omega^t \frac{L(s)}{s} ds \sim -L(t) + \omega\varphi(\omega) + \int_\omega^t \frac{L(s)}{s} ds,$$

as $t \to \infty$. Which proves (2.2) and completes the proof. $\square$

Remark 2.5. Let $L \in \mathcal{K}$. Using Remark 1.1 and (2.2), we deduce that

$$t \to \int_1^{t+1} \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_1^\infty L(s)/s\, ds$ converges, we have by (2.3) that

$$t \mapsto \int_t^\infty \frac{L(s)}{s}\, ds \in \mathcal{K}.$$

2.2. **Asymptotic behavior of some potential functions.** We are going to give estimates on the potential functions Va and $V(a\theta_\lambda^\sigma)$, where θ_λ is the function given in (1.5).

Proposition 2.6. *Let a be a function satisfying* (H1)*. Then for $x \in \mathbb{R}^n$,*

$$Va(x) \approx \psi(|x|),$$

where ψ is the function defined in $[0,\infty)$ by

$$\psi(t) = \begin{cases} \int_{t+1}^\infty L(r)/r\, dr, & \text{for } \lambda = 2, \\ \frac{L(1+t)}{(1+t)^{\lambda-2}}, & \text{for } 2 < \lambda < n, \\ \frac{1}{(1+t)^{n-2}} \int_1^{t+2} L(r)/r\, dr, & \text{for } \lambda = n, \\ \frac{1}{(1+t)^{n-2}}, & \text{for } \lambda > n. \end{cases} \tag{2.4}$$

Proof. First, we recall the following well known result. Let φ be a nonnegative radial measurable function and $x \in \mathbb{R}^n$, then we have

$$\int_{\mathbb{R}^n} \frac{\varphi(y)}{|x-y|^{n-2}} dy = c \int_0^\infty \frac{r^{n-1}}{\max(|x|,r)^{n-2}} \varphi(r) dr.$$

Now, let $\lambda \geq 2$ and $L \in \mathcal{K}$ satisfying $\int_1^\infty t^{1-\lambda} L(t) dt < \infty$ and such that

$$a(x) \approx \frac{L(1+|x|)}{(1+|x|)^\lambda}.$$

Thus

$$Va(x) \approx \int_{\mathbb{R}^n} \frac{L(1+|y|)}{(1+|y|)^\lambda} \frac{1}{|x-y|^{n-2}} dy = c_n I(|x|),$$

where I is the function defined on $[0,\infty)$ by

$$I(t) = \int_0^\infty \frac{r^{n-1} L(1+r)}{\max(t,r)^{n-2}(1+r)^\lambda} dr.$$

So to prove the result, it is sufficient to show that $I(t) \approx \psi(t)$ for $t \in [0,\infty)$. We have

$$\begin{aligned} I(t) &= \frac{1}{t^{n-2}} \int_0^1 \frac{r^{n-1} L(1+r)}{(1+r)^\lambda} dr + \frac{1}{t^{n-2}} \int_1^t \frac{r^{n-1} L(1+r)}{(1+r)^\lambda} dr + \int_t^\infty \frac{r L(1+r)}{(1+r)^\lambda} dr \\ &:= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

It is clear that for $t \geq 2$,

$$I_1(t) \approx \frac{1}{t^{n-2}}. \tag{2.5}$$

To estimate I_2 and I_3, we distinguish two cases.

