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MONOTONE ITERATIVE METHOD AND REGULAR SINGULAR NONLINEAR BVP IN THE PRESENCE OF REVERSE ORDERED UPPER AND LOWER SOLUTIONS

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Abstract. Monotone iterative technique is employed for studying the existence of solutions to the second-order nonlinear singular boundary value problem

$$-\big(p(x)y'(x)\big)' + p(x)f\big(x, y(x), p(x)y'(x)\big) = 0$$

for $0 < x < 1$ and $y'(0) = y'(1) = 0$. Here $p(0) = 0$ and $xp'(x)/p(x)$ is analytic at $x = 0$. The source function $f(x, y, py')$ is Lipschitz in py' and one sided Lipschitz in y. The initial approximations are upper solution $u_0(x)$ and lower solution $v_0(x)$ which can be ordered in one way $v_0(x) \le u_0(x)$ or the other $u_0(x) \le v_0(x)$.

1. Introduction

Recently, there have been a lot of activity as far as upper and lower solutions technique is considered (see [1, 2, 3] and the references therein). In most of the results upper and solutions are well ordered; i.e., $u_0(x) \ge v_0(x)$. But literature is not rich for the case of reverse ordered upper and lower solutions; i.e., $u_0(x) \le v_0(x)$. Though results are available for nonsingular boundary value problems ($p(x)$=constant) but singular boundary value problems require attention. The details of the work done for the nonsingular problem when upper and lower solutions are in reverse order can be seen in [2] and the references therein. To fill this gap in our recent result ([3]) we consider the singular boundary value problem of the form $-\big(x^\alpha y'(x)\big)' + x^\alpha f\big(x, y(x), x^\alpha y'(x)\big) = 0$, $\alpha \ge 1$ for $0 < x < 1$ and $y'(0) = y'(1) = 0$ with upper and lower solution in one order ($u_0 \ge v_0$) or the other ($u_0 \le v_0$). We prove the existence of the solutions under quite general conditions. This problem is simple and its advantage is that for the corresponding linear problem we obtain the solutions in terms of Bessel functions. Bessel functions have some in built simplicity which helps us in proving the results very easily. In the present paper we consider a generalized problem

$$-\big(p(x)y'(x)\big)' + p(x)f\big(x, y(x), p(x)y'(x)\big) = 0, \quad 0 < x < 1. \tag{1.1}$$

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We replace the term x^α in the boundary value problem considered in [3] with a general function $p(x)$. So in built simplicity due to Bessel functions is not there with us in this paper. Here the source function $f(x, y, py')$ is derivative dependent and boundary conditions are again of Neumann type and is written as

$$y'(0) = 0, \quad y'(1) = 0. \tag{1.2}$$

Let $p(x)$ satisfy the following conditions:

(A1) (i) $p(0) = 0$ and $p > 0$ in $(0, 1)$.
(ii) $p \in C[0, 1] \cap C^1(0, 1)$ and
(iii) for some $r > 1$, $x\frac{p'(x)}{p(x)}$ is analytic in $\{z : |z| < r\}$.
(iv) $\int_0^1 \frac{dt}{p(t)} = \infty$.

In this work we consider a computationally simple iterative scheme defined by

$$-\big(p(x)y_n'(x)\big)' + \lambda p(x)y_n(x) = -p(x)f\big(x, y_{n-1}, py_{n-1}'\big) + \lambda p(x)y_{n-1}(x) \tag{1.3}$$

$$y_n'(0) = 0, \quad y_n'(1) = 0. \tag{1.4}$$

The main aim of this work is to extend our earlier work in [3]. Now we do not know the solutions explicitly in terms of Bessel functions. Instead we have singular linear boundary-value problems which we have to analyze and properties of the solutions is to be extracted. We have arranged the paper in 4 sections. In Section 2 we discuss some elementary results, e.g., maximum principles and existence of two differential inequalities. Then using these elementary results we establish existence results for well ordered upper and lower solutions in Section 3 and for reverse ordered upper and lower solutions in Section 4. We conclude this article with some remarks.

