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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A SINGULAR SEMILINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^2$

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Abstract. Using minimax methods we study the existence and multiplicity of nontrivial solutions for a singular class of semilinear elliptic nonhomogeneous equation where the potentials can change sign and the nonlinearities may be unbounded in x and behaves like $\exp(\alpha s^2)$ when $|s| \to +\infty$. We establish the existence of two distinct solutions when the perturbation is suitable small.

1. Introduction

In this article, we consider the semilinear elliptic equation

$$-\Delta u + V(x)u = \frac{g(x)f(u)}{|x|^a} + h(x) \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $a \in [0, 2)$, the functions $V, g : \mathbb{R}^2 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous with $f(0) = 0$ and $h \in (H^1(\mathbb{R}^2))^* \equiv H^{-1}$ is a small perturbation, $h \not\equiv 0$. We are interested in finding nontrivial solutions of (1.1) when the nonlinearity $f(s)$ has the maximal growth which allows to treat (1.1) variationally in the Sobolev space $H^1(\mathbb{R}^2)$.

On the potentials we assume the hypothesis

- (V1) There exist $D > 0$ such that $V(x) \geq -D$, for all $x \in \mathbb{R}^2$;
- (V2) $\lambda_1 = \inf_{u\in E\setminus\{0\}} \|u\|_E^2/\|u\|_2^2 > 0$;

where E is the following subspace of $H^1(\mathbb{R}^2)$

$$E = \{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 \,\mathrm{d}x < \infty\},$$

which is a Hilbert space endowed with the scalar product

$$\langle u, v\rangle_E = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla v + V(x)uv]\mathrm{d}x$$

to which corresponds the norm $\|u\|_E = \langle u, u\rangle_E^{1/2}$ (see [17, Lemma 2.1 and Proposition 3.1]). Here, as usual, $H^1(\mathbb{R}^2)$ denotes the Sobolev spaces modelled in $L^2(\mathbb{R}^2)$

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with norm

$$\|u\|_{1,2} = \Big(\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2)\,\mathrm{d}x \Big)^{1/2}.$$

To ensure the continuous imbedding of E into $H^1(\mathbb{R}^2)$, we assume the condition (V2) on the first eigenvalue of the operator $A = -\Delta + V(x)$ (see [17, Proposition 2.2]).

We use the following notation: if $\Omega \subset \mathbb{R}^2$ is open and $s \geq 2$, we set

$$\nu_s(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega [|\nabla u|^2 + V(x)u^2]\,\mathrm{d}x}{\big(\int_\Omega |u|^s\,\mathrm{d}x\big)^{2/s}},$$

and we put $\nu_s(\emptyset) = \infty$. To obtain a compactness result, we shall consider the following assumptions:

(V3) $\lim_{R\to\infty} \nu_s(\mathbb{R}^2\setminus\overline{B}_R) = \infty$.

(V4) There exist a function $K(x) \in L^\infty_{\rm loc}(\mathbb{R}^2)$, with $K(x) \geq 1$, and constants $\alpha > 1$, $c_0, R_0 > 0$ such that

$$K(x) \leq c_0[1 + (V^+(x))^{1/\alpha}],$$

for all $|x| \geq R_0$, where $V^+(x) = \max_{x\in\mathbb{R}^2}\{0, V(x)\}$.

It is also well known that assumptions (V3)–(V4) imply that the imbeddings of E into $L^q(\mathbb{R}^2)$ are compact for all $2 \leq q < \infty$ (see [17, Proposition 3.1]).

Concerning the function g, we assume that it is strictly positive and does not have to be bounded in x provided that the growth of g is controlled by the growth of $V(x)$. More precisely:

(H1) There exists $a_0, b_0 > 0$ such that $a_0 \leq g(x) \leq b_0 K(x)$ for all $x \in \mathbb{R}^2$.

Moreover, we suppose that $f(s)$ satisfies the following conditions:

(H2) $\lim_{s\to 0} \frac{f(s)}{s} = 0$.

(H3) There is a number $\mu > 2$ such that for all $s \in \mathbb{R}\setminus\{0\}$

$$0 < \mu F(s) := \mu \int_0^s f(t)\mathrm{d}t \leq s f(s).$$

Motivated by Trudinger-Moser inequality (see [14, 19]) and by pioneer works of Adimurthi [1] and de Figueiredo et al. [6] we treat the so-called subcritical case, which we define next. We say that a function $f(s)$ has *subcritical growth* at $+\infty$ if for all $\beta > 0$

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\beta s^2}} = 0. \tag{1.2}$$

Throughout this paper, we denote by H^{-1} the dual space of $H^1(\mathbb{R}^2)$ with the usual norm $\|\cdot\|_{H^{-1}}$.

Next we state our existence result.

