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COMPLETE GEOMETRIC INVARIANT STUDY OF TWO CLASSES OF QUADRATIC SYSTEMS

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ABSTRACT. In this article, using affine invariant conditions, we give a complete study for quadratic systems with center and for quadratic Hamiltonian systems. There are two improvements over the results in [30] that studied centers up to GL-invariant, and over the results in [1] that classified Hamiltonian quadratic systems without invariants. The geometrical affine invariant study presented here is a crucial step toward the goal of the invariant classification of all quadratic systems according to their singularities, finite and infinite.

1. Introduction and statement of results

Let $\mathbb{R}[x, y]$ be the ring of the polynomials in the variables x and y with coefficients in $\mathbb{R}$. We consider a system of polynomial differential equations, or simply a polynomial differential system, in $\mathbb{R}^2$ defined by

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y), \end{aligned} \tag{1.1}$$

where $P, Q \in \mathbb{R}[x, y]$. We say that the maximum of the degrees of the polynomials P and Q is the degree of system (1.1). A *quadratic polynomial differential system* or simply a *quadratic system* (QS) is a polynomial differential system of degree 2. We say that the quadratic system (1.1) is *non-degenerate* if the polynomials P and Q are relatively prime or coprime; i.e., g.c.d. $(P, Q) = 1$.

During the previous one-hundred years quadratic vector fields have been investigated intensively as one of the easiest but far from trivial families of nonlinear differential systems, and more than one thousand papers have been published about these vectors fields (see for instance [24, 33, 32]). However, the problem of classifying all the quadratic vector fields (even integrable ones) remains open. For more information on the integrable differential vector fields in dimension 2, see for instance [9, 18].

Poincaré [23] defined the notion of a *center* for a real polynomial differential system in the plane (i.e. an isolated singularity surrounded by periodic orbits). The analysis of the limit cycles which bifurcate from a focus or a center of a quadratic system was made by Bautin [7], by providing the structure of the power series development of the displacement function defined near a focus or a center

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of a quadratic system. More recently the structure of this displacement function has been understood for any weak focus of a polynomial differential system. More precisely, first by using a linear change of coordinates and a rescaling of the independent variable, we transform any polynomial differential system having a weak focus or a center at the origin with eigenvalues $\pm ai \neq 0$ (i.e. having a *weak focus*) into the form

$$\begin{aligned} \dot{x} &= y + P(x,y), \\ \dot{y} &= -x + Q(x,y), \end{aligned} \tag{1.2}$$

where P and Q are polynomials without constant and linear terms. Then the *return map* $x \mapsto h(x)$ is defined for $|x| < R$, where R is a positive number sufficiently small to insure that the power series expansion of $h(x)$ at the origin is convergent. Of course, limit cycles correspond to isolated zeros of the *displacement function* $d(x) = h(x) - x$. The structure of the power series for the displacement function is given by the following restatement of Bautin's fundamental result (see [25] for more details): There exists a positive integer m and a real number $R > 0$ such that the displacement function in a neighborhood of the origin for the polynomial differential system (1.2) can be written as

$$d(x) = \sum_{j=1}^{m} v_{2j+1} x^{2j+1} \Big[\alpha_0 + \sum_{k=1}^{\infty} \alpha_k^{2j+1} x^k \Big],$$

for $|x| < R$, where the v_{2j+1}'s and the α_k^{2j+1}'s are homogeneous polynomials in the coefficients of the polynomials P and Q.

The constants $V_j = v_{2j+1}$ are called the *focus quantities* or the Poincaré-Liapunov constants. A weak focus for which $V_1 = \cdots = V_{n-1} = 0$ and $V_n \neq 0$ is a *weak focus of order n*. If all the focus quantities are zero then the weak focus is a center. Note that any weak focus has finitely many focus quantities, in our notation exactly m.

It is known that a polynomial differential system (1.2) has a center at the origin if and only if there exists a local analytic first integral of the form $H = x^2+y^2+F(x,y)$ defined in a neighborhood of the origin, where F starts with terms of order higher than 2. This result is due to Poincaré [23] (Moussu [20] gave a geometrical proof of this result). Liapunov [17] extended Poincaré result for the analytic case.

Through the coefficients of a quadratic system every one of these systems can be identified with a single point of $\mathbb{R}^{12}$. One of the first steps in a systematic study of the subclasses of QS was achieved in the determining the subclass QC of all QS having a center. Of course this problem is algebraically solvable in the sense indicated by Coppel [10], because the classification of the quadratic centers is algebraically solvable.

The phase portraits of the class QC were given by Vulpe in [30] and are here denoted by $Vul_{\#}$ using his classification. In that classification, only GL-invariants were used which implied that systems could only be characterized after displacing one center to the origin and adopting the standard normal form. Later papers related with centers provide the bifurcation diagrams for the different types of centers (see [26, 34, 22]).

The polynomial differential system (1.1) is *Hamiltonian* if there exists a polynomial $H = H(x,y)$ such that $P = \partial H/\partial y$ and $Q = -\partial H/\partial x$. Regarding Hamiltonian systems, apart from many papers using them in conservative systems, the first complete classification for quadratic systems was done in [1] where quadratic Hamiltonian systems were split into four normal forms and a bifurcation diagram was provided for each one of them. No invariants were used there. Later on in [16] the affine–invariant conditions were established but they were constructed using invariant polynomials of high degree without explicit geometrical meaning. In the later years the technique of the construction of invariant polynomials has been greatly improved. Now, with these better tools, the invariants needed are of lower degree and consistent with the set of all invariants needed to describe singular points.

The main results of this article are the following two theorems.

Theorem 1.1. *Consider a quadratic system of differential equations.*

- (i) *This system possesses a center and the configuration of all its singularities (finite and infinite) up to a congruent equivalence, given in Table 2 if and only if the corresponding affine invariant conditions described in Table 2 hold.*
- (ii) *We have a total of 41 congruently distinct configurations of singularities. For each phase portrait of quadratic systems with a center we have the following two possibilities:*
  - (a) *it corresponds to a unique configuration of singularities; there are 17 such phase portraits;*
  - (b) *it corresponds to several configurations of singularities; there are 14 such phase portraits. The richest example is Vul_2 which could occur with anyone of the 8 congruently distinct configurations 15, 17, 19, 20, 26, 37, 39, 41.*
- (iii) *The phase portrait of a system with a center corresponds to the one of 31 topologically distinct phase portraits constructed in* [30] *(except the case of a linear system) and either it is determined univocally by the respective configuration, or it is determined by the configuration and additional conditions given in Table 3. More exactly we have 35 congruently distinct configurations, each of which leads univocally to a unique phase portrait; and there are 6 configurations each of which leads to several phase portraits distinguished by the additional conditions according to Table 3.*

Tables 2 and 3 can be found in section 6.

Theorem 1.2. *Assume that a quadratic system of differential equations is Hamiltonian.*

- (i) *This system possesses the configuration of all its singularities (finite and infinite) up to a congruent equivalence, given in Table 4 if and only if the corresponding affine invariant conditions described in Table 4 hold.*
- (ii) *We have a total of 30 congruently distinct configurations of singularities. For each phase portrait of Hamiltonian quadratic systems we have the following two possibilities:*
  - (a) *it corresponds to a unique configuration of singularities; there are 23 such phase portraits;*
  - (b) *it corresponds to several configurations of singularities; there are 5 such phase portraits. The richest example is Ham_{11} which could occur with anyone of the 5 congruently distinct configurations 7,11, 20,26,28.*
- (iii) *The phase portrait of this system corresponds to the one of 28 topologically distinct phase portraits constructed in* [1] *and either it is determined univocally by the respective configuration, or it is determined by the configuration*

and additional conditions given in Table 5. More exactly we have 24 congruently distinct configurations, each of which leads univocally to a unique phase portrait; and there are 6 configurations each of which leads to several phase portraits distinguished by the additional conditions according to Table 5.

Tables 4 and 5 can be found in section 7.

The work is organized as follows. In sections 2 and 3 we introduce the notation that we use for describing the singular points. In section 4 we give some preliminary results needed for the work. In section 5 we adapt a diagram from a previous paper [5] to describe more easily the bifurcation tree of finite singularities, and we also introduce the used invariants from [30] and [16]. Finally in sections 6 and 7 we prove the main theorems of this paper.

2. Equivalence relations for singularities of planar polynomial vector fields

We first recall the topological equivalence relation as it is used in most of the literature. Two singularities p_1 and p_2 are topologically equivalent if there exist open neighborhoods N_1 and N_2 of these points and a homeomorphism $\Psi : N_1 \to N_2$ carrying orbits to orbits and preserving orientation. To reduce the number of cases, by topological equivalence we shall mean here that the homeomorphism Ψ *preserves or reverses* the orientation. In this article we use this second notion, which is sometimes used in the literature (see [12, 2]).

Polynomial vector fields can be compactified using different techniques which give a global view of the phase portraits including the trajectories close to infinity which leads to the notion of infinite singular points (for more details see, for example [19]).

Finite and infinite singular points may either be real or complex. Most of the times one only needs to observe the real ones. We point out that the sum of the multiplicities of all singular points of a quadratic system (with a finite number of singular points) is always 7. The sum of the multiplicities of the infinite singular points is always at least 3, more precisely it is always 3 plus the sum of the multiplicities of the finite points which have gone to infinity.

