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SIMPLICITY AND STABILITY OF THE FIRST EIGENVALUE OF A $(p;q)$ LAPLACIAN SYSTEM

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Abstract. This article concerns special properties of the principal eigenvalue of a nonlinear elliptic system with Dirichlet boundary conditions. In particular, we show the simplicity of the first eigenvalue of

$$\begin{gathered}-\Delta_p u = \lambda |u|^{\alpha-1}|v|^{\beta-1}v \quad \text{in } \Omega,\\ -\Delta_q v = \lambda |u|^{\alpha-1}|v|^{\beta-1}u \quad \text{in } \Omega,\\ (u,v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega),\end{gathered}$$

with respect to the exponents p and q, where Ω is a bounded domain in $\mathbb{R}^N$.

1. Preliminaries

Eigenvalue problems for p-Laplacian operators subject to Zero Dirichlet boundary conditions on a bounded domain have been studied extensively during the past two decades, and many interesting results have been obtained. Most of the investigations have relied on variational methods and deduced the existence of a principal eigenvalue as a consequence of minimization results of appropriate functionals.

In this article, we study the eigenvalue system

$$\begin{gathered}-\Delta_p u = \lambda |u|^{\alpha-1}|v|^{\beta-1}v \quad \text{in } \Omega,\\ -\Delta_q v = \lambda |u|^{\alpha-1}|v|^{\beta-1}u \quad \text{in } \Omega,\\ (u,v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega),\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p, q > 1$ and α, β are real numbers satisfying

$$\alpha > 0, \quad \beta > 0, \quad \frac{\alpha}{p} + \frac{\beta}{q} = 1. \tag{1.2}$$

We mention that problem (1.1) aries in several fields of application. For instance, in the case where $p > 2$, problem (1.1) appears in the study of non-Newtonian fluids, pseudoplastics for $1 < p < 2$, and in reaction-diffusion problems, flows through porous media, nonlinear elasticity, petroleum extraction, astronomy and glaciology for $p = 4/3$ (see [3, 5]).

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The principal eigenvalue $\lambda_1(p;q)$ of (1.1) is obtained using the Ljusternick-Schnirelman theory by minimizing the functional

$$J(u,v)=\frac{\alpha}{p}\int_\Omega |\nabla u|^p dx+\frac{\beta}{q}\int_\Omega |\nabla v|^q dx,$$

on C^1-manifold: $\{(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega);\ \Lambda(u,v)=1\}$, where

$$\Lambda(u,v)=\int_\Omega |u|^{\alpha-1}|v|^{\beta-1}uv\,dx.$$

We recall that $\lambda_1(p,q)$ can be variationally characterized as

$$\lambda_1(p,q)=\inf\{J(u,v),\ (u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega);\ \Lambda(u,v)=1\} \tag{1.3}$$

From the maximum principle of Vázquez, see [12], we deduce that the corresponding eigenpair of $\lambda_1(p;q)$; that is, $(u;v)$ is such that $u;v>0$. We call it positive eigenvector.

Definition 1.1. An open subset Ω of $\mathbb{R}^N$ is said to have the segment property if for any $x\in\partial\Omega$, there exists an open set $G_x\in\mathbb{R}^N$ with $x\in G_x$ and a pair y_x of $\mathbb{R}^N\backslash\{0\}$ such that if $z\in\overline{\Omega}\cap G_x$ and $t\in(0,1)$, then $z+ty_x\in\Omega$.

This property allows us to push the support of a function u in Ω by a translation . The following results play an important role in the proof of Theorem 2.