Theorem 1.1. *If $f(s)$ has subcritical growth at $+\infty$ and* (V1)–(V4), (H1)–(H3) *are satisfied then problem* (1.1) *has a weak solution with positive energy if $h \equiv 0$. Moreover, if $h \not\equiv 0$, there exists $\delta > 0$ such that if $\|h\|_{H^{-1}} < \delta$, problem* (1.1) *has at least two weak solutions. One of them with positive energy, while the other one with negative energy.*

The results in this paper were in part motivated by several recent papers on elliptic problems involving exponential growth. See for example de Souza [7] for the singular and homogeneous case, Giacomoni-Sreenadh [13] for the singular and nonhomogeneous case, do Ó et al. [12] and Tonkes [18] for the nonsingular and nonhomogeneous case, Cao [5], de Figueiredo et al. [6] and do Ó [11] for the nonsingular and homogeneous case. Our paper is closely related to the recent works of do Ó et al. [12] and Rabelo [15]. Indeed, we improve and complement the results in do Ó et al. [12] for the subcritical case in the sense that we use nonlinearities unbounded in x and potentials which can change sign. Moreover in [12] was studied the existence and multiplicity of weak solutions of (1.1) in terms of the Trudinger-Moser inequality for the nonsingular case. We point out that ours results are closely related with results in [3, 7, 8, 9, 10].

The proofs of our existence results rely on minimization methods in combination with the mountain-pass theorem. In the subcritical case we are able to prove that the associated functional satisfies the Palais-Smale compactness condition which allow us to obtain critical points for the functional. As a consequence we can distinguish the local minimum solution from the mountain-pass solution.

Remark 1.2. The study of such a class of problem has been motivated in part by the search for standing waves for the nonlinear Schrödinger equation (see for instance [4] and [16])

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi - G(|\psi|)\psi - e^{i\lambda t}L(x), \quad x \in \mathbb{R}^2,$$

where $\psi = \psi(t,x)$, $\psi : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$, λ is a positive constant, $W : \mathbb{R}^2 \to \mathbb{R}$ is a given potential and for suitable functions $G : \mathbb{R}^+ \to \mathbb{R},\ L : \mathbb{R}^2 \to \mathbb{R}$.

This article is organized as follows. Section 2 contains some preliminary results including a singular Trudinger-Moser inequality. In Section 3, contains the variational framework and we also check the geometric conditions of the associated functional. In Section 4, we prove some properties of the Palais-Smale sequences. Finally, in section 5 we complete the proofs of our main results.

2. Preliminary results

Let Ω be a bounded domain in $\mathbb{R}^2$, we know by the Trudinger-Moser inequality that for all $\beta > 0$ and $u \in H_0^1(\Omega)$, $e^{\beta u^2} \in L^1(\Omega)$ (see [14, 19]). Moreover, there exists a positive constant C such that

$$\sup_{u\in H_0^1(\Omega)\,:\,\|\nabla u\|_2\le 1} \int_\Omega e^{\beta u^2}\mathrm{d}x \le C|\Omega| \quad \text{if } \beta \le 4\pi,$$

where $|\Omega|$ denotes Lebesgue measure of Ω. This inequality is optimal, in the sense that for any growth $e^{\beta u^2}$ with $\beta > 4\pi$ the correspondent supremum is infinite. Adimurthi-Sandeep [2] proved a singular Trudinger-Moser inequality, which in the case $N = 2$ reads:

$$\int_\Omega \frac{e^{\beta u^2}}{|x|^a}\mathrm{d}x < \infty \quad \text{for all } u \in H_0^1(\Omega),\ \beta > 0,$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ containing the origin and $a \in [0,2)$. Moreover, there exists a positive constant $C(\beta,a)$ such that

$$\sup_{u\in H_0^1(\Omega):\|\nabla u\|_2\le 1} \int_\Omega \frac{e^{\beta u^2}}{|x|^a}\mathrm{d}x \le C(\beta,a)|\Omega| \quad \text{if and only if} \quad \beta/4\pi + a/2 \le 1. \tag{2.1}$$

Here we shall use the following extension of these results for the whole space $\mathbb{R}^2$ obtained by Giacomoni and Sreenadh in [13] (see also [7]):

Lemma 2.1. *If $\beta > 0$, $a \in [0, 2)$ and $u \in H^1(\mathbb{R}^2)$ then*

$$\int_{\mathbb{R}^2} \frac{(e^{\beta u^2} - 1)}{|x|^a} \mathrm{d}x < \infty. \tag{2.2}$$

Moreover, if $\beta/4\pi + a/2 < 1$ and $\|u\|_2 \le M$, then there exists a positive constant $C = C(\beta, M)$ such that

$$\sup_{\|\nabla u\|_2 \le 1} \int_{\mathbb{R}^2} \frac{(e^{\beta u^2} - 1)}{|x|^a} \mathrm{d}x \le C(\beta, M). \tag{2.3}$$

Our choice of the variational setting E ensures that the imbedding is continuous in $H^1(\mathbb{R}^2)$ and compact in $L^s(\mathbb{R}^2)$, for $s \ge 2$ (see [17, Lemma 2.1 and Proposition 3.1]). This lemma in [17] provides a inequality which will be needed throughout the paper:

$$\|u\|_E^2 \ge \zeta \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x, \tag{2.4}$$

for some $\zeta > 0$ and for all $u \in E$.