We use here the following terminology for singularities grouped in the families:

- We call *elemental* a singular point with its both eigenvalues not zero.
- We call *semi-elemental* a singular point with exactly one of its eigenvalues equal to zero.
- We call *nilpotent* a singular point with its eigenvalues zero but its Jacobian matrix is not identically zero.
- We call *intricate* a singular point with its Jacobian matrix identically zero.

  This notation (except "nilpotent") was proposed by Dana Schlomiuk in a personal communication in order to avoid intersection with previous well-known notations. We are grateful to her for the help.

We say that two points are *Jordan relatives* if they both belong to one of the families above.

Roughly speaking a singular point p of an analytic differential system χ is a *multiple singularity of multiplicity* m if p produces m singularities, as closed to p as we wish, in analytic perturbations χ_ϵ of this system and m is the maximum such number. In polynomial differential systems of fixed degree n we have several possibilities for obtaining multiple singularities.

(i) A finite singular point can split into several finite singularities in n-degree polynomial perturbations.
(ii) An infinite singular point could split into some finite and some infinite singularities in n-degree polynomial perturbations.
(iii) n-degree perturbations of an infinite singularity produce only infinite singular points of the systems.

To all these cases we can give a precise mathematical meaning using the notion of intersection multiplicity at a point p of two algebraic curves.

Two foci (or saddles) are *order equivalent* if their corresponding orders coincide.

Semi-elemental saddle-nodes are always topologically equivalent.

Definition 2.1. Two singularities p_1 and p_2 of two polynomial vector fields are *congruently equivalent* if and only if they are topologically equivalent, they have the same multiplicity, they are Jordan relatives and in case of foci or saddles they are order equivalent.

In this work we discuss the behavior of quadratic vector fields globally around their singularities.

Definition 2.2. Let χ_1 and χ_2 be two polynomial vector fields each having a finite number of singularities. We say that χ_1 and χ_2 have *congruent equivalent configurations of singularities* if and only if we have a bijection ϑ carrying the singularities of χ_1 to singularities of χ_2 and for every singularity p of χ_1, $\vartheta(p)$ is congruently equivalent with p.

3. The notation for singular points

In this section we present the notation that we use for describing the singular points. The complete notation for singular points will appear in our project of classification of finite and infinite singular points of all QS.

This notation used here for describing finite and infinite singular points of quadratic systems, can easily be extended to general polynomial systems.

We start by distinguishing the finite and infinite singularities denoting the first ones with lower case letters and the second with capital letters. When describing in a row both finite and infinite singular points, we will always order them first finite, latter infinite with a semicolon (';') separating them.

Starting with elemental points, we use the letters 's','S' for "saddles"; 'n', 'N' for "nodes"; 'f' for "foci" and 'c' for "centers".

An elemental singular point is called a *weak singularity* if the trace of its Jacobian is zero. It follows easily that such a singular point could be either a focus or a center or a saddle. In order to determine the stability of weak focus one needs to compute higher order terms of a certain function (see [17]). Depending on the number of the terms of this function which vanish we can determine the order of the focus. A similar technique can be used also in the case of a weak saddle.

Finite elemental foci (or saddles) are classified according to their order as weak foci (or saddles). When the trace of the Jacobian matrix evaluated at those singular points is not zero, we call them strong saddles and strong foci and we maintain the standard notations 's' and 'f.' But when the trace is zero, it is known that for quadratic systems they may have up to 3 orders plus an integrable one, which corresponds to infinite order. So, from the order 1 to order 3 we denote them by '$s^{(i)}$' and '$f^{(i)}$' where $i = 1, 2, 3$ is the order. For the integrable case, the saddle remains a topological saddle and it will be denoted by '$\$$'. In the second case we have a change in the topology of the local phase portrait which makes the singular point a center and it is denoted by 'c'.

Foci and centers cannot appear as isolated singular points at infinity and hence it is not necessary to introduce their order in this case. In case of saddles, we can have weak saddles at infinity but it is premature at this stage to describe them since the maximum order of weak singularities in cubic systems is not yet known.

All non–elemental singular points are multiple points, in the sense that when we perturb them within a nearby system they could split in at least two elemental points. For finite singular points we denote with a subindex their multiplicity as in '$\overline{s}_{(5)}$' or in '$\widehat{es}_{(3)}$' (the meaning of the '$^{-}$' and the '$\widehat{\ }$' will be explained below). The *multiplicity* of a singularity of a QS is the maximum number of singular points which can appear from this singularity when we perturb it inside the class of all QS. In order to describe the various kinds of multiplicity of infinite singular points we use the concepts and notations introduced in [28]. Thus we denote by '$\binom{a}{b}$...' the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example '$\overline{\binom{1}{1}}\ SN$' means a saddle–node at infinity produced by the collision of one finite singularity with an infinite one; '$\overline{\binom{0}{3}}\ S$' means a saddle produced by the collision of 3 infinite singularities.

Semi–elemental points can either be nodes, saddles or saddle–nodes, finite or infinite. We will denote them always with an overline, for example '$\overline{sn}$', '$\overline{s}$' and '$\overline{n}$' with the corresponding multiplicity. In the case of infinite points we will put the '$^{-}$' on top of the parenthesis of multiplicity.

Nilpotent points can either be saddles, nodes, saddle–nodes, elliptic-saddles, cusps, foci or centers. The first four of these could be at infinity. We denote the finite ones with a hat '$\widehat{\ }$' as in $\widehat{es}_{(3)}$ for a finite nilpotent elliptic-saddle of multiplicity 3, and $\widehat{cp}_{(2)}$ for a finite nilpotent cusp point of multiplicity 2. In the case of nilpotent infinite points, analogously to the case of semi-elemental points we will put the '$\widehat{\ }$' on top of the parenthesis of multiplicity. The relative position of the sectors of an infinite nilpotent point with respect to the line at infinity can produce topologically different phase portraits. This forces us to use a notation for these points similar to the notation which we will use for the intricate points.

It is known that the neighborhood of any singular point of a polynomial vector field (except foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic, hyperbolic and elliptic (see [11]). Then a reasonable way to describe intricate points and nilpotent points at infinity is to use a sequence formed by the types of their sectors. The description we give is the one which appears in the clock–wise direction once the blow–down is done. Thus in quadratic systems we have just seven possibilities for finite intricate singular points (see [3]) which are the following ones

- (a) $phpphp_{(4)}$;
- (b) $phph_{(4)}$;
- (c) $hh_{(4)}$;
- (d) $hhhhhh_{(4)}$;
- (e) $peppep_{(4)}$;

- (f) $pepe_{(4)}$;
- (g) $ee_{(4)}$.

We use lower case because of the finite nature of the singularities and add the subindex (4) since they are of multiplicity 4.

For infinite intricate and nilpotent singular points, we insert a hyphen between the sectors to split those which appear on one side of the equator of the sphere from the ones which appear in the other side. In this way we distinguish between $\binom{2}{2}PHP$-PHP and $\binom{2}{2}PPH$-PPH.

The lack of finite singular points will be encapsulated in the notation $\emptyset$. In the cases we need to point out the lack of an infinite singular point we will use the symbol $\emptyset$.

Finally there is also the possibility that we have an infinite number of finite or infinite singular points. In the first case, this means that the polynomials defining the differential system are not coprime. Their common factor may produce a real line or conic filled up with singular points, or a conic with real coefficients having only complex points.

We consider now systems which have the set of non isolated singularites located on the line at infinity. It is known that the neighborhood of infinity can be of 6 different types (see [28]) up to topological equivalence. The way to determine them comes from a study of the reduced system on the infinite local charts where the line of singularities can be removed within the chart and still a singular point may remain on the line at infinity. Thus, depending of the nature of this point, the behavior of the singularities at infinity of the original system can be denoted as $[\infty, \emptyset]$, $[\infty, N]$, $[\infty, S]$, $[\infty, C]$, $[\infty, \overline{SN}]$ or $[\infty, \widehat{ES}]$. In the families showed in this paper we will only meet the case $[\infty, S]$.

We will denote with the symbol $\ominus$ the case when the polynomials defining the system have a common factor. The symbol stands for the most generic of these cases which corresponds to a real line filled up of singular points. The degeneracy can be also be produced by a common quadratic factor which could generate any kind of conic. We will indicate each case by the following symbols

- $\ominus[|]$ for a real straight line;
- $\ominus[\cup]$ for a real parabola;
- $\ominus[||]$ for two real parallel lines;
- $\ominus[||^c]$ for two complex parallel lines;
- $\ominus[|_2]$ for a double real straight line;
- $\ominus[\ )(\ ]$ for a real hyperbola;
- $\ominus[\times]$ for two intersecting real straight lines;
- $\ominus[\circ]$ for a real circle or ellipse;
- $\ominus[©]$ for a complex conic;
- $\ominus[\cdot]$ for two complex straight lines which intersect at a real finite point.

The cases that will be considered in this paper are a subset of the previous cases.

Moreover we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity which remains on the curve defined by this common factor, we will use the symbol $\emptyset$to describe this situation. If some singular points remain we will use the corresponding notation of their types.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non isolated singularity.

In order to describe correctly the singularities at infinity, we must mention also this kind of phenomena and describe what happens to such points at infinity after the removal of the common factor. To show the existence of the common factor we will use the same symbol as before: $\ominus$, and for the type of degeneracy we use the symbols introduced above. We will use the symbol $\emptyset$ to denote the non-existence of infinite singular points after the removal of the degeneracy. There are other possibilities for a polynomial system, but this is the only one of interest in this paper.