2. Preliminaries

Let $h(x) \in C[0, 1]$ and $\lambda \in \mathbb{R}_0$, $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $A \in \mathbb{R}$ and $B \in \mathbb{R}$. Now, consider the class of linear singular problems

$$-\left(p(x)y'(x)\right)' + \lambda p(x)y(x) = p(x)h(x), \quad 0 < x < 1, \tag{2.1}$$

$$y'(0) = A, \quad y'(1) = B. \tag{2.2}$$

The corresponding homogeneous system (eigenvalue problem) is given by

$$-\big(p(x)y'(x)\big)' + \lambda p(x)y(x) = 0, \quad 0 < x < 1, \tag{2.3}$$

$$y'(0) = 0, \quad y'(1) = 0. \tag{2.4}$$

Remark 2.1. Since xp'/p is analytic at $x = 0$, the point $x = 0$ is a regular singular point of (2.3). Thus using Frobenius series method two linearly independent solutions can be computed (see [4, 5, Lemma 2]). If $p = x^\alpha$ these Frobenius series solutions can be written in terms of Bessel functions.

It is easy to verify that all the eigenvalues of the Sturm-Liouville problem (2.3)–(2.4) are real, negative and simple.

The solution of the nonhomogeneous problem (2.1)-(2.2) can be written as

$$\begin{aligned} w(x) = z_1(x)\Big[\int_0^x \frac{p(t)h(t)z_0(t)}{W_p(z_1, z_0)}dt + \frac{A}{z_1'(0)}\Big] \\ + z_0(x)\Big[\int_x^1 \frac{p(t)h(t)z_1(t)}{W_p(z_1, z_0)}dt + \frac{B}{z_0'(1)}\Big] \end{aligned} \tag{2.5}$$

where $z_0(x,\lambda)$ is the solution of

$$\begin{gathered}-\big(p(x)z_0'(x)\big)' + \lambda p(x)z_0(x) = 0, \quad 0 < x < 1, \\ z_0(0) = 1, \quad z_0'(0) = 0,\end{gathered} \tag{2.6}$$

and $z_1(x,\lambda)$ is the solution of

$$\begin{gathered}-\big(p(x)z_1'(x)\big)' + \lambda p(x)z_1(x) = 0, \quad 0 < x < 1, \\ z_1(1) = 1, \quad z_1'(1) = 0\end{gathered} \tag{2.7}$$

and $W_p(z_1,z_0) = p(t)\,(z_1z_0' - z_1'z_0)$. By replacing x with $1-x$ in (2.6) it is easy to verify that

$$z_1(x) = z_0(1-x)$$

for both positive and negative values of λ.

Remark 2.2. We have z_0 and z_1 as two linearly independent solutions (Frobenius Series Solution) of (2.3). Eigenvalues of the eigenvalue problem (2.3)–(2.4) are the zeros of $z_0'(1,\lambda)$. $z_0'(1,\lambda)$ is an analytic function of λ so its zeros are isolated and they all will be negative. Let them be $-\lambda_0, -\lambda_1, -\lambda_2, \dots$ such that $\lambda_i > 0$ for $i = 0,1,2,\dots$. Where $-\lambda_0$ is the first negative zero of $z_0'(1,\lambda)$ or in other words first negative eigenvalue of (2.3)–(2.4).

Since $z_0(x,\lambda)$ does not change its sign for $-\lambda_0 < \lambda < 0$ and $z_0(0,\lambda) = 1$ therefore $z_0(x,\lambda) > 0$ for all $x \in [0,1]$ and for all $-\lambda_0 < \lambda < 0$.

Remark 2.3. Using (2.6) and the fact that $z_1(x) = z_0(1-x)$ it is easy to prove that if $\lambda > 0$ then for all $x \in (0,1]$, $z_0(x) > 1$ and $z_0'(x) > 0$ and for all $x \in [0,1)$ we have $z_1(x) > 1$ and $z_1'(x) < 0$.

Remark 2.4. Using Remark 2.2, $z_1(x) = z_0(1-x)$ and the differential equation (2.7) it is easy to prove that if $-\lambda_0 < \lambda < 0$ for all $x \in [0,1)$, $z_0(x) > 0$ and $z_1'(x) > 0$ and for all $x \in (0,1]$ we have $z_0'(x) < 0$ and $z_1(x) > 0$.

Remark 2.5. Let $\lambda > 0$ and $h \in C[0,1]$. If $h \geq 0$ (*or* $h \leq 0$) then

$$\int_0^x \frac{p(t)h(t)z_0(t)}{W_p(z_1,z_0)}dt \quad \text{and} \quad \int_x^1 \frac{p(t)h(t)z_1(t)}{W_p(z_1,z_0)}dt$$

are non-negative (or non-positive).