Lemma 1.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the segment property. If $u\in W^{1,p}(\Omega)\cap W_0^{1,s}(\Omega)$ for some $s\in(1,p)$, then $u\in W_0^{1,p}(\Omega)$.*

2. Simplicity

Firstly we introduce

$$\begin{aligned} A_p(u,\varphi)&=\int_\Omega |\nabla u|^p dx+(p-1)\int_\Omega |\nabla\varphi|^p\Big(\frac{|u|}{\varphi}\Big)^p dx\\ &\quad -p\int_\Omega \frac{|u|^{p-2}u}{\varphi^{p-1}}|\nabla\varphi|^{p-2}\nabla\varphi\nabla u dx\\ &=\int_\Omega |\nabla u|^p dx+\int_\Omega \frac{\Delta_p\varphi}{\varphi^{p-1}}|u|^p dx. \end{aligned} \tag{2.1}$$

Lemma 2.1 ([4])**.** *For all $(u,\varphi)\in(W_0^{1,p}(\Omega)\cap C^{1,\gamma}(\Omega))^2$ with $\varphi>0$ in Ω and $\gamma\in(0,1)$, we have $A_p(u,\varphi)$; i.e.,*

$$\int_\Omega |\nabla u|^p dx\ge\int_\Omega \frac{\Delta_p\varphi}{\varphi^{p-1}}|u|^p dx,$$

and if $A_p(u,\varphi)=0$ there is $c\in\mathbb{R}$ such that $u=c\varphi$.

Proof. Using Young's inequality (since $\frac{1}{p}+\frac{p-1}{p}=1$) we can write, for $\epsilon>0$,

$$\begin{aligned} \nabla u|\nabla\varphi|^{p-2}\nabla\varphi\frac{u|u|^{p-2}}{\varphi^{p-1}}&\le|\nabla u||\nabla\varphi|^{p-1}\Big(\frac{|u|}{\varphi}\Big)^{p-1}\\ &\le\frac{\epsilon^p}{p}|\nabla u|^p+\frac{p-1}{p\epsilon^p}|\frac{u}{\varphi}|^p|\nabla\varphi|^p. \end{aligned} \tag{2.2}$$

By choosing $\epsilon=1$ and integration over Ω, we have

$$p\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u\Big(\frac{u|u|^{p-2}}{\varphi^{p-1}}\Big)dx\le\int_\Omega |\nabla u|^p dx+(p-1)\int_\Omega |\frac{u}{\varphi}|^p|\nabla\varphi|^p dx. \tag{2.3}$$

Therefore, we conclude that $A_p(u,\varphi) \geq 0$.

If $A_p(u,\varphi) = 0$, then we obtain

$$p\int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\nabla u\Big(\frac{u|u|^{p-2}}{\varphi^{p-1}}\Big)dx = \int_\Omega |\nabla u|^p dx + (p-1)\int_\Omega |\frac{u}{\varphi}|^p|\nabla\varphi|^p dx. \tag{2.4}$$

Letting $\epsilon = 1$ in (2.2), we obtain

$$\int_\Omega \nabla u|\nabla\varphi|^{p-2}\nabla\varphi\frac{u|u|^{p-2}}{\varphi^{p-1}}dx = \int_\Omega |\nabla u||\nabla\varphi|^{p-1}\Big(\frac{|u|}{\varphi}\Big)^{p-1}dx. \tag{2.5}$$

Combining the two inequalities, we deduce that $|\nabla u| = |\big(\frac{u}{\varphi}\big)\nabla\varphi|$, it follows that $\nabla u = \eta\big(\frac{u}{\varphi}\big)\nabla\varphi$, where $|\eta| = 1$. On the other hand, $A_p(u,\varphi) = 0$ implies that $\eta = 1$ and $\nabla(\frac{u}{\varphi}) = 0$; that is, there is $c \in \mathbb{R}$ such that $u = c\varphi$. This completes the proof. $\square$

Theorem 2.2. *Let $\lambda_1(p,q)$ be defined by* (1.3)*, then $\lambda_1(p,q)$ is simple.*