4. Some preliminary results

Consider real quadratic systems of the form

$$\begin{aligned} \frac{dx}{dt} &= p_0 + p_1(x,y) + p_2(x,y) \equiv P(x,y), \\ \frac{dy}{dt} &= q_0 + q_1(x,y) + q_2(x,y) \equiv Q(x,y), \end{aligned} \tag{4.1}$$

with homogeneous polynomials p_i and q_i $(i = 0, 1, 2)$ of degree i in x, y, where

$$p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y, \quad p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,$$
$$q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y, \quad q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2.$$

Let $\tilde{a} = (a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})$ be the 12-tuple of the coefficients of systems (4.1) and denote $\mathbb{R}[\tilde{a}, x, y] = \mathbb{R}[a_{00}, \dots, b_{02}, x, y]$.

4.1. Number and types of weak singularities of quadratic systems.

A complete characterization of the finite weak singularities of quadratic systems via invariant theory was done in [31], where the next result is proved.

Proposition 4.1. *Consider a non-degenerate quadratic system* (4.1).

- (a) *If $\mathcal{T}_4 \neq 0$ then this system has no weak singularity.*
- (b) *If $\mathcal{T}_4 = 0$ and $\mathcal{T}_3 \neq 0$ then the system has exactly one weak singularity. Moreover this singularity is either a weak focus (respectively a weak saddle) of the indicated order below, or a center (respectively an integrable saddle) if and only if $\mathcal{T}_3\mathcal{F} < 0$ (respectively $\mathcal{T}_3\mathcal{F} > 0$) and the following corresponding condition holds*
  - (b_1) $f^{(1)}$ *(respectively $s^{(1)}$)* $\Leftrightarrow \mathcal{F}_1 \neq 0$;
  - (b_2) $f^{(2)}$ *(respectively $s^{(2)}$)* $\Leftrightarrow \mathcal{F}_1 = 0$, $\mathcal{F}_2 \neq 0$;
  - (b_3) $f^{(3)}$ *(respectively $s^{(3)}$)* $\Leftrightarrow \mathcal{F}_1 = \mathcal{F}_2 = 0$, $\mathcal{F}_3\mathcal{F}_4 \neq 0$;
  - (b_4) c *(respectively $\$$)* $\Leftrightarrow \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0$.
- (c) *If $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\mathcal{T}_2 \neq 0$, then the system could possess two and only two weak singularities and none of them is of order* 2 *or* 3. *Moreover this system possesses two weak singularities, which are of the types indicated below, if and only if $\mathcal{F} = 0$ and one of the following conditions holds*
  - (c_1) $s^{(1)}, s^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0$, $\mathcal{T}_2 < 0$, $\mathcal{B} \leq 0$, $\mathcal{H} > 0$;
  - (c_2) $s^{(1)}, f^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0$, $\mathcal{T}_2 > 0$, $\mathcal{B} < 0$;
  - (c_3) $f^{(1)}, f^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0$, $\mathcal{T}_2 < 0$, $\mathcal{B} < 0$, $\mathcal{H} < 0$;
  - (c_4) $\$, \$ \Leftrightarrow \mathcal{F}_1 = 0$, $\mathcal{T}_2 < 0$, $\mathcal{B} < 0$, $\mathcal{H} > 0$;
  - (c_5) $\$, c \Leftrightarrow \mathcal{F}_1 = 0$, $\mathcal{T}_2 > 0$, $\mathcal{B} < 0$;

(c_6) $c, c \Leftrightarrow \mathcal{F}_1 = 0,\ \mathcal{T}_2 < 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0.$

(d) *If $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$ and $\mathcal{T}_1 \neq 0$, then the system could possess one and only one weak singularity (which is of order 1). Moreover this system has one weak singularity of the type indicated below if and only if $\mathcal{F} = 0$ and one of the following conditions holds*

(d_1) $s^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0;$

(d_2) $f^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0.$

(e) *If $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0$ and $\sigma(a, x, y) \neq 0$, then the system could possess one and only one weak singularity. Moreover this system has one weak singularity, which is of the type indicated below, if and only if one of the following conditions holds*

(e_1) $s^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0;$

(e_2) $f^{(1)} \Leftrightarrow \mathcal{F}_1 \neq 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0;$

(e_3)

$$\$ \Leftrightarrow \begin{cases} [\alpha]\ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} > 0,\ \text{or} \\ [\beta]\ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 > 0,\ \text{or} \\ [\gamma]\ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mu_0 = 0,\ K(\mu_2^2 + \mu_3^2) \neq 0,\ \text{or} \\ [\delta]\ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = K = 0,\ \mu_2 \mathcal{G} \neq 0,\ \text{or} \\ [\varepsilon]\ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mathcal{B}_4 = K = \mu_2 = 0,\ \mu_3 \neq 0; \end{cases}$$

(e_4)

$$c \Leftrightarrow \begin{cases} [\alpha]\ \mathcal{F}_1 = 0,\ \mathcal{F} = 0,\ \mathcal{B} < 0,\ \mathcal{H} < 0,\ \text{or} \\ [\beta]\ \mathcal{F}_1 = 0,\ \mathcal{H} = \mathcal{B}_1 = 0,\ \mathcal{B}_2 < 0. \end{cases}$$

(f) *If $\sigma(\tilde{a}, x, y) = 0$, then the system is Hamiltonian and it possesses i (with $1 \leq i \leq 4$) weak singular points of the types indicated below if and only if one of the following conditions holds*

(f_1) $\$, \$, \$, c \Leftrightarrow \mu_0 < 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;$

(f_2) $\$, \$, c, c \Leftrightarrow \mu_0 > 0,\ \mathbf{D} < 0,\ \mathbf{R} > 0,\ \mathbf{S} > 0;$

(f_3) $\$, \$, c \Leftrightarrow \mu_0 = 0,\ \mathbf{D} < 0,\ \mathbf{R} \neq 0;$

(f_4)

$$\$, \$ \Leftrightarrow \begin{cases} [\alpha]\ \mu_0 < 0,\ \mathbf{D} > 0,\ \text{or} \\ [\beta]\ \mu_0 < 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0,\ \text{or} \\ [\gamma]\ \mu_0 = \mathbf{R} = 0,\ \mathbf{P} \neq 0,\ \mathbf{U} > 0,\ K \neq 0; \end{cases}$$

(f_5)

$$\$, c \Leftrightarrow \begin{cases} [\alpha]\ \mu_0 > 0,\ \mathbf{D} > 0,\ \text{or} \\ [\beta]\ \mu_0 > 0,\ \mathbf{D} = 0,\ \mathbf{T} < 0,\ \text{or} \\ [\gamma]\ \mu_0 = \mathbf{R} = 0,\ \mathbf{P} \neq 0,\ \mathbf{U} > 0,\ K = 0; \end{cases}$$

(f_6)

$$\$ \Leftrightarrow \begin{cases} [\alpha]\ \mu_0 < 0,\ \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0,\ \text{or} \\ [\beta]\ \mu_0 = 0,\ \mathbf{D} > 0,\ \mathbf{R} \neq 0,\ \text{or} \\ [\gamma]\ \mu_0 = 0,\ \mathbf{D} = 0,\ \mathbf{PR} \neq 0,\ \text{or} \\ [\delta]\ \mu_0 = \mathbf{R} = \mathbf{P} = 0,\ \mathbf{U} \neq 0; \end{cases}$$

(f_7) $c \Leftrightarrow \mu_0 > 0,\ \mathbf{D} = \mathbf{T} = \mathbf{P} = 0,\ \mathbf{R} \neq 0.$

The invariant polynomials used in the above theorem are constructed as follows

$$
\begin{aligned}
\mathcal{F}_1(\tilde{a}) &= A_2,\\
\mathcal{F}_2(\tilde{a}) &= -2A_1^2A_3 + 2A_5(5A_8+3A_9) + A_3(A_8-3A_{10}+3A_{11}+A_{12})\\
&\quad - A_4(10A_8-3A_9+5A_{10}+5A_{11}+5A_{12}),\\
\mathcal{F}_3(\tilde{a}) &= -10A_1^2A_3 + 2A_5(A_8-A_9) - A_4(2A_8+A_9+A_{10}+A_{11}+A_{12})\\
&\quad + A_3(5A_8+A_{10}-A_{11}+5A_{12}),\\
\mathcal{F}_4(\tilde{a}) &= 20A_1^2A_2 - A_2(7A_8-4A_9+A_{10}+A_{11}+7A_{12}) + A_1(6A_{14}-22A_{15})\\
&\quad - 4A_{33}+4A_{34},\\
\mathcal{F}(\tilde{a}) &= A_7, \quad \mathcal{B}(\tilde{a}) = -(3A_8+2A_9+A_{10}+A_{11}+A_{12}),\\
\mathcal{H}(\tilde{a}) &= -(A_4+2A_5), \quad \mathcal{G}(\tilde{a},x,y) = \widetilde{M}+32\widetilde{H},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{B}_1(\tilde{a}) &= \Big\{\big(T_7,D_2\big)^{(1)}\big[12D_1T_3+2D_1^3+9D_1T_4+36\big(T_1,D_2\big)^{(1)}\big] - 2D_1\big(T_6,D_2\big)^{(1)}\\
&\quad \times\big[D_1^2+12T_3\big] + D_1^2\big[D_1\big(T_8,C_1\big)^{(2)} + 6\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{B}_2(\tilde{a}) &= \Big\{\big(T_7,D_2\big)^{(1)}\big[8T_3\big(T_6,D_2\big)^{(1)} - D_1^2\big(T_8,C_1\big)^{(2)} - 4D_1\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\\
&\quad + \Big[\big(T_7,D_2\big)^{(1)}\Big]^2(8T_3-3T_4+2D_1^2)\Big\}/384,\\
\mathcal{B}_3(\tilde{a},x,y) &= -D_1^2(4D_2^2+T_8+4T_9) + 3D_1D_2(T_6+4T_7) - 24T_3(D_2^2-T_9),\\
\mathcal{B}_4(\tilde{a},x,y) &= D_1(T_5+2D_2C_1) - 3C_2(D_1^2+2T_3).
\end{aligned}
$$