Remark 2.6. Let $-\lambda_0 < \lambda < 0$ and $h \in C[0,1]$. If $h \geq 0$ (or $h \leq 0$) then

$$\int_0^x \frac{p(t)h(t)z_0(t)}{W_p(z_1,z_0)}dt \quad \text{and} \quad \int_x^1 \frac{p(t)h(t)z_1(t)}{W_p(z_1,z_0)}dt$$

are non-positive (or non-negative).

Proposition 2.7 (Maximum Principle)**.** *Let $\lambda > 0$. If $A \leq 0$, $B \geq 0$ (or $A \geq 0$, $B \leq 0$) and $h \in C[0,1]$ such that $h \geq 0$ (or $h \leq 0$), then $w(x) \geq 0$ (or $w(x) \leq 0$), where $w(x)$ is the solution of* (2.1)-(2.2).

Proposition 2.8 (Anti-maximum Principle)**.** *Let $-\lambda_0 < \lambda < 0$. If $A \leq 0$, $B \geq 0$ (or $A \geq 0$, $B \leq 0$) and $h \in C[0,1]$ such that $h \geq 0$ (or $h \leq 0$), then $w(x) \leq 0$ (or $w(x) \geq 0$), where $w(x)$ is the solution of* (2.1)-(2.2).

Now we derive conditions on λ which will help us to prove the monotonicity of the solutions generated by iterative scheme (1.3)-(1.4).

Lemma 2.9. *Let M and $N \in \mathbb{R}^+$. If $\lambda > 0$ such that*
$$\lambda \geq M\Big(1 - N\int_0^1 p(t)dt\Big)^{-1},$$
then for all $x \in [0,1]$,
$$(M-\lambda)z_0(x) + Np(x)z_0'(x) \leq 0. \tag{2.8}$$

Proof. Integrating (2.6) from 0 to x and using that $z_0'(x) > 0$ in $(0,1]$ we obtain
$$p(x)z_0'(x) \leq \lambda z_0(x)\int_0^1 p(t)dt.$$
Therefore we obtain $(M-\lambda)z_0(x) + Np(x)z_0'(x) \leq (M-\lambda)z_0 + N\lambda z_0(x)\int_0^1 p(t)dt$. Hence (2.8) will hold if $(M-\lambda) + N\lambda\int_0^1 p(t)dt \leq 0$. Hence the result. □

Lemma 2.10. *Let M and $N \in \mathbb{R}^+$. If $-\lambda_0 < \lambda < 0$ is such that*
$$-\Big(\int_0^1 \frac{1}{p(x)}\int_0^x p(t)\,dt\,dx\Big)^{-1} < \lambda < -M$$
and
$$(M+\lambda)\Big(1 + \lambda\int_0^1 \frac{1}{p(x)}\int_0^x p(t)\,dt\,dx\Big) - N\lambda\int_0^1 p(x)dx \leq 0$$
then for all $x \in [0,1]$,
$$(M+\lambda)z_0(x) - Np(x)z_0'(x) \leq 0. \tag{2.9}$$

Proof. Using (2.6) and Remark 2.4 it can be deduced that $z_0(x)$ and $p(x)z_0'(x)$ are decreasing functions of x for $-\lambda_0 < \lambda < 0$, thus
$$(M+\lambda)z_0(x) - Np(x)z_0'(x) \leq (M+\lambda)z_0(1) - Np(1)z_0'(1).$$
Now using (2.6) we obtain
$$-p(1)z_0'(1) \leq (-\lambda)\int_0^1 p(x)dx \quad \text{and} \quad z_0(1) > 1 + \lambda\int_0^1 \frac{1}{p(x)}\int_0^x p(t)\,dt\,dx.$$
Which completes the proof. □

3. Well-ordered upper and lower solutions

Let us define upper and lower solutions:
A function $u_0 \in C^2[0,1]$ is an upper solution of (1.1)-(1.2) if
$$\begin{gathered} -(pu_0')' + p(x)f(x,u_0,pu_0') \geq 0, \quad 0 < x < 1; \\ u_0'(0) \leq 0 \leq u_0'(1). \end{gathered} \tag{3.1}$$
A function $v_0 \in C^2[0,1]$ is a lower solution of (1.1)-(1.2) if
$$\begin{gathered} -(pv_0')' + p(x)f(x,v_0,pv_0') \leq 0, \quad 0 < x < 1; \\ v_0'(0) \geq 0 \geq v_0'(1). \end{gathered} \tag{3.2}$$

Now, for every n, problem (1.3)-(1.4) has a unique solution y_{n+1} given by (2.5) with $h(x) = -f(x,y_n',py_n') + \lambda y_n$, $A = 0$ and $B = 0$.