Lemma 2.2. *Let $\beta > 0$ and $r \ge 1$. Then for each $\theta > r$ there exists a positive constant $C = C(\theta)$ such that for all $s \in \mathbb{R}$*

$$(e^{\beta s^2} - 1)^r \le C(e^{\theta\beta s^2} - 1).$$

In particular, for $r \in [1, \alpha)$, we have that $K(x)^r \frac{(e^{\beta u^2} - 1)^r}{|x|^a}$ belongs to $L^1(\mathbb{R}^2)$ for all $u \in H^1(\mathbb{R}^2)$.

Proof. The proof of the inequality above is a consequence of L'Hospital Rule (see [12, Lemma 2.2] for a proof). Now, as $K(x) \in L^\infty_{\rm loc}(\mathbb{R}^2)$, for $R > 1$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)^r \frac{(e^{\beta u^2} - 1)^r}{|x|^a} \, \mathrm{d}x \\
&\le C_1 \int_{|x| \le R} \frac{(e^{\beta u^2} - 1)^r}{|x|^a} \, \mathrm{d}x + \int_{|x| > R} K(x)^r (e^{\beta u^2} - 1)^r \, \mathrm{d}x \\
&\le C_2 \int_{|x| \le R} \frac{(e^{\theta\beta u^2} - 1)}{|x|^a} \, \mathrm{d}x + C_3 \int_{|x| > R} K(x)^r (e^{\theta\beta u^2} - 1) \, \mathrm{d}x.
\end{aligned}$$

From Lemma 2.1 it follows that the first term is integrable. To estimate the other term, we note that

$$\int_{|x| > R} K(x)^r (e^{\theta\beta u^2} - 1) \, \mathrm{d}x = \sum_{m=1}^{\infty} \frac{(\theta\beta)^m}{m!} \int_{|x| > R} K(x)^r |u|^{2m} \, \mathrm{d}x.$$

By (V4) and Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)^r |u|^{2m} \, \mathrm{d}x \\
&\le C_4 \|u\|_{2m}^{2m} + C_5 \int_{|x| > R_0} \left(V^+(x)\right)^{r/\alpha} |u|^{2m} \, \mathrm{d}x
\end{aligned}$$

$$\le C_4\|u\|_{2m}^{2m} + C_5\Big[\int_{|x|>R_0} V^+(x)|u|^2\,\mathrm{d}x\Big]^{r/\alpha}\Big[\int_{|x|>R_0} |u|^{2(m\alpha-r)/(\alpha-r)}\Big]^{(\alpha-r)/\alpha}.$$

By (V2) and the continuous imbedding $E \hookrightarrow L^s(\mathbb{R}^2)$, for all $s \ge 2$, we can conclude that

$$\int_{\mathbb{R}^2} K(x)^r |u|^{2m}\,\mathrm{d}x \le C\|u\|_E^{2m}.$$

Thus, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)^r \frac{(e^{\beta u^2}-1)^r}{|x|^a}\,\mathrm{d}x \\
&\le C_1 \int_{|x|\le R} \frac{(e^{\theta\beta u^2}-1)}{|x|^a}\,\mathrm{d}x + C\sum_{m=1}^{\infty}\frac{1}{m!}\left(\theta\beta\|u\|_E^2\right)^m \qquad (2.5)\\
&\le C_1 \int_{|x|\le R} \frac{(e^{\theta\beta u^2}-1)}{|x|^a}\,\mathrm{d}x + C\left[\exp\left(\theta\beta\|u\|_E^2\right)-1\right] < \infty,
\end{aligned}$$

which completes the proof. □

Corollary 2.3. *If $v \in E$, $\beta > 0$, $q > 0$ and $\|v\|_E \le M$ with $\frac{\beta M^2}{4\pi\zeta} + \frac{a}{2} < 1$, then there exists $C = C(\beta, M, q, \zeta) > 0$ such that*

$$\int_{\mathbb{R}^2} K(x)|v|^q \frac{(e^{\beta v^2}-1)}{|x|^a}\,\mathrm{d}x \le C\|v\|_E^q.$$

Proof. By Hölder inequality,

$$\int_{\mathbb{R}^2} K(x)|v|^q \frac{(e^{\beta v^2}-1)}{|x|^a}\,\mathrm{d}x \le \|v\|_{qs}^q\Big[\int_{\mathbb{R}^2} K(x)^r \frac{(e^{\beta v^2}-1)^r}{|x|^{ar}}\mathrm{d}x\Big]^{1/r}, \qquad (2.6)$$

where $r > 1$ is close to 1 and $s = r/(r-1)$. Now, we consider $\theta > r$ close to r such that $\frac{\theta\beta M^2}{4\pi\zeta} + \frac{ar}{2} < 1$. By (2.5) and Lemma 2.1, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)|v|^q \frac{(e^{\beta v^2}-1)}{|x|^a}\,\mathrm{d}x \\
&\le \Big\{C_1\int_{|x|\le R} \frac{[e^{\frac{\theta\beta M^2}{\zeta}\left(\frac{v}{\|\nabla v\|_2}\right)^2}-1]}{|x|^{ar}}\,\mathrm{d}x + C_2\left[\exp\left(\theta\beta M^2\right)-1\right]\Big\}^{1/r}\|v\|_{qs}^q \qquad (2.7)\\
&\le C_3\|v\|_E^q.
\end{aligned}$$

□

To show that the weak limit of a Palais-Smale sequence in E is a weak solution of (1.1) we will use the following convergence result, which is a version of Lemma 2.1 in [6].