Here by $(f,g)^{(k)}$ is denoted the differential operator called *transvectant of index k* (see [21]) of two polynomials $f,\ g\in\mathbb{R}[\tilde{a},x,y]$

$$
(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h\partial y^{k-h}},
$$

and $A_i(\tilde{a})$ are the elements of the minimal polynomial basis of affine invariants up to degree 12 (containing 42 elements) constructed in [8]. We have applied here only the following elements (keeping the notation from [8])

$$
\begin{gathered}
A_1 = \tilde{A}, \quad A_2 = (C_2,D)^{(3)}/12,\\
A_3 = \big(\big((C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, \quad A_4 = (\widetilde{H},\widetilde{H})^{(2)},\\
A_5 = (\widetilde{H},\widetilde{K})^{(2)}/2, \quad A_6 = (\widetilde{E},\widetilde{H})^{(2)}/2,\\
A_7 = \big((C_2,\widetilde{E})^{(2)},D_2\big)^{(1)}/8, \quad A_8 = \big((\widetilde{D},\widetilde{H})^{(2)},D_2\big)^{(1)}/8,\\
A_9 = \big(\big((\widetilde{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48, \quad A_{10} = \big((\widetilde{D},\widetilde{K})^{(2)},D_2\big)^{(1)}/8,\\
A_{11} = (\widetilde{F},\widetilde{K})^{(2)}/4, \quad A_{12} = (\widetilde{F},\widetilde{H})^{(2)}/4,\\
A_{14} = (\widetilde{B},C_2)^{(3)}/36, \quad A_{15} = (\widetilde{E},\widetilde{F})^{(2)}/4,\\
A_{33} = \big(\big(\big((\widetilde{D},D_2)^{(1)},\widetilde{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128,\\
A_{34} = \big(\big(\big((\widetilde{D},\widetilde{D})^{(2)},D_2\big)^{(1)},\widetilde{K}\big)^{(1)},D_2\big)^{(1)}/64,
\end{gathered}
$$

where

$$
\begin{aligned}
\tilde{A} &= \big(C_1, T_8 - 2T_9 + D_2^2\big)^{(2)}/144,\\
\widetilde{D} &= \Big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6 - (C_1, T_5)^{(1)} + 6D_1(C_1D_2 - T_5)\\
&\quad - 9D_1^2C_2\Big]/36,\\
\widetilde{E} &= \Big[D_1(2T_9 - T_8) - 3\,(C_1, T_9)^{(1)} - D_2(3T_7 + D_1D_2)\Big]/72,\\
\widetilde{F} &= \Big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0\,(D_2, T_9)^{(1)} - 9D_2^2T_4 + 288D_1\widetilde{E}\\
&\quad - 24(C_2, \widetilde{D})^{(2)} + 120(D_2, \widetilde{D})^{(1)} - 36C_1\,(D_2, T_7)^{(1)} + 8D_1\,(D_2, T_5)^{(1)}\Big]/144,\\
\widetilde{B} &= \Big\{16D_1\,(D_2, T_8)^{(1)}\,(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0\,(D_2, T_9)^{(1)}\,\big(3D_1D_2\\
&\quad - 5T_6 + 9T_7\big) + 2\,(D_2, T_9)^{(1)}\,\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32\,(C_0, T_5)^{(1)}\big)\\
&\quad + 6\,(D_2, T_7)^{(1)}\,[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
&\quad + C_2(9T_4 + 96T_3)] + 6\,(D_2, T_6)^{(1)}\,[32C_0T_9 - C_1(12T_7 + 52D_1D_2) - 32C_2D_1^2]\\
&\quad + 48D_2\,(D_2, T_1)^{(1)}\,\big(2D_2^2 - T_8\big) - 32D_1T_8\,(D_2, T_2)^{(1)} + 9D_2^2T_4\,(T_6 - 2T_7)\\
&\quad - 16D_1\,(C_2, T_8)^{(1)}\,\big(D_1^2 + 4T_3\big) + 12D_1\,(C_1, T_8)^{(2)}\,(C_1D_2 - 2C_2D_1)\\
&\quad + 6D_1D_2T_4\,\big(T_8 - 7D_2^2 - 42T_9\big) + 12D_1\,(C_1, T_8)^{(1)}\,(T_7 + 2D_1D_2)\\
&\quad + 96D_2^2\Big[D_1\,(C_1, T_6)^{(1)} + D_2\,(C_0, T_6)^{(1)}\Big] - 16D_1D_2T_3\,\big(2D_2^2 + 3T_8\big)\\
&\quad - 4D_1^3D_2\,\big(D_2^2 + 3T_8 + 6T_9\big) + 6D_1^2D_2^2\,(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^83^3),\\
\widetilde{K} &= (T_8 + 4T_9 + 4D_2^2)/72 \equiv \Big(p_2(x,y), q_2(x,y)\Big)^{(1)}/4,\\
\widetilde{H} &= (-T_8 + 8T_9 + 2D_2^2)/72,\\
\widetilde{M} &= (C_2, C_2)^{(2)} = 2\,\mathrm{Hess}\,\big(C_2(x,y)\big)
\end{aligned}
$$

and

$$
\begin{gathered}
T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)},\\
T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \quad T_6 = (C_1, C_2)^{(2)},\\
T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}
\end{gathered}
\tag{4.2}
$$

are the GL-comitants constructed by using the following five polynomials, basic ingredients in constructing invariant polynomials for systems (4.1)

$$
\begin{gathered}
C_i(\tilde{a}, x, y) = yp_i(x,y) - xq_i(x,y), \ (i = 0, 1, 2),\\
D_i(\tilde{a}, x, y) = \frac{\partial p_i}{\partial x} + \frac{\partial q_i}{\partial y}, \ (i = 1, 2).
\end{gathered}
\tag{4.3}
$$

The affine invariants $\mathcal{T}_j$ $(j = 1, 2, 3, 4)$ which are responsible for the number of vanishing traces of the finite singularities (see [31]) are constructed as follows.

We consider the polynomial $\sigma(\tilde{a}, x, y)$ which is an affine comitant of systems (4.1)

$$
\sigma(\tilde{a}, x, y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \sigma_0(\tilde{a}) + \sigma_1(\tilde{a}, x, y) \ (\equiv D_1(\tilde{a}) + D_2(\tilde{a}, x, y)),
$$

and the differential operator $\mathcal{L} = x \cdot \mathbf{L}_2 - y \cdot \mathbf{L}_1$ (see [5]) acting on $\mathbb{R}[\tilde{a}, x, y]$, where

$$\begin{aligned}\mathbf{L}_1 &= 2a_{00}\frac{\partial}{\partial a_{10}} + a_{10}\frac{\partial}{\partial a_{20}} + \frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{10}} + b_{10}\frac{\partial}{\partial b_{20}} + \frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}};\\ \mathbf{L}_2 &= 2a_{00}\frac{\partial}{\partial a_{01}} + a_{01}\frac{\partial}{\partial a_{02}} + \frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}} + 2b_{00}\frac{\partial}{\partial b_{01}} + b_{01}\frac{\partial}{\partial b_{02}} + \frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}}.\end{aligned}$$

Applying the differential operators $\mathcal{L}$ and $(*,*)^{(k)}$ (i.e. transvectant of index k) in [31] is defined the following polynomial function (named *trace function*)

$$\mathfrak{T}(w) = \sum_{i=0}^{4} \frac{1}{(i!)^2}\Big(\sigma_1^i, \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0)\Big)^{(i)} w^{4-i} = \sum_{i=0}^{4} \mathcal{G}_i\, w^{4-i}, \tag{4.4}$$

where the coefficients $\mathcal{G}_i(\tilde{a}) = \frac{1}{(i!)^2}(\sigma_1^i, \mu_i)^{(i)},\ i = 0,1,2,3,4\ \big(\mathcal{G}_0(\tilde{a}) \equiv \mu_0(\tilde{a})\big)$ are GL-invariants.

Finally using the function $\mathfrak{T}(w)$ the following four needed affine invariants $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$, $\mathcal{T}_1$ are constructed [31]

$$\mathcal{T}_{4-i}(\tilde{a}) = \frac{1}{i!}\,\frac{d^i\mathfrak{T}}{dw^i}\Big|_{w=\sigma_0}, \quad i = 0,1,2,3 \quad \big(\mathcal{T}_4 \equiv \mathfrak{T}(\sigma_0)\big),$$

which are basic schematic affine invariants for the characterization of weak singularities via invariant polynomials (see Proposition 4.1).

The invariant polynomials $\mathbf{D}, \mathbf{P}, \mathbf{R}, \mathbf{S}, \mathbf{T}, \mathbf{U}$ and $\mathbf{V}$ are defined in the next section (see (5.2)).