In this section we show that for the proposed scheme (1.3)-(1.4) a good choice of λ is possible so that the solutions generated by the approximation scheme converge monotonically to solutions of (1.1)-(1.2). We require a number of results.

Lemma 3.1. *Let $\lambda > 0$. If u_n is an upper solution of* (1.1)-(1.2) *and u_{n+1} is defined by* (1.3)-(1.4) *then $u_{n+1} \le u_n$.*

Proof. Let $w_n = u_{n+1} - u_n$, then
$$-(pw_n')' + \lambda p w_n = (pu_n')' - pf(x, u_n, pu_n') \le 0,$$
$$w_n'(0) \ge 0, \quad w_n'(1) \le 0,$$
and using Proposition 2.7 we have $u_{n+1} \le u_n$. □

For the next proposition we use the following assumptions:

(H1) there exists upper solution (u_0) and lower solution (v_0) in $C^2[0,1]$ such that $v_0 \le u_0$ for all $x \in [0,1]$;

(H2) the function $f : D \to \mathbb{R}$ is continuous on
$$D := \{(x, y, py') \in [0,1] \times R \times R : v_0 \le y \le u_0\};$$

(H3) there exists $M \ge 0$ such that for all $(x, \tau, pv'), (x, \sigma, pv') \in D$,
$$f(x, \tau, pv') - f(x, \sigma, pv') \ge M(\tau - \sigma), \quad (\tau \le \sigma);$$

(H4) there exist $N \ge 0$ such that for all $(x, u, pv_1'), (x, u, pv_2') \in D$,
$$|f(x, u, pv_1') - f(x, u, pv_2')| \le N|pv_2' - pv_1'|.$$

Proposition 3.2. *Assume* (H1)–(H4)*, and let $\lambda > 0$ such that*
$$\lambda \ge M\big(1 - N\int_0^1 p(t)dt\big)^{-1}.$$
Then the functions u_{n+1} defined recursively by (1.3)-(1.4) *are such that for all $n \in \mathbb{N}$,*

(i) *u_n is an upper solution of* (1.1)-(1.2)*.*

(ii) *$u_{n+1} \le u_n$.*

Proof. We prove the claims by the principle of mathematical induction. Since u_0 is an upper solution and by Lemma 3.1 $u_0 \ge u_1$, therefore both the claims are true for $n = 0$.

Further, let the claims be true for $n - 1$; i.e., u_{n-1} is an upper solution and $u_{n-1} \ge u_n$. Now we are required to prove that u_n is an upper solution and $u_{n+1} \le u_n$. To prove this let $w = u_n - u_{n-1}$, then we have
$$-(pu_n')' + pf(x, u_n, pu_n') \ge p[(M - \lambda)w - N(\operatorname{sign} w')pw'].$$
Thus to prove that u_n is an upper solution we require to prove that
$$(M - \lambda)w - N(\operatorname{sign} w')pw' \ge 0. \tag{3.3}$$
Now, since w satisfies
$$-(pw')' + \lambda pw = (pu_{n-1}')' - pf(x, u_{n-1}, pu_{n-1}') \le 0, \quad w'(0) \ge 0,\ w'(1) \le 0,$$
from Proposition 2.7 we have $w \le 0$ for $\lambda > 0$. Now, putting the value of w from (2.5) in (3.3) and in view of $h = (pu_{n-1}')' - pf(x, u_{n-1}, pu_{n-1}') \le 0$ we deduce that to prove (3.3) it is sufficient to prove that
$$(M - \lambda)z_0 - N(sign\ w')pz_0' \le 0$$
and
$$(M - \lambda)z_1 - N(sign\ w')pz_1' \le 0$$

for all $x \in [0,1]$. Since $z_1 = z_0(1-x)$, using Remark 2.5, above inequalities will be true if for all $x \in [0,1]$ we have

$$(M-\lambda)z_0(x) + Np(x)z_0'(x) \le 0,$$

and which is true (by Lemma 2.9). Therefore (3.3) holds and hence u_n is an upper solution.