Case 1: $\lambda > 2$. Using Lemma 2.3 (ii) and Remark 2.2, for $t \geq 2$ we have

$$I_3(t) \approx \int_t^\infty r^{1-\lambda} L(r) dr \approx \frac{L(t)}{t^{\lambda-2}}. \tag{2.6}$$

• If $2 < \lambda < n$, then applying again Remark 2.2, we have $\int_1^\infty r^{n-1-\lambda} L(r) dr = \infty$ and $\int_1^t r^{n-1-\lambda} L(r) dr \sim t^{2-\lambda} L(t)$, as $t \to \infty$. So by Lemma 2.3 (ii), for $t \geq 2$ we obtain

$$I_2(t) \approx \frac{1}{t^{n-2}} \int_1^t r^{n-1-\lambda} L(r) dr \approx \frac{L(t)}{t^{\lambda-2}}.$$

Then by (2.5), (2.6) and (2.1), for $t \geq 2$ we have

$$I(t) \approx \frac{1}{t^{n-2}} + \frac{L(t)}{t^{\lambda-2}} \approx \frac{L(t)}{t^{\lambda-2}}.$$

Now, since the functions $t \to I(t)$ and $t \to \frac{L(1+t)}{(1+t)^{\lambda-2}}$ are positive and continuous in $[0,\infty)$, for $t \geq 0$ we obtain

$$I(t) \approx \frac{L(1+t)}{(1+t)^{\lambda-2}}.$$

• If $\lambda > n$, then applying Remark 2.2, we have $\int_1^t r^{n-1-\lambda} L(r) dr < \infty$. So by Lemma 2.3 (ii), for $t \geq 2$, we obtain

$$I_2(t) \approx \frac{1}{t^{n-2}} \int_1^t r^{n-1-\lambda} L(r) dr \approx \frac{1}{t^{n-2}}.$$

This together with (2.5), (2.6) and (2.1) implies that for $t \geq 2$,

$$I(t) \approx \frac{1}{t^{n-2}}.$$

Then by the same argument as above, we deduce that for $t \geq 0$,

$$I(t) \approx \frac{1}{(1+t)^{n-2}}.$$

• If $\lambda = n$, then using (2.5), (2.6) and (2.2), for $t \geq 2$, we have

$$I(t) \approx \frac{1}{t^{n-2}}(1 + \int_1^t \frac{L(r)}{r} dr + L(t)) \approx \frac{1}{t^{n-2}} \int_1^t \frac{L(r)}{r} dr.$$

So for $t \geq 0$, we obtain

$$I(t) \approx \frac{1}{(1+t)^{n-2}} \int_1^{t+2} \frac{L(r)}{r} dr.$$

Case 2: $\lambda = 2$. By Remark 2.2, for $t \geq 2$, we have $I_2(t) \approx L(t)$. So for $t \geq 2$, we have

$$I(t) \approx \frac{1}{t^{n-2}} + L(t) + \int_t^\infty \frac{L(r)}{r} dr.$$

Hence using (2.1) and (2.3), for $t \geq 2$, we have

$$I(t) \approx \int_t^\infty \frac{L(r)}{r} dr.$$

So for $t \geq 0$, we obtain

$$I(t) \approx \int_{t+1}^\infty \frac{L(r)}{r} dr.$$

This completes the proof. $\square$

The following Proposition plays a key role in this paper.

Proposition 2.7. *Let a be a function satisfying (H1) and let θ_λ be the function given by* (1.5)*. Then for $x \in \mathbb{R}^n$,*

$$V(a\theta_\lambda^\sigma)(x) \approx \theta_\lambda(x).$$

Proof. Let $\lambda \geq 2$ and $L \in \mathcal{K}$ satisfying $\int_1^\infty t^{1-\lambda} L(t) dt < \infty$ and such that

$$a(x) \approx \frac{L(1+|x|)}{(1+|x|)^\lambda}.$$

Then for every $x \in \mathbb{R}^n$, we have

$$a(x)\theta_\lambda^\sigma(x) \approx h(x) := \begin{cases} \frac{L(1+|x|)}{(1+|x|)^2}\big(\int_{|x|+1}^\infty \frac{L(t)}{t} dt\big)^{\sigma/(1-\sigma)}, & \lambda = 2, \\ \frac{(L(1+|x|))^{1/(1-\sigma)}}{(1+|x|)^{(\lambda-2\sigma)/(1-\sigma)}}, & 2 < \lambda < n-(n-2)\sigma, \\ \frac{L(1+|x|)}{(1+|x|)^n}\big(\int_1^{|x|+2} \frac{L(t)}{t} dt\big)^{\sigma/(1-\sigma)}, & \lambda = n-(n-2)\sigma, \\ \frac{L(1+|x|)}{(1+|x|)^{\lambda+(n-2)\sigma}}, & \lambda > n-(n-2)\sigma. \end{cases}$$