In what follows we also need the following invariant polynomials:

$$\begin{gathered}B_3(\tilde{a}, x, y) = (C_2, \widetilde{D})^{(1)} = \text{Jacob}\left(C_2, \widetilde{D}\right),\\ B_2(\tilde{a}, x, y) = (B_3, B_3)^{(2)} - 6B_3(C_2, \widetilde{D})^{(3)},\\ B_1(\tilde{a}) = \text{Res}_x\left(C_2, \widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2, B_3)^{(4)},\\ B_4(\tilde{a}, x, y) = -\big((\widetilde{D}, \widetilde{H})^{(2)}, \widetilde{H}\big)^{(1)} \times (\widetilde{D}, \widetilde{H})^{(2)},\\ B_5(\tilde{a}, x, y) = D_2\big[((C_2, D_2)^{(1)}, D_2)^{(1)} - 3(C_2, \widetilde{K})^{(2)}\big],\\ B_6(\tilde{a}, x, y) = C_1^2 - 4C_0C_2.\end{gathered} \tag{4.5}$$

and

$$\begin{gathered}\eta(\tilde{a}) = \text{Discrim}\big(C_2(x,y)\big) = (\widetilde{M}, \widetilde{M})^{(2)}/384,\\ \widetilde{N}(\tilde{a}, x, y) = 4\widetilde{K} - 4\widetilde{H};\quad \widetilde{R}(\tilde{a}, x, y) = \tilde{L} + 32\widetilde{K},\\ \theta(\tilde{a}) = \text{Discrim}\big(\widetilde{N}(\tilde{a}, x, y)\big) = -(\widetilde{N}, \widetilde{N})^{(2)}/2,\\ \tilde{L}(\tilde{a}, x, y) = 16\widetilde{K} - 32\widetilde{H} - \widetilde{M};\quad \theta_1(\tilde{a}) = 16\eta - 2\theta - 16\mu_0.\end{gathered} \tag{4.6}$$

4.2. **Number and multiplicities of the finite singularities of quadratic systems.** The conditions for the number and multiplicities of the finite singularities of quadratic systems were first constructed in [5].

We shall use here the notion of *zero–cycle* in order to describe the number and multiplicity of singular points of a quadratic system. This notion as well as the notion of *divisor*, were used for classification purposes of planar quadratic differential systems by Pal and Schlomiuk [27], Llibre and Schlomiuk [19], Schlomiuk and Vulpe [28] and by Artes and Llibre and Schlomiuk [2].

Definition 4.2. We consider formal expressions $\mathcal{D} = \sum n(w)w$ where $n(w)$ is an integer and only a finite number of $n(w)$ are nonzero. Such an expression is called a *zero–cycle of* $\mathbf{P}_2(\mathbb{C})$ if all w appearing in $\mathcal{D}$ are points of $\mathbf{P}_2(\mathbb{C})$. We call *degree* of the zero–cycle $\mathcal{D}$ the integer $\deg(\mathcal{D}) = \sum n(w)$. We call *support* of $\mathcal{D}$ the set $\operatorname{supp}(\mathcal{D})$ of w's appearing in $\mathcal{D}$ such that $n(w) \neq 0$.

We note that $\mathbf{P}_2(\mathbb{C})$ denotes the complex projective space of dimension 2. For a system (S) belonging to the family (4.1) we denote $\nu(P,Q) = \{w \in \mathbb{C}_2 : P(w) = Q(w) = 0\}$ and we define the zero-cycle $\mathcal{D}_S(P,Q) = \sum_{w\in\nu(P,Q)} I_w(P,Q)w$, where $I_w(P,Q)$ is the intersection number or multiplicity of intersection of P and Q at w. It is clear that for a non–degenerate quadratic system $\deg(\mathcal{D}_S) \leq 4$ as well as $\operatorname{supp}(\mathcal{D}_S) \leq 4$. For a degenerate system the zero–cycle $\mathcal{D}_S(P,Q)$ is undefined.

Using the affine invariant polynomials

$$\mu_0(\tilde{a}), \quad \mathbf{D}(\tilde{a}), \quad \mathbf{R}(\tilde{a},x,y), \quad \mathbf{S}(\tilde{a},x,y), \quad \mathbf{T}(\tilde{a},x,y), \quad \mathbf{U}(\tilde{a},x,y), \quad \mathbf{V}(\tilde{a},x,y) \tag{4.7}$$

(the construction of these polynomials will be discussed further), in [5] the next proposition was proved.

Proposition 4.3 ([5])**.** *The form of the divisor $\mathcal{D}_S(P,Q)$ for non-degenerate quadratic systems* (4.1) *is determined by the corresponding conditions indicated in Table 3, where we write $p+q+r^c+s^c$ if two of the finite points, i.e. r^c, s^c, are complex but not real.*

TABLE 1.

No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria	No.	Zero–cycle $\mathcal{D}_S(P,Q)$	Invariant criteria
1	$p+q+r+s$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} > 0, \mathbf{S} > 0$	10	$p+q+r$	$\mu_0 = 0, \mathbf{D} < 0, \mathbf{R} \neq 0$
2	$p+q+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} > 0$	11	$p+q^c+r^c$	$\mu_0 = 0, \mathbf{D} > 0, \mathbf{R} \neq 0$
3	$p^c+q^c+r^c+s^c$	$\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{R} \leq 0$ / $\mu_0 \neq 0, \mathbf{D} < 0, \mathbf{S} \leq 0$	12	$2p+q$	$\mu_0 = \mathbf{D} = 0, \mathbf{PR} \neq 0$
4	$2p+q+r$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} < 0$	13	$3p$	$\mu_0 = \mathbf{D} = \mathbf{P} = 0, \mathbf{R} \neq 0$
5	$2p+q^c+r^c$	$\mu_0 \neq 0, \mathbf{D} = 0, \mathbf{T} > 0$	14	$p+q$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} > 0$
6	$2p+2q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} > 0$	15	p^c+q^c	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} < 0$
7	$2p^c+2q^c$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{PR} < 0$	16	$2p$	$\mu_0 = \mathbf{R} = 0, \mathbf{P} \neq 0, \mathbf{U} = 0$
8	$3p+q$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = 0, \mathbf{R} \neq 0$	17	p	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} \neq 0$
9	$4p$	$\mu_0 \neq 0, \mathbf{D} = \mathbf{T} = 0, \mathbf{P} = \mathbf{R} = 0$	18	0	$\mu_0 = \mathbf{R} = \mathbf{P} = 0, \mathbf{U} = 0, \mathbf{V} \neq 0$

5. The global diagram for the finite singularities of quadratic systems. Some needed invariants

We note that the polynomials (4.7) were constructed in [5] (see also [3]) using the basic ingredients (4.3) in constructing invariant polynomials for systems (4.1) and applying the differential operator $(*,*)^{(k)}$ (i.e. transvectant of index k).

Here we shall use the new expressions for the polynomials (4.7) (constructed in [31]), which are equivalent to the old ones but make more transparent their geometry and allow us to observe the dynamic of the finite singularities. More exactly we shall use the polynomials $\mu_0(\tilde{a})$ and $\mu_i(\tilde{a},x,y)$ constructed in [5] as follows

$$\begin{aligned} \mu_0(\tilde{a}) &= \operatorname{Res}_x\big(p_2(x,y),q_2(x,y)\big)/y^4,\\ \mu_i(\tilde{a},x,y) &= \frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\quad i=1,\dots,4, \end{aligned} \tag{5.1}$$

where $\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0))$. Their geometrical meaning is revealed in the following two lemmas.

Lemma 5.1 ([5])**.** *The total multiplicity of all finite singularities of a quadratic system* (4.1) *equals k if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_k(\tilde{a},x,y)\neq 0$. Moreover a system* (4.1) *is degenerate if and only if $\mu_i(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ for every $i=0,1,2,3,4$.*

Lemma 5.2 ([6])**.** *The point $M_0(0,0)$ is a singular point of multiplicity k ($1\le k\le 4$) for a quadratic system* (4.1) *if and only if for every $i\in\{0,1,\dots,k-1\}$ we have $\mu_{4-i}(\tilde{a},x,y)=0$ in $\mathbb{R}[x,y]$ and $\mu_{4-k}(\tilde{a},x,y)\neq 0$.*

Using the invariant polynomials μ_i ($i=0,1,\dots,4$) in [31] the polynomials (4.7) are constructed as follows

$$\begin{aligned} \mathbf{D} &= \Big[3\big((\mu_3,\mu_3)^{(2)},\mu_2\big)^{(2)}-\big(6\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\ \mathbf{P} &= 12\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\\ \mathbf{R} &= 3\mu_1^2-8\mu_0\mu_2,\\ \mathbf{S} &= \mathbf{R}^2-16\mu_0^2\mathbf{P},\\ \mathbf{T} &= 18\mu_0^2(3\mu_3^2-8\mu_2\mu_4)+2\mu_0(2\mu_2^3-9\mu_1\mu_2\mu_3+27\mu_1^2\mu_4)-\mathbf{PR},\\ \mathbf{U} &= \mu_3^2-4\mu_2\mu_4,\\ \mathbf{V} &= \mu_4. \end{aligned} \tag{5.2}$$

Considering these expressions we have the next remark.