Now applying Lemma 3.1 we deduce that $u_{n+1} \le u_n$. This completes the proof. □

Similarly we can prove the following two results (Lemma 3.3, Proposition 3.4) for lower solutions.

Lemma 3.3. *Let $\lambda > 0$. If v_n is a lower solution of* (1.1)-(1.2) *and v_{n+1} is defined by* (1.3)-(1.4) *then $v_n \le v_{n+1}$.*

Proposition 3.4. *Assume that* (H1)–(H4) *hold and let $\lambda > 0$ be such that $\lambda \ge M\big(1 - N\int_0^1 p(t)dt\big)^{-1}$. Then the functions v_{n+1} defined recursively by* (1.3)-(1.4) *are such that for all $n \in \mathbb{N}$,*

(i) *v_n is a lower solution of* (1.1)-(1.2).
(ii) *$v_n \le v_{n+1}$.*

In the next result we prove that upper solution u_n is larger than lower solution v_n for all n.

Proposition 3.5. *Assume that* (H1)–(H4) *hold and let $\lambda > 0$ such that $\lambda \ge M\big(1 - N\int_0^1 p(t)dt\big)^{-1}$ and for all $x \in [0,1]$*

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \ge 0.$$

Then for all $n \in \mathbb{N}$, the functions u_n and v_n defined recursively by (1.3)-(1.4) *satisfy $v_n \le u_n$.*

Proof. We define a function

$$h_i(x) = f(x, v_i pv_i') - f(x, u_i, pu_i') + \lambda(u_i - v_i), \quad i \in \mathbb{N}.$$

It is easy to see that for all $i \in \mathbb{N}$, $w_i = u_i - v_i$ satisfies the differential equation

$$\begin{aligned} -(pw_i')' + \lambda p w_i &= p\{f(x, v_{i-1}, pv_{i-1}') - f(x, u_{i-1}, pu_{i-1}') + \lambda(u_{i-1} - v_{i-1})\} \\ &= ph_{i-1}\,. \end{aligned}$$

To prove this proposition we use again the principle of mathematical induction. For $i = 1$ we have $h_0 \ge 0$ and w_1 is the solution of (2.1)-(2.2) with $A = 0$ and $B = 0$. Using Proposition 2.7 we deduce that $w_1 \ge 0$; i.e., $u_1 \ge v_1$.

Now, let $n \ge 2$, $h_{n-2} \ge 0$ and $u_{n-1} \ge v_{n-1}$, then we are required to prove that $h_{n-1} \ge 0$ and $u_n \ge v_n$. First we show that for all $x \in [0,1]$ the function h_{n-1} is non-negative. Indeed we have

$$\begin{aligned} h_{n-1} &= f(x, v_{n-1}, pv_{n-1}') - f(x, u_{n-1}, pu_{n-1}') + \lambda(u_{n-1} - v_{n-1}) \\ &\ge -[(M-\lambda)w_{n-1} + N(sign\ w_{n-1}')pw_{n-1}']. \end{aligned}$$

Here w_{n-1} is a solution of (2.1) with $h(x) = h_{n-2} \ge 0$, $A = 0$ and $B = 0$. Arguments similar to the Proposition 3.2 can be used to prove that $h_{n-1} \ge 0$. Now, we have $h_{n-1} \ge 0$, $w_n'(0) = 0$ and $w_n'(1) = 0$ thus from Proposition 2.7 we deduce that $w_n \ge 0$, i.e., $u_n \ge v_n$. □

For the next lemma we use the assumption

(H5) For all $(x, u, pu') \in D$, $|f(x, u, pu')| \le \varphi(|pu'|)$ where $\varphi : [0, \infty) \to (0, \infty)$ is continuous and satisfies
$$\int_0^\infty \frac{ds}{\varphi(s)} > \int_0^1 p(x)dx.$$

Lemma 3.6. *If $f(x, u, pu')$ satisfies* (H1), (H2), (H5)*, then there exists $R_0 > 0$ such that any solution of*
$$-(pu')' + pf(x, u, pu') \ge 0, \quad 0 < x < 1, \quad u'(0) = 0 = u'(1)$$
with $u \in [v_0, u_0]$ for all $x \in [0, 1]$, satisfies $\|pu'\|_\infty < R_0$.