We point out that $h(x) = \frac{\widetilde{L}(1+|x|)}{(1+|x|)^\mu}$, where $\mu \geq 2$. Moreover, due to Lemma 2.3 and Remark 2.5, we deduce that $\widetilde{L} \in \mathcal{K}$ and satisfies $\int_1^\infty t^{1-\mu}\widetilde{L}(t) dt < \infty$. Hence, it follows by Proposition 2.6, that

$$V(a\theta_\lambda^\sigma)(x) \approx V(h)(x) \approx \widetilde{\psi}(|x|), \quad \text{in } \mathbb{R}^n,$$

where $\widetilde{\psi}$ is the function defined by (2.4) by replacing L by $\widetilde{L}$ and λ by μ. This completes the proof by a simple calculus. $\square$

3. Proof of Theorem 1.4

Let a be a function satisfying (H1). The main idea is to find a subsolution and a supersolution of problem (1.3) of the form $cV(a\phi^\sigma)$, where $c > 0$ and $\phi(x) = \frac{L_0(1+|x|)}{(1+|x|)^\beta}$, which will satisfy necessarily

$$V(a\phi^\sigma) \approx \phi. \tag{3.1}$$

So, the choice of the real β and the function L_0 in $\mathcal{K}$ is such that (3.1) is satisfied. Setting $\phi(x) = \theta_\lambda(x)$, where θ_λ is the function given by (1.5), we have by Proposition 2.7 that the function θ_λ satisfies (3.1).

Let $v := V(a\theta_\lambda^\sigma)$ and let $M > 1$ be such that

$$\frac{1}{M} v \leq \theta_\lambda \leq Mv.$$

Which implies that for $\sigma < 1$,

$$\frac{v^\sigma}{M^{|\sigma|}} \leq \theta_\lambda^\sigma \leq M^{|\sigma|} v^\sigma.$$

Put $c := M^{|\sigma|/(1-\sigma)}$, then it is easy to verify that $\underline{u} = \frac{1}{c} v$ and $\overline{u} = cv$ are respectively a subsolution and a supersolution of problem (1.3).

Now, since $c > 1$, we get $\underline{u} \leq \overline{u}$ on $\mathbb{R}^n$ and thanks to the method of sub and supersolution, it follows that problem (1.3) has a solution u satisfying $\underline{u} \leq u \leq \overline{u}$, in $\mathbb{R}^n$.

Finally, by using Remark 1.3 (ii) and Proposition 2.7, we deduce that the unique classical positive solution of problem (1.3) satisfies (1.4). This completes the proof.

4. Applications

4.1. **First application.** Let $\sigma < 1$ and a be a positive function in $C^\alpha_{\rm loc}(\mathbb{R}^n)$ satisfying for $x \in \mathbb{R}^n$

$$a(x) \approx \frac{1}{(1+|x|)^\lambda} \prod_{k=1}^m (\log_k(w(1+|x|)))^{-\lambda_{\mathbf{k}}},$$

where $m \in \mathbb{N}^*$ and w is a positive constant large enough. The real numbers λ and λ_k, $1 \leq k \leq m$, satisfy one of the following two conditions

- $\lambda > 2$ and $\lambda_k \in \mathbb{R}$ for $1 \leq k \leq m$.
- $\lambda = 2$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{l-1} = 1$, $\lambda_l > 1$, $\lambda_k \in \mathbb{R}$ for $l+1 \leq k \leq m$.