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Then for any sequence (u_n) in $L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$,*

$$\frac{g(x)f(u_n)}{|x|^a} \in L^1(\Omega) \quad \text{and} \quad \int_\Omega \frac{g(x)|f(u_n)u_n|}{|x|^a}\,\mathrm{d}x \le C_1,$$

up to a subsequence we have

$$\frac{g(x)f(u_n)}{|x|^a} \to \frac{g(x)f(u)}{|x|^a} \quad \text{in } L^1(\Omega).$$

Proof. It suffices to prove
$$\int_\Omega \frac{|g(x)f(u_n)|}{|x|^a}\,\mathrm{d}x \to \int_\Omega \frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x.$$
Since $u, g(x)f(u)/|x|^a \in L^1(\Omega)$, for each $\epsilon > 0$ there is a $\delta > 0$ such that for any measurable subset $A \subset \Omega$,
$$\int_A |u|\,\mathrm{d}x < \epsilon \quad \text{and} \quad \int_A \frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x < \epsilon \quad \text{if } |A| \le \delta. \tag{2.8}$$
Next using the fact that $u \in L^1(\Omega)$ we find $M_1 > 0$ such that
$$|\{x \in \Omega : |u(x)| \ge M_1\}| \le \delta. \tag{2.9}$$
Let $M = \max\{M_1, C_1/\epsilon\}$. We write
$$\Big|\int_\Omega \frac{|g(x)f(u_n)|}{|x|^a}\,\mathrm{d}x - \int_\Omega \frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x\Big| \le I_{1,n} + I_{2,n} + I_{3,n},$$
where
$$I_{1,n} = \int_{[|u_n| \ge M]} \frac{|g(x)f(u_n)|}{|x|^a}\,\mathrm{d}x,$$
$$I_{2,n} = \Big|\int_{[|u_n| < M]} \frac{|g(x)f(u_n)|}{|x|^a}\,\mathrm{d}x - \int_{[|u| < M]} \frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x\Big|,$$
$$I_{3,n} = \int_{[|u| \ge M]} \frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x.$$
Now we estimate each integral separately.
$$I_{1,n} = \int_{[|u_n| \ge M]} \frac{|g(x)f(u_n)|}{|x|^a}\,\mathrm{d}x = \int_{[|u_n| \ge M]} \frac{|g(x)f(u_n)u_n|}{|u_n||x|^a}\,\mathrm{d}x \le \frac{C_1}{M} \le \epsilon.$$
From (2.8) and (2.9), we have $I_{3,n} \le \epsilon$.

Next we claim $I_{2,n} \to 0$ as $n \to +\infty$. Indeed,
$$\begin{aligned} I_{2,n} \le & \Big|\int_\Omega \frac{\mathcal{X}_{[|u_n| < M]}(|g(x)f(u_n)| - |g(x)f(u)|)}{|x|^a}\,\mathrm{d}x\Big| \\ & + \Big|\int_\Omega \frac{(\mathcal{X}_{[|u_n| < M]} - \mathcal{X}_{[|u| < M]})|g(x)f(u)|}{|x|^a}\,\mathrm{d}x\Big| \end{aligned}$$
and $g_n(x) = \mathcal{X}_{[|u_n| < M]}(|g(x)f(u_n)| - |g(x)f(u)|) \to 0$ almost everywhere in Ω. Moreover,
$$|g_n(x)| \le \begin{cases} |g(x)f(u)| & \text{if } |u_n(x)| \ge M, \\ C + |g(x)f(u)| & \text{if } |u_n(x)| < M, \end{cases}$$
where $C = \sup\{|g(x)f(t)| : (x,t) \in \overline{\Omega} \times [-M, M]\}$. So, by the Lebesgue dominated convergence theorem, we obtain
$$\Big|\int_\Omega \frac{\mathcal{X}_{[|u_n| < M]}(|g(x)f(u_n)| - |g(x)f(u)|)}{|x|^a}\,\mathrm{d}x\Big| \to 0 \quad \text{as } n \to \infty.$$
Moreover,
$$\{x \in \Omega : |u_n(x)| < M\} \setminus \{x \in \Omega : |u(x)| < M\} \subset \{x \in \Omega : |u(x)| \ge M\}.$$

Hence by (2.8),

$$\Big|\int_{\Omega}\frac{(\mathcal{X}_{[|u_n|<M]}-\mathcal{X}_{[|u|<M]})|g(x)f(u)|}{|x|^a}\,\mathrm{d}x\Big|\le\int_{[|u|\ge M]}\frac{|g(x)f(u)|}{|x|^a}\,\mathrm{d}x<\epsilon,$$

which completes the proof. □

3. The variational framework

We now consider the functional I given by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[|\nabla u|^2+V(x)u^2\right]\mathrm{d}x-\int_{\mathbb{R}^2}\frac{g(x)F(u)}{|x|^a}\mathrm{d}x-\int_{\mathbb{R}^2}h(x)u\mathrm{d}x. \tag{3.1}$$