Remark 5.3. If $\mu_0=0$ then the condition $\mathbf{R}=0$ (respectively $\mathbf{R}=\mathbf{P}=0$; $\mathbf{R}=\mathbf{P}=\mathbf{U}=0$; $\mathbf{R}=\mathbf{P}=\mathbf{U}=\mathbf{V}=0$) is equivalent to $\mu_1=0$ (respectively $\mu_1=\mu_2=0$; $\mu_1=\mu_2=\mu_3=0$; $\mu_1=\mu_2=\mu_3=\mu_4=0$).

On the other hand, considering Lemma 5.1 we deduce that the invariant polynomials μ_i ($i=0,1,\dots,4$) are responsible for the number of finite singularities which have coalesced with infinite ones. So taking into account the remark above and Proposition 4.3 we could present a diagram, which is equivalent to Table 1. So we get the next result.

Theorem 5.4. *The number and multiplicities of the finite singular points (described by the divisor $\mathcal{D}_S(P,Q)$) for non-degenerate quadratic systems* (4.1) *is given by the diagram presented in Figure 1.*

We are interested in a global characterization of the singularities (finite and infinite) of the family of quadratic systems. More precisely we would like to extend the diagram of Figure 1 adding the infinite singularities (their number and multiplicities) and then including the types of all these singularities. Moreover we wish to distinguish the weak singularities (if it is the case) as well as their order. This is one of the motivations why we consider again the class of quadratic systems with

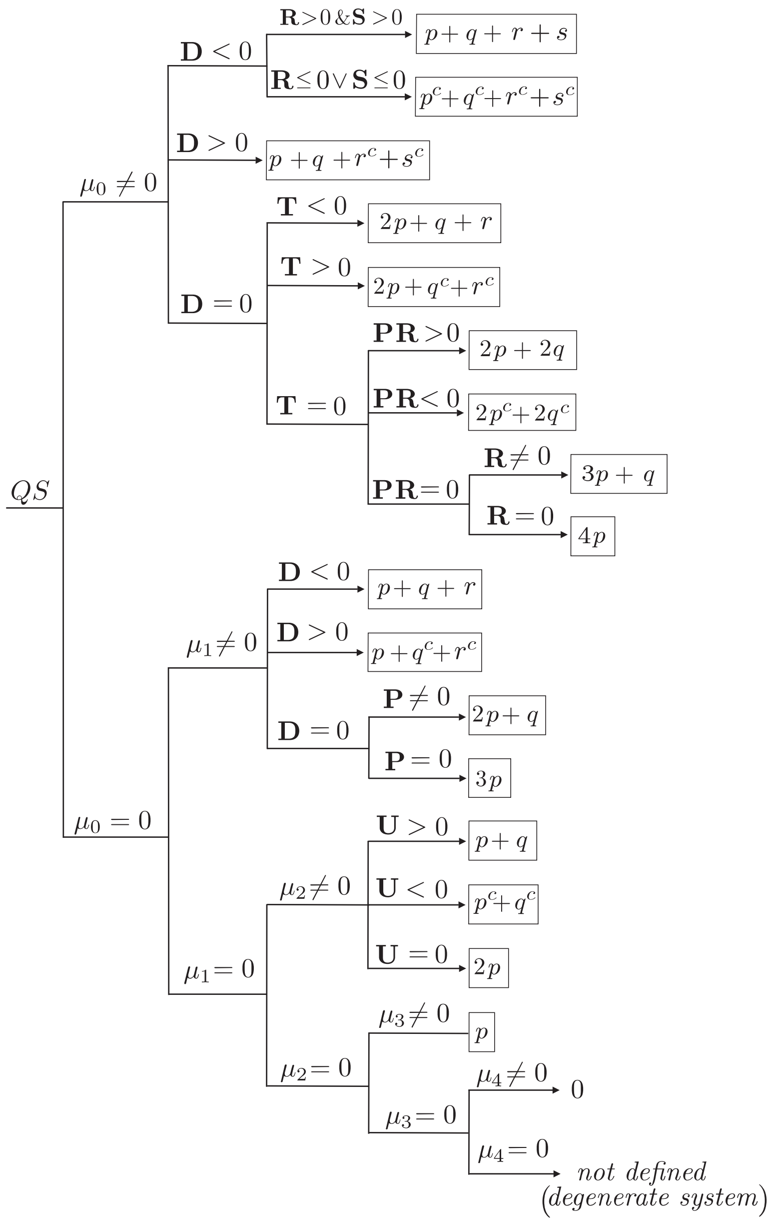

Figure 1. Diagram for Finite Singularities of Quadratic systems

centers as well as the class of Hamiltonian systems, the topological classifications of which could be found in articles [15, 13, 14] and [16], respectively.

On the other hand we would like to reveal the main affine invariant polynomials associated to the singularities of quadratic systems, having a transparent geometrical meaning. And it is clear that all the conditions we need have to be based on the invariant polynomials contained in the diagram of Figure 1.

Thus in this article new geometrical more transparent affine invariant conditions for distinguishing topological phase portraits of the two mentioned families of quadratic systems are simultaneously constructed. For this purpose we need the following GL-invariant polynomials constructed in tensorial form in [29] (we keep the respective notations)

$$\begin{gathered}
I_1 = a^{\alpha}_{\alpha}, \quad I_2 = a^{\alpha}_{\beta} a^{\beta}_{\alpha}, \quad I_3 = a^{\alpha}_{p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\gamma} \varepsilon^{pq}, \quad I_4 = a^{\alpha}_{p} a^{\beta}_{\beta q} a^{\gamma}_{\alpha\gamma} \varepsilon^{pq}, \\
I_5 = a^{\alpha}_{p} a^{\beta}_{\gamma q} a^{\gamma}_{\alpha\beta} \varepsilon^{pq}, \quad I_6 = a^{\alpha}_{p} a^{\beta}_{\gamma} a^{\gamma}_{\alpha q} a^{\delta}_{\beta\delta} \varepsilon^{pq}, \quad I_7 = a^{\alpha}_{pr} a^{\beta}_{\alpha q} a^{\gamma}_{\beta s} a^{\delta}_{\gamma\delta} \varepsilon^{pq} \varepsilon^{rs}, \\
I_8 = a^{\alpha}_{pr} a^{\beta}_{\alpha q} a^{\gamma}_{\delta s} a^{\delta}_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs}, \quad I_9 = a^{\alpha}_{pr} a^{\beta}_{\beta q} a^{\gamma}_{\gamma s} a^{\delta}_{\beta\gamma} \varepsilon^{pq} \varepsilon^{rs}, \\
I_{10} = a^{\alpha}_{p} a^{\beta}_{\delta} a^{\gamma}_{\nu} a^{\delta}_{\alpha q} a^{\nu}_{\beta\gamma} \varepsilon^{pq}, \quad I_{13} = a^{\alpha}_{p} a^{\beta}_{qr} a^{\gamma}_{\gamma s} a^{\delta}_{\alpha\beta} a^{\nu}_{\delta\nu} \varepsilon^{pq} \varepsilon^{rs}, \qquad (5.3)\\
I_{16} = a^{\alpha}_{p} a^{\beta}_{r} a^{\gamma}_{\delta} a^{\delta}_{\alpha q} a^{\nu}_{\beta s} a^{\tau}_{\gamma\mu} a^{\mu}_{\nu\tau} \varepsilon^{pq} \varepsilon^{rs}, \quad I_{18} = a^{\alpha} a^{q} a^{p}_{\alpha} \varepsilon_{pq}, \\
I_{19} = a^{\alpha} a^{\beta}_{\gamma} a^{\gamma}_{\alpha\beta}, \quad I_{21} = a^{\alpha} a^{\beta} a^{q} a^{p}_{\alpha\beta} \varepsilon_{pq}, \quad I_{23} = a^{\alpha} a^{\beta} a^{\gamma}_{\alpha\delta} a^{\delta}_{\beta\gamma}, \\
I_{24} = a^{\alpha} a^{\beta}_{\delta} a^{\gamma}_{\alpha} a^{\delta}_{\beta\gamma}, \quad I_{28} = a^{\alpha} a^{\beta} a^{\gamma}_{\delta} a^{\delta}_{\gamma\mu} a^{\mu}_{\alpha\beta}, \quad I_{30} = a^{\alpha} a^{\beta}_{p} a^{\gamma}_{\beta q} a^{\delta}_{\gamma\mu} a^{\mu}_{\alpha\delta} \varepsilon^{pq},
\end{gathered}$$

$$\begin{gathered}
I_{33} = a^{\alpha} a^{\beta} a^{\gamma} a^{\delta}_{\alpha\beta} a^{\mu}_{\gamma\nu} a^{\nu}_{\delta\mu}, \quad I_{35} = a^{\alpha} a^{\beta}_{p} a^{\gamma}_{\alpha} a^{\delta}_{\beta q} a^{\mu}_{\gamma\nu} a^{\nu}_{\delta\mu} \varepsilon^{pq}, \\
K_1 = a^{\alpha}_{\alpha\beta} x^{\beta}, \quad K_2 = a^{p}_{\alpha} x^{\alpha} x^{q} \varepsilon_{pq}, \quad K_3 = a^{\alpha}_{\beta} a^{\beta}_{\alpha\gamma} x^{\gamma}, \quad K_5 = a^{p}_{\alpha\beta} x^{\alpha} x^{\beta} x^{q} \varepsilon_{pq}, \\
K_7 = a^{\alpha}_{\beta\gamma} a^{\beta}_{\alpha\delta} x^{\gamma} x^{\delta} \quad K_{11} = a^{p}_{\alpha} a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} x^{q} \varepsilon_{pq}, \quad K_{12} = a^{\alpha}_{\beta} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\delta\mu} x^{\delta} x^{\mu}, \qquad (5.4)\\
K_{14} = a^{\alpha}_{p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\delta} a^{\delta}_{\gamma\mu} x^{\mu} \varepsilon^{pq}, \quad K_{21} = a^{p} x^{q} \varepsilon_{pq}, \quad K_{22} = a^{\alpha} a^{p}_{\alpha} x^{q} \varepsilon_{pq}, \\
K_{23} = a^{p} a^{q}_{\alpha\beta} x^{\alpha} x^{\beta} \varepsilon_{pq}, \quad K_{27} = a^{\alpha} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\beta\delta} x^{\delta}, \quad K_{31} = a^{\alpha} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\beta\delta} a^{\delta}_{\mu\nu} x^{\mu} x^{\nu},
\end{gathered}$$