Using Theorem 1.4, we deduce that problem (1.3) has a unique classical positive solution u in $\mathbb{R}^n$ satisfying

(i) If $\lambda = 2$, then for $x \in \mathbb{R}^n$

$$u(x) \approx (\log_l w(1+|x|))^{(1-\lambda_l)/(1-\sigma)} \prod_{k=l+1}^m (\log_k w(1+|x|))^{-\lambda_k/(1-\sigma)}.$$

(ii) If $2 < \lambda < n - \sigma(n-2)$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \frac{1}{(1+|x|)^{(\lambda-2)/(1-\sigma)}} \prod_{k=1}^m (\log_k w(1+|x|))^{-\lambda_k/(1-\sigma)}.$$

(iii) If $\lambda = n - \sigma(n-2)$ and $\lambda_1 = \lambda_2 = \dots = \lambda_m = 1$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \frac{1}{(1+|x|)^{n-2}} (\log_{m+1} w(1+|x|))^{1/(1-\sigma)}.$$

(iv) If $\lambda = n - \sigma(n-2)$ and $\lambda_1 = \lambda_2 = \dots = \lambda_{l-1} = 1$, $\lambda_l < 1$, $\lambda_k \in \mathbb{R}$, for $l+1 \le k \le m$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \frac{1}{(1+|x|)^{n-2}} (\log_l w(1+|x|))^{(1-\lambda_l)/(1-\sigma)} \prod_{k=l+1}^{m} (\log_k w(1+|x|))^{-\lambda_k/(1-\sigma)}.$$

(v) If $\lambda = n - \sigma(n-2)$ and $\lambda_1 = \lambda_2 = \dots = \lambda_{l-1} = 1$, $\lambda_l > 1$, $\lambda_k \in \mathbb{R}$, for $l+1 \le k \le m$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \frac{1}{(1+|x|)^{n-2}}.$$

(vi) If $\lambda > n + \sigma(n-2)$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \frac{1}{(1+|x|)^{n-2}}.$$

4.2. **Second application.** Let a be a function satisfying (H1) and let $\sigma, \beta < 1$. We are interested in the problem

$$\begin{gathered} -\Delta u + \frac{\beta}{u}|\nabla u|^2 = a(x)u^\sigma \quad \text{in } \mathbb{R}^n, \\ u > 0, \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0. \end{gathered} \tag{4.1}$$

Put $v = u^{1-\beta}$, then by a simple calculus, we obtain that v satisfies

$$\begin{gathered} -\Delta v = (1-\beta)a(x)v^{\frac{\sigma-\beta}{1-\beta}} \quad \text{in } \mathbb{R}^n, \\ v > 0 \quad \text{in } \mathbb{R}^n, \\ \lim_{|x|\to\infty} v(x) = 0. \end{gathered} \tag{4.2}$$

Applying Theorem 1.4 to problem (4.2), we obtain that there exists a unique solution v such that

$$v(x) \approx \widetilde{\theta}_\lambda(x) := \begin{cases} \left(\int_{|x|+1}^\infty \frac{L(s)}{s} ds\right)^{(1-\beta)/(1-\sigma)}, & \text{if } \lambda = 2, \\ \frac{(L(1+|x|))^{(1-\beta)/(1-\sigma)}}{(1+|x|)^{(\lambda-2)/(1-\sigma)}}, & \text{if } 2 < \lambda < n-(n-2)\frac{\sigma-\beta}{1-\beta}, \\ \frac{1}{(1+|x|)^{n-2}}\left(\int_1^{|x|+2} \frac{L(s)}{s} ds\right)^{\frac{1-\beta}{1-\sigma}}, & \text{if } \lambda = n-(n-2)\frac{\sigma-\beta}{1-\beta}, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{if } \lambda > n-(n-2)\frac{\sigma-\beta}{1-\beta}. \end{cases}$$