Under our assumptions we have that I is well-defined and is C^1 on E. Indeed, by (H_2), given $\varepsilon>0$ there exists $\delta>0$ such that $|f(s)|\le\varepsilon|s|$ always that $|s|<\delta$. On the other hand, for $\beta>0$ we have that there exists $C>0$ such that $|f(s)|\le C(e^{\beta s^2}-1)$ for all $s\ge\delta$. Thus

$$|f(s)|\le\varepsilon|s|+C_1(e^{\beta s^2}-1), \tag{3.2}$$

for all $s\in\mathbb{R}$. By (H1), (H3), (V4) and Hölder inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{g(x)F(u)}{|x|^a}\,\mathrm{d}x&\le\varepsilon\int_{\mathbb{R}^2}\frac{K(x)u^2}{|x|^a}\,\mathrm{d}x+C_2\int_{\mathbb{R}^2}\frac{K(x)|u|(e^{\beta u^2}-1)}{|x|^a}\,\mathrm{d}x\\&\le C_1\int_{|x|\le1}\frac{u^2}{|x|^a}\,\mathrm{d}x+\varepsilon\int_{|x|>1}K(x)u^2\,\mathrm{d}x\\&\quad+C_2\|u\|_s\Big[\int_{\mathbb{R}^2}K(x)^r\frac{(e^{\beta u^2}-1)^r}{|x|^{ar}}\,\mathrm{d}x\Big]^{1/r},\end{aligned}$$

where $r\in[1,\alpha)$ and $s=r/(r-1)$, with $ar<2$. Considering the continuous imbedding $E\hookrightarrow L^s_{K(x)}(\mathbb{R}^2)$ for $s\ge2$, $a\in[0,2)$ and Lemma 2.2, it follows that $g(x)F(u)/|x|^a\in L^1(\mathbb{R}^2)$ which implies that I is well defined.

Next, we show that I is in C^1 on E. Indeed, letting $N(u)=\int_{\mathbb{R}^2}g(x)F(u)/|x|^a\,\mathrm{d}x$, we have by dominated convergence theorem that

$$\begin{aligned}\langle I'(u),\phi\rangle&=\langle u,\phi\rangle_E-\lim_{t\to0}\frac{1}{t}[N(u+t\phi)-N(u)]-\int_{\mathbb{R}^2}h(x)\phi\,\mathrm{d}x\\&=\langle u,\phi\rangle_E-\int_{\mathbb{R}^2}\frac{g(x)f(u)\phi}{|x|^a}\,\mathrm{d}x-\int_{\mathbb{R}^2}h(x)\phi\,\mathrm{d}x,\end{aligned}$$

for all $\phi\in E$. As $I'(u)$ is linear and bounded, it suffices to show that the Gateaux derivative of I is continuous. It is clear that the first and last term are C^1. Hence, it remains to prove that N is C^1. Let $u_n\to u$ in E. By Proposition 2.7 in [12], there exists a subsequence (u_{n_k}) in E and $\ell(x)\in H^1(\mathbb{R}^2)$ such that $u_{n_k}(x)\to u(x)$ and $|u_{n_k}(x)|\le\ell(x)$ almost everywhere in $\mathbb{R}^2$. Given $\xi\in E$, we define

$$H_{n_k}(x)=\frac{g(x)f(u_{n_k}(x))\xi(x)}{|x|^a}.$$

Then

$$H_{n_k}(x)\to H(x)=\frac{g(x)f(u(x))\xi(x)}{|x|^a}\quad\text{almost everywhere in }\mathbb{R}^2.$$

Using (3.2) and Lemma 2.1, we obtain that $H_{n_k}(x)$ is integrable, it follows by dominated convergence theorem that

$$\lim_{k\to\infty}\int_{\mathbb{R}^2} H_{n_k}(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} H(x)\,\mathrm{d}x.$$

Thus, for each $\xi \in E$ with $\|\xi\|_E = 1$, we obtain

$$\begin{aligned}\lim_{k\to\infty}\|N'(u_{n_k}) - N'(u)\|_{E^*} &= \lim_{k\to\infty}\sup_{\|\xi\|_E=1}|\langle N'(u_{n_k}) - N'(u), \xi\rangle| \\ &= \sup_{\|\xi\|_E=1}\lim_{k\to\infty}\int_{\mathbb{R}^2}\frac{g(x)[f(u_{n_k}) - f(u)]\xi}{|x|^a}\,\mathrm{d}x = 0\end{aligned}$$

and the proof is complete.

The geometric conditions of the mountain-pass theorem for the functional I are established by our next two lemmas.