where $\varepsilon_{11} = \varepsilon_{22} = \varepsilon^{11} = \varepsilon^{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = \varepsilon^{12} = -\varepsilon^{21} = 1$. We note that the expressions for the above invariants are associated to the tensor notation for quadratic systems (4.1) (see [29])

$$\begin{gathered}
\frac{dx^j}{dt} = a^j + a^j_{\alpha} x^{\alpha} + a^j_{\alpha\beta} x^{\alpha} x^{\beta}, \quad (j, \alpha, \beta = 1, 2); \\
a^1 = a_{00}, \quad a^1_1 = a_{10}, \quad a^1_2 = a_{01}, \quad a^1_{11} = a_{20}, \quad a^1_{22} = a_{02}, \\
a^2 = b_{00}, \quad a^2_1 = b_{10}, \quad a^2_2 = b_{01}, \quad a^2_{11} = b_{20}, \quad a^2_{22} = b_{02}, \\
a^1_{12} = a^1_{21} = a_{11}, \quad a^2_{12} = a^2_{21} = b_{11}.
\end{gathered}$$

6. The family of quadratic systems with centers

The proof of Theorem 1.1 is based on the classification of the family of quadratic systems with centers given in [30]. Using the expressions (5.3) of [30] we get the following GL-invariants (we keep the respective notations adding only the "hat")

$$\begin{gathered}
\hat{\alpha} = I_2^2 I_8 - 28 I_2 I_5^2 + 6 I_5 I_{10}, \quad \hat{\beta} = 4 I_4^2 - 3 I_2 I_9 - 4 I_3 I_4, \\
\hat{\gamma} = \frac{3}{I_4^2}(2 I_3 I_4 + I_2 I_9), \quad \hat{\delta} = 27 I_8 - I_9 - 18 I_7, \qquad (6.1)\\
\hat{\xi} = I_2 I_5 (I_2 I_5 + 2 I_{10}) - 4 I_{10}^2 - I_2^3 I_8.
\end{gathered}$$

We note that in [30] the expressions of the invariants $\hat{\xi}$ and $\hat{\delta}$ are used directly, but we set these notations for compactness.

According to [30] (see Lemmas 2-5) the next result follows.

TABLE 2.

Conditions for the existence of a center [statement (b) of Proposition 4.1]: $\mathcal{T}_3 \neq 0,\ \mathcal{T}_4 = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0,\ \mathcal{T}_3\mathcal{F} < 0,\ (b_4)$					
Additional conditions for configurations				*Configuration of singularities*	No.
$\mu_0 \neq 0$	$\mathbf{D} < 0$	$\widetilde{K} < 0$		$c, s, s, s;\ N, N, N$	1
		$\widetilde{K} > 0$		$c, s, n, n;\ S, S, N$	2
	$\mathbf{D} > 0$	$\eta < 0$		$c, f;\ S$	3
		$\eta > 0$		$c, n;\ S, S, N$	4
	$\mathbf{D} = 0$			$c, n, \overline{sn}_{(2)};\ S, \overline{\binom{0}{2}}\ SN$	5
$\mu_0 = 0$				$c, s;\ N, \overline{\binom{1}{1}}\ SN, \overline{\binom{1}{1}}\ SN$	6
Conditions for the existence of a center [statement (c) of Proposition 4.1]: $\mathcal{T}_4 = \mathcal{T}_3 = 0,\ \mathcal{T}_2 \neq 0,\ (c_5) \cup (c_6)$					
$\mu_0 < 0$	$\mathbf{D} < 0$	$\eta < 0$		$c, \$, n, n;\ S$	7
		$\eta > 0$	$\widetilde{K} < 0$	$c, \$, s, s;\ N, N, N$	8
			$\widetilde{K} > 0$	$c, \$, n, n;\ S, S, N$	9
		$\eta = 0$		$c, \$, n, n;\ \overline{\binom{0}{3}}\ S$	10
	$\mathbf{D} > 0$	$\eta < 0$		$c, c;\ S$	11
		$\eta > 0$		$c, c;\ S, S, N$	12
		$\eta = 0$		$c, c;\ \overline{\binom{0}{3}}\ S$	13
$\mu_0 > 0$	$\mathbf{D} < 0$			$c, c, s, s;\ N$	14
	$\mathbf{D} > 0$	$\eta < 0$		$c, \$;\ N$	15
		$\eta > 0$		$c, \$;\ S, N, N$	16
		$\eta = 0$		$c, \$;\ \overline{\binom{0}{3}}\ N$	17
$\mu_0 = 0$	$\eta < 0$	$\widetilde{K} \neq 0$	$\mu_2 < 0$	$c, c;\ \overline{\binom{2}{1}}\ S$	18
			$\mu_2 > 0$	$c, \$;\ \overline{\binom{2}{1}}\ N$	19
		$\widetilde{K} = 0$		$c, \$;\ N$	20
	$\eta > 0$	$\widetilde{K} \neq 0$		$c, \$;\ \overline{\binom{2}{1}}\ S, N, N$	21
		$\widetilde{K} = 0$		$c, \$;\ N, \overline{\binom{1}{1}}\ SN, \overline{\binom{1}{1}}\ SN$	22

Proposition 6.1. *Assume that for a quadratic system with the singular point* $(0,0)$ *the conditions* $I_1 = I_6 = 0$ *and* $I_2 < 0$ *hold. Then this system has a center at* $(0,0)$ *and via a linear transformation could be brought to one of the canonical forms below if and only if the respective additional GL-invariant conditions hold*

$$(S_1^{(c)}) \quad \begin{cases} \dot{x} = -y + gx^2 - xy, \ (g \neq 0), \\ \dot{y} = x + x^2 + 3gxy - 2y^2, \end{cases}$$

$$\iff I_3 I_{13} \neq 0,\ 5I_3 - 2I_4 = 13I_3 - 10I_5 = 0;$$

$$(S_2^{(c)}) \quad \begin{cases} \dot{x} = y + 2nxy, \ (wm \neq 0), \\ \dot{y} = -x + lx^2 + 2mxy - ly^2, \end{cases} \quad \iff I_3 = 0,\ I_{13} \neq 0;$$

Table 2 (*continued*)

Conditions for the existence of a center [statement (e) of Proposition 4.1]: $\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = 0$, $\sigma \neq 0$, (e_4)					
Additional conditions for configurations				*Configuration of singularities*	No.
$\mu_0 < 0$	$\eta < 0$			$c, \widehat{es}_{(3)}\,;\ S$	23
	$\eta > 0$			$c, \widehat{es}_{(3)}\,;\ S, S, N$	24
	$\eta = 0$			$c, \widehat{es}_{(3)}\,;\ \overline{\binom{0}{3}}\ S$	25
$\mu_0 > 0$				$c, \overline{s}_{(3)}\,;\ N$	26
$\mu_0 = 0$	$\mu_1 \neq 0$	$\mathbf{D} < 0$		$c, s, s\,;\ N, \widehat{\binom{1}{2}} P\ E\ P - H$	27
		$\mathbf{D} > 0$	$\widetilde{N} \leq 0$	$c\,;\ S, \widehat{\binom{1}{2}} P\ E\ P - H$	28
			$\widetilde{N} > 0$, $\tilde{L} < 0$	$c\,;\ S, \widehat{\binom{1}{2}} P\ E\ P - P\ H\ P$	29
			$\widetilde{N} > 0$, $\tilde{L} > 0$	$c\,;\ N, \widehat{\binom{1}{2}} H - H\ H\ H$	30
			$\widetilde{N} > 0$, $\tilde{L} = 0$	$c\,;\ [\infty, S]$	31
	$\mu_1 = 0$	$\mu_3 \neq 0$	$\tilde{L} \neq 0$	$c\,;\ N, \binom{3}{2} H - H\ H\ H$	32
			$\tilde{L} = 0$	$c\,;\ \overline{\binom{2}{1}}\ S, \widehat{\binom{1}{2}} P\ E\ P - H$	33
		$\mu_3 = 0$		$c, (\ominus [\vert\vert]; \emptyset);\ (\ominus [\vert\vert]; \emptyset)$	34
Conditions for the existence of a center [statement (f) of Proposition 4.1]: Hamiltonian systems $\Rightarrow \sigma = 0$, (f_1)–(f_3), (f_5), (f_7)					
$\mu_0 < 0$				$c, \$, \$, \$;\ N, N, N$	35
$\mu_0 > 0$	$\mathbf{D} < 0$			$c, c, \$, \$;\ N$	36
	$\mathbf{D} > 0$			$c, \$;\ N$	37
	$\mathbf{D} = 0$	$\mathbf{T} \neq 0$		$c, \$, \widehat{cp}_{(2)}\,;\ N$	38
		$\mathbf{T} = 0$		$c, \widehat{s}_{(3)}\,;\ N$	39
$\mu_0 = 0$	$\mu_1 \neq 0$			$c, \$, \$;\ N, \widehat{\binom{1}{2}} PEP - H$	40
	$\mu_1 = 0$			$c, \$;\ \widehat{\binom{2}{3}}\ N$	41

$$(S_3^{(c)}) \quad \begin{cases} \dot{x} = y + 2(1-e)xy, \\ \dot{y} = -x + dx^2 + ey^2, \end{cases} \quad \Longleftrightarrow I_{13} = 0,\ I_4 \neq 0;$$

$$(S_4^{(c)}) \quad \begin{cases} \dot{x} = y + 2cxy + by^2, \\ \dot{y} = -x - ax^2 - cy^2,\ c \in \{0, 1/2\} \end{cases} \quad \Longleftrightarrow I_4 = 0;$$

where $w = m^2(2n-l) - (n-l)^2(2n+l)$.