Consequently, we deduce that (4.1) has a unique positive solution u satisfying

$$\begin{aligned} u(x) &\approx \left(\widetilde{\theta}_\lambda(x)\right)^{1/(1-\beta)} \\ &= \begin{cases} \left(\int_{|x|+1}^\infty \frac{L(s)}{s} ds\right)^{1/(1-\sigma)}, & \text{if } \lambda = 2, \\ (1+|x|)^{\frac{2-\lambda}{(1-\sigma)(1-\beta)}} (L(1+|x|))^{1/(1-\sigma)}, & \text{if } 2 < \lambda < n-(n-2)\frac{\sigma-\beta}{1-\beta}, \\ (1+|x|)^{(2-n)/(1-\beta)} \left(\int_1^{|x|+2} \frac{L(s)}{s} ds\right)^{1/(1-\sigma)}, & \text{if } \lambda = n-(n-2)\frac{\sigma-\beta}{1-\beta}, \\ (1+|x|)^{(2-n)/(1-\beta)}, & \text{if } \lambda > n-(n-2)\frac{\sigma-\beta}{1-\beta}. \end{cases} \end{aligned}$$

4.3. **Third application.** Let a be a function satisfying (H1) and L be a function in $\mathcal{K}$ such that

$$a(x) \approx \frac{L(1+|x|)}{(1+|x|)^{\lambda}}.$$

Let $b \in C^{\gamma}_{\rm loc}(\mathbb{R}^n)$, $0<\gamma<1$ satisfying for $x \in \mathbb{R}^n$,

$$b(x) \approx \frac{L_1(1+|x|)}{(1+|x|)^{\mu}},$$

where $\mu \in \mathbb{R}$ and $L_1 \in \mathcal{K}$. Let $\sigma, \beta < 1$ and $p \in \mathbb{R}$. We are interested in the system

$$\begin{gathered} -\Delta u = a(x)u^{\sigma} \quad \text{in } \mathbb{R}^n, \\ -\Delta v = b(x)u^{p}v^{\beta} \quad \text{in } \mathbb{R}^n, \\ u, v > 0 \quad \text{in } \mathbb{R}^n, \ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0. \end{gathered} \tag{4.3}$$

By Theorem 1.4, it follows that there exists a unique classical solution u to (1.3) satisfying (1.4). So, we distinguish the following cases.

Case 1: $\lambda = 2$. By hypothesis (H1), we have $\int_1^{\infty} \frac{L(t)}{t} dt < \infty$ and using estimates (1.5), we deduce that

$$b(x)u^p(x) \approx \frac{L_1(1+|x|)}{(1+|x|)^{\mu}} \Big(\int_{|x|+1}^{\infty} \frac{L(s)}{s} ds \Big)^{p/(1-\sigma)} := \frac{L_2(1+|x|)}{(1+|x|)^{\mu}}.$$

It is obvious to see by Lemma 2.3 and Remark 2.5 that $L_2 \in \mathcal{K}$. Now suppose that $\mu \geq 2$ and $\int_1^{\infty} t^{1-\mu} L_2(t) dt < \infty$. Then applying Theorem 1.4, we conclude that (4.3) has a unique classical solution (u,v) such that $u(x) \approx \theta_{\lambda}(x)$ and

$$v(x) \approx \begin{cases} \big(\int_{|x|+1}^{\infty} \frac{L_2(s)}{s} ds \big)^{1/(1-\beta)}, & \text{if } \mu = 2, \\ \frac{(L_2(1+|x|))^{1/(1-\beta)}}{(1+|x|)^{(\mu-2)/(1-\beta)}}, & \text{if } 2 < \mu < n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}} \big(\int_1^{|x|+2} \frac{L_2(s)}{s} ds \big)^{1/(1-\beta)}, & \text{if } \mu = n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{if } \mu > n-(n-2)\beta. \end{cases}$$

Case 2: $2 < \lambda < n-(n-2)\sigma$. Put $\gamma = \mu + \frac{\lambda-2}{1-\sigma} p$. From the estimates (1.5), we deduce that

$$b(x)u^p(x) \approx \frac{L_1(1+|x|)(L(1+|x|))^{p/(1-\sigma)}}{(1+|x|)^{\gamma}} := \frac{L_2(1+|x|)}{(1+|x|)^{\gamma}}.$$