Lemma 3.1. *Suppose that (V1)-(V2), (V4), (H1)-(H3) and* (1.2) *are satisfied. Then there exists $\delta > 0$ such that for each $h \in H^1(\mathbb{R}^2)$ with $\|h\|_{H^{-1}} < \delta$, there exists $\rho_h > 0$ such that*

$$I(u) > 0 \quad \textit{whenever} \quad \|u\|_E = \rho_h.$$

Proof. In the same manner that (3.2) was obtained, we can see that

$$|f(s)| \le \varepsilon|s| + C_1|s|^q(e^{\beta s^2} - 1), \tag{3.3}$$

with $q > 2$. Thus, considering the continuous imbedding $E \hookrightarrow L^s_{K(x)}(\mathbb{R}^2)$ for $s \ge 2$ (see [17, Proposition 3.1]), we obtain for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}I(u) &\ge \frac{1}{2}\|u\|_E^2 - \varepsilon\int_{\mathbb{R}^2}\frac{K(x)u^2}{|x|^a}\,\mathrm{d}x - C_1\int_{\mathbb{R}^2}\frac{K(x)|u|^{q+1}(e^{\beta u^2} - 1)}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)u\,\mathrm{d}x \\ &\ge \big(\frac{1}{2} - \varepsilon\big)\|u\|_E^2 - C_1\int_{\mathbb{R}^2}\frac{K(x)|u|^{q+1}(e^{\beta u^2} - 1)}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)u\,\mathrm{d}x\end{aligned}$$

and since $\frac{\beta\sigma^2}{4\pi\zeta} + \frac{a}{2} < 1$ if $\|u\|_E < \sigma$ is sufficiently small, we can apply Corollary 2.3 to conclude that

$$I(u) \ge \big(\frac{1}{2} - \varepsilon\big)\|u\|_E^2 - C\|u\|_E^{q+1} - \|h\|_{H^{-1}}\|u\|_E.$$

Thus there exists $\rho_h > 0$ such that $I(u) > 0$ whenever $\|u\|_E = \rho_h$ and $\|h\|_{H^{-1}}$ is sufficiently small. Indeed, for $\varepsilon > 0$ sufficiently small and $q > 2$, we may choose $\rho_h > 0$ such that

$$\big(\frac{1}{2} - \varepsilon\big)\rho_h - C_1\rho_h^q > 0.$$

Thus, for $\|h\|_{H^{-1}}$ sufficiently small there exists $\rho_h > 0$ such that $I(u) > 0$ if $\|u\|_E = \rho_h$. ∎

Lemma 3.2. *Assume that* (H1), (H3) *and* (1.2) *are satisfied. Then there exists $e \in E$ with $\|e\|_E > \rho_h$ such that*

$$I(e) < \inf_{\|u\|=\rho_h} I(u).$$

Proof. Let $u \in E\backslash\{0\}$ with compact support and $u \geq 0$. Integrating (H3) we obtain that there exist $c, d > 0$ such that

$$F(s) \geq cs^{\mu} - d$$

for all $s \in \mathbb{R}$. Thus, denoting $K = supp(u)$ and using (H1), we have that

$$\begin{aligned} I(tu) &\leq \frac{t^2}{2}\|u\|_E^2 - ct^{\mu}\int_K \frac{g(x)u^{\mu}}{|x|^a}\,\mathrm{d}x + d\int_K \frac{g(x)}{|x|^a}\,\mathrm{d}x - t\int_{\mathbb{R}^2} h(x)u\,\mathrm{d}x \\ &\leq \frac{t^2}{2}\|u\|_E^2 - C_1t^{\mu}\int_K \frac{u^{\mu}}{|x|^a}\,\mathrm{d}x + C_2(|K|) - t\int_{\mathbb{R}^2} h(x)u\,\mathrm{d}x, \end{aligned}$$

for all $t > 0$, which implies that $I(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t large enough, the proof is complete. □

To find an appropriate ball to use a minimization argument we need the following result.

Lemma 3.3. *If $f(s)$ satisfies* (1.2) *and $h \neq 0$, there exist $\eta > 0$ and $v \in E$ with $\|v\|_E = 1$ such that $I(tv) < 0$ for all $0 < t < \eta$. In particular,*

$$\inf_{\|u\|\leq\eta} I(u) < 0.$$

Proof. For each $h \in H^{-1}$, by applying the Riesz representation theorem in the space E, the problem

$$-\Delta v + V(x)v = h, \quad x \in \mathbb{R}^2$$

has a unique weak solution v in E. Thus,

$$\int_{\mathbb{R}^2} h(x)v\,\mathrm{d}x = \|v\|_E^2 > 0 \quad \text{for each } h \neq 0.$$

Since $f(0) = 0$, by continuity, it follows that there exists $\eta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}I(tv) = t\|v\|_E^2 - \int_{\mathbb{R}^2} \frac{g(x)f(tv)v}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)v\,\mathrm{d}x < 0,$$

for all $0 < t < \eta$. Using that $I(0) = 0$, it must hold that $I(tv) < 0$ for all $0 < t < \eta$. □

4. Palais-Smale sequences

To prove that a Palais-Smale sequence converges to a solution of problem (1.1) we need to establish the following lemma.