Proof. Let (u,v) and (φ,ψ) be two eigenvectors associated with $\lambda_1(p,q)$. We show that there exist real numbers k_1, k_2 such that $u = k_1\varphi$ and $v = k_2\psi$. Using Young's inequality, by (1.2) and the definition of $\lambda_1(p,q)$, we can write

$$\begin{aligned}
J(\varphi,\psi) &= \lambda_1(p,q)\Lambda(\varphi,\psi)\\
&\leq \lambda_1(p,q)\int_\Omega u^\alpha v^\beta\frac{|\varphi|^\alpha|\psi|^\beta}{u^\alpha v^\beta}dx\\
&\leq \lambda_1(p,q)\int_\Omega u^\alpha v^\beta\Big[\frac{\alpha}{p}\frac{|\varphi|^p}{u^p} + \frac{\beta}{q}\frac{|\psi|^q}{v^q}\Big]dx\\
&\leq \lambda_1(p,q)\int_\Omega \Big[\frac{\alpha}{p}\frac{u^{\alpha-1}v^\beta}{u^{p-1}}|\varphi|^p + \frac{\beta}{q}\frac{u^\alpha v^{\beta-1}}{v^{q-1}}|\psi|^q\Big]dx\\
&\leq \frac{\alpha}{p}\int_\Omega\frac{-\Delta_p u}{u^{p-1}}|\varphi|^p dx + \frac{\beta}{q}\int_\Omega\frac{-\Delta_q v}{v^{q-1}}|\psi|^q dx.
\end{aligned}$$

Due to Lemma 2.1, we obtain

$$J(u,v) = \frac{\alpha}{p}\int_\Omega\frac{-\Delta_p u}{u^{p-1}}|\varphi|^p dx + \frac{\beta}{q}\int_\Omega\frac{-\Delta_q v}{v^{q-1}}|\psi|^q dx.$$

Thus

$$\int_\Omega |\nabla\varphi|^p dx = \int_\Omega\frac{-\Delta_p u}{u^{p-1}}|\varphi|^p dx, \quad \int_\Omega |\nabla\psi|^q dx = \int_\Omega\frac{-\Delta_q v}{v^{q-1}}|\psi|^q dx.$$

By Lemma 2.1, there exist real numbers k_1 and k_2 such that $u = k_1\varphi$ and $v = k_2\psi$ and the theorem follows. $\square$

3. Stability

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the segment property. Then, the function $(p,q) \to \lambda_1(p,q)$ is continuous from $I_{\alpha,\beta}$ to $\mathbb{R}^+$, where*

$$I_{\alpha,\beta} = \{(p,q) \in (1,+\infty)\times(1,+\infty) \text{ such that } (1.2) \text{ is satisfied}\}.$$

Proof. Let $(t_n)_{n\geq1}$, $t_n = (p_n,q_n)$ be a sequence in $I_{\alpha,\beta}$ converging at $t = (p,q) \in I_{\alpha,\beta}$. We will prove that

$$\lim_{n\to\infty}\lambda_1(p_n,q_n) = \lambda_1(p,q).$$

Indeed, let $(\varphi,\psi)\in C_0^\infty(\Omega)\times C_0^\infty(\Omega)$ such that $\Lambda(\varphi,\psi)>0$; hence,

$$\lambda_1(p_n,q_n)\le \frac{\frac{\alpha}{p_n}\|\nabla\varphi\|_{p_n}^{p_n}+\frac{\beta}{q_n}\|\nabla\psi\|_{q_n}^{q_n}}{\Lambda(\varphi,\psi)},$$

since $\lambda_1(p_n,q_n)$ is the infimum. Letting n tend to infinity, we deduce from Lebesgue's theorem

$$\limsup_{n\to\infty}\lambda_1(p_n,q_n)\le \frac{\frac{\alpha}{p}\|\nabla\varphi\|_p^p+\frac{\beta}{q}\|\nabla\psi\|_q^q}{\Lambda(\varphi,\psi)}.$$