Proof. Consider an interval $[x, x_0] \subset [0, 1]$ such that for all $s \in [x, x_0)$,
$$u'(s) < 0 \quad \text{and} \quad u'(x_0) = 0.$$
Now using (H5) we have
$$(pu')' \le p\varphi(|pu'|)$$
and after integrating it from x to x_0 and using $(H5)$ we have $-pu' \le R_0$. Similarly for the interval $[x_0, x]$ we have $pu' \le R_0$. Thus $\|pu'\|_\infty \le R_0$. □

In the same way we can prove the following result for lower solutions.

Lemma 3.7. *If $f(x, v, pv')$ satisfies* (H1), (H2), (H5)*, then there exists $R_0 > 0$ such that any solution of*
$$-(pv')' + pf(x, v, pv') \le 0, \quad 0 < x < 1, \quad v'(0) = 0 = v'(1)$$
with $v \in [v_0, u_0]$ for all $x \in [0, 1]$, satisfies $\|pv'\|_\infty < R_0$.

Now we are in a situation to prove the our final result for the case when upper and lower solutions are well ordered.

Theorem 3.8. *Assume* (H1)–(H5) *hold. Let $\lambda > 0$ be such that*
$$\lambda \ge M\Big(1 - N\int_0^1 p(t)dt\Big)^{-1}$$
and for all the $x \in [0, 1]$,
$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \ge 0.$$
Then the sequences u_n and v_n defined by (1.3)–(1.4) *converge monotonically to solutions $\widetilde{u}(x)$ and $\widetilde{v}(x)$ of* (1.1)-(1.2)*. Any solution $z(x)$ of* (1.1)-(1.2) *in D satisfies*
$$\widetilde{v}(x) \le z(x) \le \widetilde{u}(x).$$

Proof. Using Lemma 3.1 to Lemma 3.7 and Proposition 3.2 to Proposition 3.5 we deduce that sequences $\{u_n\}$ and $\{v_n\}$ are monotonic ($u_0 \ge u_1 \ge u_2 \dots \ge u_n \ge v_n \dots \ge v_2 \ge v_1 \ge v_0$) and are bounded by v_0 and u_0 in $C[0, 1]$ and by Dini's theorem they converge uniformly to $\widetilde{u}$ and $\widetilde{v}$ (say). We can also deduce that the sequences $\{pu_n'\}$ and $\{pv_n'\}$ are uniformly bounded and equi-continuous in $C[0, 1]$ and by Arzela-Ascoli theorem there exists uniformly convergent subsequences $\{pu_{n_k}'\}$ and $\{pv_{n_k}'\}$ in $C[0, 1]$. It is easy to observe that $u_n \to \widetilde{u}$ and $v_n \to \widetilde{v}$ implies $pu_n' \to p\widetilde{u}'$ and $p\widetilde{v}_n' \to p\widetilde{v}'$.

Solution of (1.3)-(1.4) is given by (2.5) where $h(x) = -pf(x, y_{n-1}, py_{n-1}') + \lambda p y_{n-1}$. Since the sequences are uniformly convergent taking limit as $n \to \infty$ we obtain $\widetilde{u}$ and $\widetilde{v}$ as the solutions of the nonlinear boundary value problem (1.1)-(1.2). Any solution $z(x)$ in D plays the role of u_0. Hence $z(x) \geq \widetilde{v}(x)$. Similarly one concludes that $z(x) \leq \widetilde{u}(x)$. □

Remark 3.9. When the source function is derivative independent; i.e., $N = 0$. In this case we can choose $\lambda = M$.

4. Upper and lower solutions in reverse order

In this section we consider the case when the upper and lower solutions are in reverse order; i.e., $u_0(x) \leq v_0(x)$. For this we require opposite one-sided Lipschitz condition and we assume that

(F1) there exists upper solution (u_0) and lower solution (v_0) in $C^2[0,1]$ such that $u_0 \leq v_0$ for all $x \in [0,1]$;

(F2) the function $f : D_0 \to \mathbb{R}$ is continuous on
$$D_0 := \{(x, y, py') \in [0,1] \times R \times R : u_0 \leq y \leq v_0\};$$