Lemma 4.1. *Assume* (H3) *and that $f(s)$ satisfies* (1.2). *Let (u_n) in E such that $I(u_n) \to c$ and $I'(u_n) \to 0$. Then $\|u_n\|_E \leq C$,*

$$\int_{\mathbb{R}^2} \frac{g(x)|f(u_n)u_n|}{|x|^a}\,\mathrm{d}x \leq C \quad \textit{and} \quad \int_{\mathbb{R}^2} \frac{g(x)F(u_n)}{|x|^a}\,\mathrm{d}x \leq C.$$

Proof. Let $(u_n) \subset E$ be a sequence such that $I(u_n) \to c$ and $I'(u_n) \to 0$, that is, for any $\varphi \in E$,

$$\frac{1}{2}\|u_n\|_E^2 - \int_{\mathbb{R}^2} \frac{g(x)F(u_n)}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)u_n\,\mathrm{d}x = c + \delta_n \tag{4.1}$$

and

$$\Big|\int_{\mathbb{R}^2} [\nabla u_n\nabla\varphi + V(x)u_n\varphi]\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{g(x)f(u_n)\varphi}{|x|^a}\,\mathrm{d}x - \int_{\mathbb{R}^2} h(x)\varphi\,\mathrm{d}x\Big| \leq \varepsilon_n\|\varphi\|_E, \tag{4.2}$$

where $\delta_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Taking $\varphi = u_n$ in (4.2) and using (H3), we have

$$\begin{aligned}&\mu(c+\delta_n) + \varepsilon_n \|u_n\|_E + (\mu - 1)\int_{\mathbb{R}^2} h(x)u_n \,\mathrm{d}x \\ &\geq (\frac{\mu}{2} - 1)\|u_n\|_E^2 - \int_{\mathbb{R}^2} \frac{g(x)[\mu F(u_n) - f(u_n)u_n]}{|x|^a}\,\mathrm{d}x \\ &\geq (\frac{\mu}{2} - 1)\|u_n\|_E^2.\end{aligned}$$

Consequently, $\|u_n\|_E \leq C$ and by (4.1) and (4.2), we obtain

$$\int_{\mathbb{R}^2} \frac{g(x)F(u_n)}{|x|^a}\,\mathrm{d}x \leq C \quad \text{and} \quad \int_{\mathbb{R}^2} \frac{g(x)|f(u_n)u_n|}{|x|^a}\,\mathrm{d}x \leq C.$$

□

Corollary 4.2. *Let (u_n) a Palais-Smale sequence for I. Then (u_n) has a subsequence, still denoted by (u_n), which is weakly convergent to a weak solution of (1.1).*

Proof. Using Lemma 4.1, up to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in E. Now, from (4.2), taking the limit and using Lemma 2.4, we have

$$\int_{\mathbb{R}^2} (\nabla u \nabla \varphi + V(x)u\varphi)\mathrm{d}x - \int_{\mathbb{R}^2} \frac{g(x)f(u)}{|x|^a}\varphi \mathrm{d}x - \int_{\mathbb{R}^2} h(x)\varphi \mathrm{d}x = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Since $C_0^\infty(\mathbb{R}^2)$ is dense in E, we conclude that u is a weak solution of (1.1). □

5. Proof of Theorem 1.1

Let (u_n) in E such that $I(u_n) \to c$ and $I'(u_n) \to 0$. We will use the Mountain-Pass Theorem to obtain a nontrivial solution of (1.1). Since

$$\|u_n - u\|_E^2 = \langle I'(u_n) - I'(u), u_n - u\rangle + \int_{\mathbb{R}^2} \frac{g(x)[f(u_n) - f(u)]}{|x|^a}(u_n - u)\,\mathrm{d}x,$$

we have that the Palais-Smale condition is satisfied if

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \frac{g(x)[f(u_n) - f(u)]}{|x|^a}(u_n - u)\,\mathrm{d}x = 0.$$

By (3.2) and Hölder inequality, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^2} \frac{g(x)[f(u_n) - f(u)]}{|x|^a}|u_n - u|\,\mathrm{d}x \\ &\leq C_1 \int_{\mathbb{R}^2} K(x)|x|^{-a}(|u_n| + |u|)|u_n - u|\,\mathrm{d}x \\ &\quad + C_2 \int_{\mathbb{R}^2} K(x)\big[\frac{(e^{\beta u_n^2} - 1)}{|x|^a} + \frac{(e^{\beta u^2} - 1)}{|x|^a}\big]|u_n - u|\,\mathrm{d}x \\ &\leq C_1 \int_{\mathbb{R}^2} K(x)|x|^{-a}(|u_n| + |u|)|u_n - u|\,\mathrm{d}x \\ &\quad + C_2\|u_n - u\|_s \Big\{\int_{\mathbb{R}^2} K(x)^r \big[\frac{(e^{\beta u_n^2} - 1)^r}{|x|^{ar}} + \frac{(e^{\beta u^2} - 1)^r}{|x|^{ar}}\big]\,\mathrm{d}x\Big\}^{1/r},\end{aligned}$$

with $r > 1$ close to 1 such that $ar < 2$ and $s = r/(r-1)$. Since $f(s)$ has subcritical growth and $E \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $s \geq 2$, the second term converges to zero.