Then

$$\limsup_{n\to\infty}\lambda_1(p_n,q_n)\le\lambda_1(p,q). \tag{3.1}$$

On the other hand, let $\{(p_{n_k},q_{n_k})\}_{k\ge1}$ be a subsequence of $(t_n)_n$ such that

$$\lim_{k\to\infty}\lambda_1(p_{n_k},q_{n_k})=\liminf_{n\to\infty}\lambda_1(p_n,q_n).$$

Let us fix $\epsilon_0>0$ small enough, so that for all $\epsilon\in(o,\epsilon_0)$, we have

$$1<\min(p-\epsilon,q-\epsilon), \tag{3.2}$$

$$\max(p+\epsilon,q+\epsilon)<\min((p-\epsilon)^*,(q-\epsilon)^*). \tag{3.3}$$

For each $k\in\mathbb{N}$, let $(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})})\in W_0^{1,p_{n_k}}(\Omega)\times W_0^{1,q_{n_k}}(\Omega)$ be a principal eigenfunction of $(S_{p_{n_k},q_{n_k}})$ related with $\lambda_1(p_{n_k},q_{n_k})$. Then, by Holder's inequality, for $\epsilon\in(0,\epsilon_0)$, the following inequalities hold:

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\epsilon}\le\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p_{n_k}}|\Omega|^{\frac{p_{n_k}-p+\epsilon}{p_{n_k}(p-\epsilon)}}, \tag{3.4}$$

$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\epsilon}\le\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q_{n_k}}|\Omega|^{\frac{q_{n_k}-q+\epsilon}{q_{n_k}(q-\epsilon)}}. \tag{3.5}$$

Combining these two inequalities and using the variational characterization of λ_1, we have

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\epsilon}\le\Big\{\frac{p_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\alpha}\Big\}^{\frac{1}{p_{n_k}}}|\Omega|^{\frac{p_{n_k}-p+\epsilon}{p_{n_k}(p-\epsilon)}} \tag{3.6}$$

$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\epsilon}\le\Big\{\frac{q_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\beta}\Big\}^{\frac{1}{q_{n_k}}}|\Omega|^{\frac{q_{n_k}-q+\epsilon}{q_{n_k}(q-\epsilon)}}. \tag{3.7}$$

Therefore, via (3.2) and (3.3), for a subsequence

$$\begin{gathered}(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})})\rightharpoonup(u,v)\quad\text{weakly in } W_0^{1,p-\epsilon}(\Omega)\times W_0^{1,q-\epsilon}(\Omega),\\ (u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})})\to(u,v)\quad\text{strongly in } L^{p+\epsilon}(\Omega)\times L^{q+\epsilon}(\Omega)\end{gathered}$$

Passing to the limit in (3.6) and (3.7), respectively as $k\to\infty$ and as $\epsilon\to\infty$, we have

$$\begin{gathered}\|\nabla u\|_p^p\le\frac{p}{\alpha}\lim_{k\to\infty}\lambda_1(p_{n_k},q_{n_k})<\infty,\\ \|\nabla v\|_q^q\le\frac{q}{\beta}\lim_{k\to\infty}\lambda_1(p_{n_k},q_{n_k})<\infty.\end{gathered}$$

Then

$$u\in W_0^{1,p-\epsilon}(\Omega)\cap W^{1,p}(\Omega)=W_0^{1,p}(\Omega),\quad v\in W_0^{1,q-\epsilon}(\Omega)\cap W^{1,q}(\Omega)=W_0^{1,q}(\Omega),$$

because Ω satisfies the segment property.