Obviously by Lemma 2.3 we have that $L_2 \in \mathcal{K}$. Now suppose that $\gamma \geq 2$ and $\int_1^{\infty} t^{1-\gamma} L_2(t) dt < \infty$. Then applying Theorem 1.4, we conclude that system (4.3) has a unique classical solution (u,v) such that $u(x) \approx \theta_{\lambda}(x)$ and

$$v(x) \approx \begin{cases} \big(\int_{|x|+1}^{\infty} \frac{L_2(s)}{s} ds \big)^{1/(1-\beta)}, & \text{if } \gamma = 2, \\ \frac{L_2((1+|x|))^{1/(1-\beta)}}{(1+|x|)^{(\gamma-2)/(1-\beta)}}, & \text{if } 2 < \gamma < n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}} \big(\int_1^{|x|+2} \frac{L_2(s)}{s} ds \big)^{1/(1-\beta)}, & \text{if } \gamma = n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{if } \gamma > n-(n-2)\beta. \end{cases}$$

Case 3: $\lambda = n-(n-2)\sigma$. We have

$$b(x)u^p(x) \approx \frac{L_1(1+|x|)}{(1+|x|)^{\mu+(n-2)p}} \Big(\int_1^{|x|+2} \frac{L(s)}{s} ds \Big)^{p/(1-\sigma)} := \frac{L_2(1+|x|)}{(1+|x|)^{\mu+(n-2)p}}.$$

By Lemma 2.3 and Remark 2.5, obviously we have that $L_2 \in \mathcal{K}$. Now suppose that $\mu + (n-2)p \geq 2$ and $\int_1^\infty t^{1-\mu-(n-2)p} L_2(t)dt < \infty$. Then applying Theorem 1.4, we conclude that (4.3) has a unique classical solution (u, v) such that $u(x) \approx \theta_\lambda(x)$ and

$$v(x) \approx \begin{cases} \big(\int_{|x|+1}^\infty \frac{L_2(s)}{s} ds\big)^{1/(1-\beta)}, & \text{if } \mu + (n-2)p = 2, \\ \frac{(L_2(1+|x|))^{1/(1-\beta)}}{(1+|x|)^{\frac{\mu+(n-2)p}{1-\beta}}}, & \text{if } 2 < \mu + (n-2)p < n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}} \big(\int_1^{|x|+2} \frac{L_2(s)}{s} ds\big)^{1/(1-\beta)}, & \text{if } \mu + (n-2)p = n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{if } \mu + (n-2)p > n-(n-2)\beta. \end{cases}$$

Case 4: $\lambda > n - (n-2)\sigma$. We have

$$b(x)u^p(x) \approx \frac{L_1(1+|x|)}{(1+|x|)^{n-2+\mu}}.$$

Suppose that $n - 2 + \mu \geq 2$ and $\int_1^\infty t^{1-(n-2+\mu)} L_1(t)dt < \infty$. Then applying Theorem 1.4, we conclude that (4.3) has a unique classical solution (u, v) such that $u(x) \approx \theta_\lambda(x)$ and

$$v(x) \approx \begin{cases} \big(\int_{|x|+1}^\infty \frac{L_1(s)}{s} ds\big)^{1/(1-\beta)}, & \text{if } n-2+\mu = 2, \\ \frac{(L_1(1+|x|))^{1/(1-\beta)}}{(1+|x|)^{(\mu+n-4)/(1-\beta)}}, & \text{if } 2 < n-2+\mu < n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}} \big(\int_1^{|x|+2} \frac{L_1(s)}{s} ds\big)^{1/(1-\beta)}, & \text{if } n-2+\mu = n-(n-2)\beta, \\ \frac{1}{(1+|x|)^{n-2}}, & \text{if } n-2+\mu > n-(n-2)\beta. \end{cases}$$

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