Now, to estimate the other term we will use Hölder inequality, Young inequality and that $\|u_n\|_E \leq C$, thus we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)|x|^{-a}(|u_n| + |u|)|u_n - u| \,\mathrm{d}x \\
&\leq \sqrt{2}\Big(\int_{\mathbb{R}^2} \frac{K(x)|u_n|^2}{|x|^a} \,\mathrm{d}x + \int_{\mathbb{R}^2} \frac{K(x)|u|^2}{|x|^a} \,\mathrm{d}x\Big)^{1/2} \Big(\int_{\mathbb{R}^2} \frac{K(x)|u_n - u|^2}{|x|^a} \,\mathrm{d}x\Big)^{1/2} \\
&\leq C_1 \left\{ C_2\|u_n - u\|_s^2 + \int_{\mathbb{R}^2} K(x)|u_n - u|^2 \,\mathrm{d}x \right\}^{1/2}.
\end{aligned} \tag{5.1}$$

Using (V4), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)|u_n - u|^2 \,\mathrm{d}x \\
&= \int_{|x|\leq R_0} K(x)|u_n - u|^2 \,\mathrm{d}x + \int_{|x|>R_0} K(x)|u_n - u|^2 \,\mathrm{d}x \\
&\leq \max_{|x|\leq R_0} \{K(x)\} \int_{|x|\leq R_0} |u_n - u|^2 \,\mathrm{d}x \\
&\quad + \int_{|x|>R_0} c_0[1 + (V^+(x))^{1/\alpha}]|u_n - u|^2 \,\mathrm{d}x \\
&\leq C\big\{\|u_n - u\|_2^2 + \int_{|x|>R_0} V^+(x)^{1/\alpha}|u_n - u|^2 \,\mathrm{d}x\big\}.
\end{aligned} \tag{5.2}$$

By Hölder inequality, we obtain

$$\begin{aligned}
&\int_{|x|>R_0} V^+(x)^{1/\alpha}|u_n - u|^2 \,\mathrm{d}x \\
&\leq \Big[\int_{|x|>R_0} V^+(x)|u_n - u|^2 \,\mathrm{d}x\Big]^{1/\alpha} \Big[\int_{|x|>R_0} |u_n - u|^{(2\alpha-2)/(\alpha-1)} \,\mathrm{d}x\Big]^{(\alpha-1)/\alpha}
\end{aligned} \tag{5.3}$$

and by (V1), we have

$$\begin{aligned}
&\int_{|x|>R_0} V^+(x)|u_n - u|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^2} V(x)|u_n - u|^2 \,\mathrm{d}x - \int_{|x|\leq R_0} V(x)|u_n - u|^2 \,\mathrm{d}x - \int_{|x|>R_0} V^-(x)|u_n - u|^2 \,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^2} \big[|\nabla(u_n - u)|^2 + V(x)|u_n - u|^2\big] \,\mathrm{d}x.
\end{aligned} \tag{5.4}$$

From (5.3), (5.4) in (5.2) and using (V3), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} K(x)|u_n - u|^2 \,\mathrm{d}x \\
&\leq C\big\{\|u_n - u\|_2^2 + \big(\|u_n - u\|_E^2 + D\|u_n - u\|_2^2\big)^{1/\alpha} \|u_n - u\|_{(2\alpha-2)/(\alpha-1)}^{(2\alpha-2)/\alpha}\big\} \\
&\leq C\big\{\|u_n - u\|_2^2 + \big(1 + \frac{D}{\lambda_1}\big)^{1/\alpha}\|u_n - u\|_E^{2/\alpha}\|u_n - u\|_{(2\alpha-2)/(\alpha-1)}^{(2\alpha-2)/\alpha}\big\}.
\end{aligned} \tag{5.5}$$

Thus, by (5.1),

$$\begin{aligned}&\int_{\mathbb{R}^2} K(x)|x|^{-a}(|u_n|+|u|)|u_n-u|\,\mathrm{d}x\\&\leq C_1\big\{C_2\|u_n-u\|_s^2+\|u_n-u\|_2^2+C_3\|u_n-u\|_E^{2/\alpha}\|u_n-u\|_2^{2(\alpha-1)/\alpha}\big\}^{1/2}.\end{aligned}$$

By compact embedding of E in $L^s(\mathbb{R}^2)$ for any $s\geq 2$, we obtain

$$\int_{\mathbb{R}^2} K(x)|x|^{-a}(|u_n|+|u|)|u_n-u|\,\mathrm{d}x\to 0\quad\text{as } n\to+\infty.$$

Hence the Palais-Smale condition is satisfied. Therefore, the functional I has a critical point u_M at minimax level

$$c_M=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))>0,$$
$$\Gamma=\{\gamma\in C(E,\mathbb{R}):\gamma(0)=0,\ \gamma(1)=e\}.$$

On the other hand, if $h\not\equiv 0$, then we obtain a second solution of (1.1) with negative energy. Indeed, let ρ_h be as in Lemma 3.1. Since $\overline{B}_{\rho_h}$ is a complete metric space with the metric given by norm of E, convex and the functional I is of class C^1 and bounded below on $\overline{B}_{\rho_h}$, it follows by Ekeland variational principle that there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n)\to c_0=\inf_{\|u\|_E\leq\rho_h}I(u)\quad\text{and}\quad\|I'(u_n)\|_{E'}\to 0. \tag{5.6}$$

We now apply the argument above again to conclude that (1.1) possesses a solution u_0 such that $I(u_0)=c_0<0$.

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