(F3) there exists $M \geq 0$ such that for all $(x, \widetilde{\tau}, pv'),\ (x, \widetilde{\sigma}, pv') \in D_0$,
$$f(x, \widetilde{\sigma}, pv') - f(x, \widetilde{\tau}, pv') \geq -M(\widetilde{\sigma} - \widetilde{\tau}), \quad (\widetilde{\tau} \leq \widetilde{\sigma});$$

(F4) there exist $N \geq 0$ such that for all $(x, u, pv_1'), (x, u, pv_2') \in D_0$,
$$|f(x, u, pv_1') - f(x, u, pv_2')| \leq N|pv_2' - pv_1'|.$$

Here we define the approximation scheme by (1.3)-(1.4) and use Anti-maximum principle. We make a good choice of λ so that the sequences thus generated converge to the solution of the nonlinear problem. Similar to the Section 3 we require the following Lemmas and Propositions.

Lemma 4.1. *Let $-\lambda_0 < \lambda < 0$. If u_n is an upper solution of* (1.1)-(1.2) *and u_{n+1} is defined by* (1.3)-(1.4) *then $u_{n+1} \geq u_n$.*

Proof. Let $w_n = u_{n+1} - u_n$, then
$$-(pw_n')' + \lambda p w_n = (pu_n')' - pf(x, u_n, pu_n') \leq 0,$$
$$w_n'(0) \geq 0, \quad w_n'(1) \leq 0,$$
and using Proposition 2.7 we have $u_{n+1} \geq u_n$. □

Proposition 4.2. *Assume that* (F1)–(F4) *hold. Let $-\lambda_0 < \lambda < 0$ be such that $M + \lambda \leq 0$ and $(M + \lambda)\big(1 + \lambda \int_0^1 \frac{1}{p(x)} \int_0^x p(t)\,dt\,dx\big) - N\lambda \int_0^1 p(x)dx \leq 0$. Then the functions u_{n+1} defined recursively by* (1.3)-(1.4) *are such that for all $n \in \mathbb{N}$,*

(i) *u_n is an upper solution of* (1.1)-(1.2).

(ii) *$u_{n+1} \geq u_n$.*

Proof. Using Remark 2.4, Remark 2.6, Lemma 2.10, Lemma 4.1 and on the lines of the proof of Proposition 3.2 this proposition can be deduced easily. □

In the same way we can prove the following results for the lower solutions.

Lemma 4.3. *Let $-\lambda_0 < \lambda < 0$. If v_n is a lower solution of* (1.1)-(1.2) *and v_{n+1} is defined by* (1.3)-(1.4) *then $v_n \geq v_{n+1}$.*

Proposition 4.4. *Assume that* (F1)–(F4) *hold. Let $-\lambda_0 < \lambda < 0$ be such that $M + \lambda \leq 0$ and $(M + \lambda)\big(1 + \lambda \int_0^1 \frac{1}{p(x)} \int_0^x p(t)\,dt\,dx\big) - N\lambda \int_0^1 p(x)dx \leq 0$. Then the functions v_{n+1} defined recursively by* (1.3)-(1.4) *are such that for all $n \in \mathbb{N}$,*

(i) v_n *is a lower solution of* (1.1)-(1.2).
(ii) $v_n \geq v_{n+1}$.

In the next result we prove that lower solution v_n is larger than upper solution u_n for all n.

Proposition 4.5. *Assume that* (F1)–(F4) *hold. Let* $-\lambda_0 < \lambda < 0$ *be such that* $M + \lambda \leq 0$ *and*

$$(M+\lambda)\Big(1+\lambda\int_0^1 \frac{1}{p(x)}\int_0^x p(t)\,dt\,dx\Big) - N\lambda\int_0^1 p(x)dx \leq 0$$

and for all $x \in [0,1]$

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \geq 0.$$

Then for all $n \in \mathbb{N}$*, the functions* u_n *and* v_n *defined recursively by* (1.3)-(1.4) *satisfy* $v_n \geq u_n$.