On the other hand, from the variational characterization of $\lambda_1(p_{n_k}, q_{n_k})$, (3.4), (3.5), and using the weak lower semi continuity of the norm; it follows that

$$\frac{1}{|\Omega|^{\frac{\epsilon}{p-\epsilon}}}\frac{\alpha}{p}\|\nabla u\|_{p-\epsilon}^{p}+\frac{1}{|\Omega|^{\frac{\epsilon}{q-\epsilon}}}\frac{\beta}{q}\|\nabla v\|_{q-\epsilon}^{q}\le\lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k}). \tag{3.8}$$

Letting $\epsilon \to 0^+$ in (3.8), the Fatou lemma yields

$$\frac{\alpha}{p}\|\nabla u\|_p^p+\frac{\beta}{q}\|\nabla v\|_q^q\le\lim_{k\to\infty}\lambda_1(p_{n_k},q_{n_k}).$$

Since $\Lambda(u_{(p_{n_k},q_{n_k})}, v_{(p_{n_k},q_{n_k})}) = 1$ via compactness of Λ, (u, v) is admissible in the variational characterization of $\lambda_1(p, q)$; hence

$$\lambda_1(p,q)\le\lim_{k\to\infty}\lambda_1(p_{n_k},q_{n_k})=\liminf_{n\to\infty}\lambda_1(p_n,q_n).$$

This and (3.1) will complete the proof. Observe that the segment property is used only to prove that

$$\lambda_1(p,q)\le\liminf_{n\to\infty}\lambda_1(p_n,q_n).$$

$\square$

References

[1] A. Annane; *Simplicité et isolation de la première valeur proper du p-Laplacian avec poids,* C. R. Acad. Sci. Paris, Sér. I Math., 305 (1987), no. 16, 725-728.

[2] R. A. Adams; *Sobolev Spaces*, Academic Press, New York, 1975.

[3] J. I. Diaz; *Nonlinear Partial Differential Equations and Free Boundaries* Vol. I. Elliptic Equations, Research Notes in mathematics. vol. 106, Pitman, Massachusetts, 1985.

[4] A. El Khalil, M. Ouanan, A. Touzani; *Bifurcation of nonlinear elliptic system from the first eigenvalue,* Electron. J. Qual. Theory Differ. Equ. 2003 (2003), no. 21, 1-18.

[5] A. El Khalil; *Autour de la première courbe propre du p-Laplacien,* Thèse de Doctorat, 1999.

[6] A. El Khalil, S. El Manouni, M. Ouanan; *Simplicity and stability of the first eigenvalue of a nonlinear elliptic system,* International Journal of Mathematical Sciences, 10 (2005) 1555-1563.

[7] A. El Khalil, M. Ouanan, A. Touzani; *Stability of the principal eigenvalue of a nonlinear elliptic system,* Electron. Jouranl of Diff. Equ., Conference 11, (2004), 129-134.

[8] A. Ghanmi, H. Mâagli, V. Rădulescu, N. Zeddini; *Large and bounded solutions for a class of nonlinear Schröödinger stationary systems*, Anal. Appl. (Singap.) 7 (2009), no. 4, 391-404.

[9] A. Kristály, V. Rădulescu, Cs. Varga; *Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, Cambridge, 2010.

[10] P. Lindqvist; *On nonlinear Rayleigh quotients*, Potential Analysis 2, (1993), 199-218.

[11] P. Lindqvist; *On a nonlinear eigenvalue problem*, Ber. Univer. Jyväskylä Math. Inst. 68, (1995), 33-54.

[12] J. L. Vazquez; *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. 12 (1984), no. 3, 191-202.

[13] E. Zeidler; *Nonlinear Functional Analysis and Applications*, vol. 3, Variational Methods and Optimization, Springer, Berlin, 1985.

Addendum posted by the editor on July 21, 2016

A reader informed the editors that the results in this article are a direct transfer of the results in reference [6]; $(0, \infty)$ is considered in [1], while $(-1, \infty)$ is considered here. The reader also informed us that most of the material was copied from [6] without mentioning the source; which seems to be plagiarism.

The authors response was: "Carrying the results from $(0, \infty)$ to $(-1, \infty)$ is an important step. Some modifications were made in the second part of the proof of Theorem 3.1 for our model" and "We should have stated that our proofs are based on proofs in [6]".

The author's reason for copying without mentioning the source was "We mentioned [1] in the references but we should have mentioned it explicitly in the text."

End of addendum.

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