Now similar to the Lemma 3.6 and Lemma 3.7 we state the following two results. These results establish a bound on $p(x)u'(x)$ and $p(x)v'(x)$. We use the assumption

(F5) For all $(x, u, pu') \in D_0$, $|f(x, u, pu')| \leq \varphi(|pu'|)$ where $\varphi : [0,\infty) \to (0,\infty)$ is continuous and satisfies

$$\int_0^\infty \frac{ds}{\varphi(s)} > \int_0^1 p(x)dx.$$

Lemma 4.6. *If* $f(x, u, pu')$ *satisfies* (F1), (F2), (F5)*, then there exists* $R_0 > 0$ *such that any solution of*

$$-(pu')' + pf(x, u, pu') \geq 0, \quad 0 < x < 1, \quad u'(0) = 0 = u'(1)$$

with $u \in [u_0, v_0]$ *for all* $x \in [0,1]$*, satisfies* $\|pu'\|_\infty < R_0$.

Lemma 4.7. *If* $f(x, v, pv')$ *satisfies* (F1), (F2),(F5)*, then there exists* $R_0 > 0$ *such that any solution of*

$$-(pv')' + pf(x, v, pv') \leq 0, \quad 0 < x < 1, \quad v'(0) = 0 = v'(1)$$

with $v \in [u_0, v_0]$ *for all* $x \in [0,1]$*, satisfies* $\|pv'\|_\infty < R_0$.

Finally we arrive at the theorem similar to the Theorem 3.8.

Theorem 4.8. *Assume* (F1)–(F5) *hold. Let* $-\lambda_0 < \lambda < 0$ *be such that* $M + \lambda \leq 0$ *and* $(M+\lambda)\big(1+\lambda\int_0^1 \frac{1}{p(x)}\int_0^x p(t)\,dt\,dx\big) - N\lambda\int_0^1 p(x)dx \leq 0$ *and for all* $x \in [0,1]$,

$$f(x, v_0, pv_0') - f(x, u_0, pu_0') + \lambda(u_0 - v_0) \geq 0.$$

Then the sequences u_n *and* v_n *defined by* (1.3)–(1.4) *converge monotonically to solutions* $\widetilde{u}(x)$ *and* $\widetilde{v}(x)$ *of* (1.1)-(1.2). *Any solution* $z(x)$ *of* (1.1)-(1.2) *in* D_0 *satisfies*

$$\widetilde{u}(x) \leq z(x) \leq \widetilde{v}(x).$$

Proof. Using Lemma 4.1 to Lemma 4.7 and Proposition 4.2 to Proposition 4.5 we deduce that

$$u_0 \leq u_1 \leq u_2 \leq \dots \leq u_n \leq v_n \dots \leq v_1 \leq v_0.$$

Now similar to the proof of Theorem 3.8 the result of this theorem can be deduced. □

Remark 4.9. When the source function is derivative independent; i.e., $N = 0$. In this case we can choose $\lambda = -M$.

Conclusion. This work fills the gap existing in the literature for reverse ordered upper and lower solutions. Some new existence results have been established. This work also generalize our earlier work [3]. We establish existence results under quite general conditions on $p(x)$ and $f(x,y,py')$. In this work we do not have Bessel functions and therefore we have to analyze the differential equation with general function $p(x)$. We prove some differential inequalities which enables us to prove the monotonicity of the sequences $\{u_n\}$ and $\{v_n\}$. As future scope of the present work we can further consider the following differential equation and generalize the present work even further.

$$-\big(p(x)y'(x)\big)' + q(x)f\big(x, y(x), p(x)y'(x)\big) = 0, \quad 0 < x < 1.$$

Here $q(x)$ is an integrable function on $[0,1]$ such that $q(x) > 0$ in $(0,1)$.

References

[1] D. O'Regan, M. A. El-Gebeily; Existence, upper and lower solutions and quasilinearization for singular differential equations, *IMA J. Appl. Math.*, Vol. 73 (2008) 323–344.

[2] M. Cherpion, C. D. Coster, P. Habets; A constructive monotone iterative method for second-order BVP in the presence of lower and upper solutions, *Appl. Math. Comp.*, Vol 123 (2001) 75-91.

[3] A. K. Verma; The monotone iterative method and zeros of Bessel functions for nonlinear singular derivative dependent BVP in the presence of upper and lower solutions, *Nonlinear Analysis*, Vol. 74, Issue 14, (2011) 4709-4717.

[4] R. K. Pandey; On a class of weakly regular singular two point boundary value problems II, *Journal of Differential Equation*, Vol. 127 (1996) 110-123.

[5] R. K. Pandey; On a class of regular singular two point boundary value problems, *J. Math. Anal. and Appl.*, Vol. 208 (1997) 388-403.

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