Proof of Theorem 1.1. We shall consider each one of the systems $(S_1^{(c)}) - (S_4^{(c)})$ and will compare the GL-invariant conditions [30] with the affine invariant ones given by Tables 2 and 3.

6.1. **The family of systems $(S_1^{(c)})$.** For these systems we calculate the respective GL-invariants

$$I_{13} = 125g(1+g^2)/8, \quad I_3 = 5(1+g^2)/2 \tag{6.2}$$

TABLE 3.

Configuration	Phase portrait	Configuration	Phase portrait	Configuration	Phase portrait
1	Vul_{10}	15	Vul_{2}	30	Vul_{13}
2	Vul_{27}	16	Vul_{19}	31	Vul_{15}
3	Vul_{30}	17	Vul_{2}	32	Vul_{13}
4	Vul_{32}	18	Vul_{20}	33	Vul_{12}
5	Vul_{31}	19	Vul_{2}	34	Vul_{29}
6	Vul_{17}	20	Vul_{2}	35	Vul_{11} if $B_1 \neq 0$
7	Vul_{25}	21	Vul_{19}		Vul_{9} if $B_1 = 0,\ B_3B_4 < 0$
8	Vul_{9} if $B_3B_5 < 0$	22	Vul_{18} if $B_3B_5 < 0$		Vul_{8} if $B_1 = 0,\ B_3B_4 > 0$
	Vul_{8} if $B_3B_5 > 0$		Vul_{16} if $B_3B_5 > 0$		Vul_{10} if $B_1 = B_3 = 0$
	Vul_{10} if $B_3 = 0$		Vul_{17} if $B_3 = 0$	36	Vul_{4} if $B_1 \neq 0$
9	Vul_{28} if $B_3B_5 < 0$	23	Vul_{22} if $\theta_1 < 0$		Vul_{3} if $B_1 = 0$
	Vul_{26} if $B_3B_5 > 0$		Vul_{23} if $\theta_1 \geq 0$	37	Vul_{2}
	Vul_{27} if $B_3 = 0$	24	Vul_{24}	38	Vul_{7}
10	Vul_{25}	25	Vul_{23}	39	Vul_{2}
11	Vul_{20}	26	Vul_{2}	40	Vul_{6} if $B_1 \neq 0$
12	Vul_{21}	27	Vul_{5}		Vul_{5} if $B_1 = 0$
13	Vul_{20}	28	Vul_{12}	41	Vul_{2}
14	Vul_{3}	29	Vul_{14}		

and the affine invariant polynomials

$$\begin{gathered} \mathcal{T}_4 = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_4 = 0, \quad \mathcal{T}_3 = -125g(1+g^2), \quad \mathcal{F}_3 = 625g^2(1+g^2)^2, \\ \mu_0 = -2(1+g^2), \quad \mathbf{D} = 5184g^2(1+g^2), \quad \eta = 4(1+g^2). \end{gathered} \tag{6.3}$$

According to [30] in the case $I_{13} \neq 0$ the phase portrait of systems $(S_1^{(c)})$ is given by Vul_{32} and hence we have the configuration of the singularities $c, n;\ S, S,\ N$.

On the other hand the condition $I_{13} \neq 0$ implies $\mathcal{T}_3 \neq 0$ and as $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0$, the conditions provided by the statement (b_4) of Proposition 4.1 are satisfied. Moreover, as $\mu_0 \neq 0$, $\mathbf{D} > 0$ and $\eta > 0$, we obtain the respective conditions given by Table 2 (row No. 4).

6.2. **The family of systems $(S_2^{(c)})$.** In this case calculations yield:

$$\begin{gathered} I_{13} = m\big[m^2(2n-l) - (n-l)^2(2n+l)\big], \quad I_9 - I_8 = 4ln(l^2+m^2+n^2), \\ \hat{\delta} = -8(l+2n)^2(l^2+m^2+2ln). \end{gathered} \tag{6.4}$$

According to [30] (see Lemma 4) the phase portrait (and this yields the respective configuration of singularities) of a system from the family $(S_2^{(c)})$ is determined by the following GL-invariant conditions, respectively

$$\begin{array}{llllll} I_9 - I_8 > 0 & \Leftrightarrow & Vul_{10} & \Rightarrow & c, s, s, s; N, N, N; \\ I_9 - I_8 = 0 & \Leftrightarrow & Vul_{17} & \Rightarrow & c, s;\ N, \overline{\binom{1}{1}}\ SN, \overline{\binom{1}{1}}\ SN; \\ I_9 - I_8 < 0,\ \hat{\delta} < 0 & \Leftrightarrow & Vul_{27} & \Rightarrow & c, s, n, n; S, S, N; \\ I_9 - I_8 < 0,\ \hat{\delta} > 0 & \Leftrightarrow & Vul_{30} & \Rightarrow & c, f; S; \\ I_9 - I_8 < 0,\ \hat{\delta} = 0 & \Leftrightarrow & Vul_{31} & \Rightarrow & c, n, \overline{sn}_{(2)}; N, \overline{\binom{0}{2}}\ SN. \end{array} \tag{6.5}$$

On the other hand calculations yield

$$\begin{gathered}\mathcal{T}_4 = \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 0, \quad \mathcal{T}_3\mathcal{F} = -8m^2\big[m^2(2n-l) - (n-l)^2(2n+l)\big]^2, \\ \mu_0 = -4l^2n^2, \quad \mathbf{D} = -192(l+2n)^2(l^2+m^2+2ln) = -48\eta, \quad \widetilde{K} = -4ln(x^2+y^2).\end{gathered} \tag{6.6}$$

We observe that the condition $I_{13} \neq 0$ implies $\mathcal{T}_3\mathcal{F} < 0$ and as $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0$ we conclude that the conditions provided by statement (b_4) of Proposition 4.1 are fulfilled. Moreover comparing (6.4) and (6.6) if $\mathbf{D}\mu_0 \neq 0$ we obtain

$$\operatorname{sign}(I_9 - I_8) = -\operatorname{sign}(\widetilde{K}), \quad \operatorname{sign}(\hat{\delta}) = \operatorname{sign}(\mathbf{D}) = -\operatorname{sign}(\eta),$$

and $I_9 - I_8 = 0$ (respectively $\hat{\delta} = 0$) if and only if $\mu_0 = 0$ (respectively $\mathbf{D} = 0$). So taking into consideration that the condition $\widetilde{K} < 0$ implies $\mathbf{D} < 0$, we obviously arrive to the conditions provided by Table 2 (the case of statement (b) of Proposition 4.1).

6.3. **The family of systems** $(S_3^{(c)})$**.** For these systems calculations yield

$$\begin{gathered}\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_1 = \mathcal{F} = \mathcal{F}_1 = 0, \quad \mathcal{T}_2 = 4d(d+2-2e), \\ \mathcal{B} = -2, \quad \mathcal{H} = 4d(1-e), \quad \sigma = 2y.\end{gathered} \tag{6.7}$$

Therefore according to Proposition 4.1 for a system $(S_3^{(c)})$ could be satisfied either the conditions of statement (c) (if $\mathcal{T}_2 \neq 0$) or of statement (e) (if $\mathcal{T}_2 = 0$). We shall consider each one of these possibilities.

6.3.1. *The case* $\mathcal{T}_2 \neq 0$. Following [30, Lemma 3] for systems $(S_3^{(c)})$ we calculate

$$I_4 = -1, \quad I_3 = -(d+e), \quad I_9 = d, \quad \hat{\beta} = 2(d+2-2e), \quad \hat{\gamma} = 6e, \tag{6.8}$$

and therefore the condition $\mathcal{T}_2 \neq 0$ is equivalent to $I_9\hat{\beta} \neq 0$. It was above mentioned that the conditions of statement (c) (Proposition 4.1) are satisfied in this case; i.e., there should be two weak singularities on the phase plane of these systems. So according to [30] (see the proof of Lemma 3) the phase portrait (and this yields the respective configuration of singularities) of a system from the family $(S_3^{(c)})$ with the condition $I_9\hat{\beta} \neq 0$ is determined by the following GL-invariant conditions

$$\begin{gathered}
\hat{\gamma}I_9 > 0,\ \hat{\beta}\hat{\gamma} < 0 \Leftrightarrow Vul_3 \Rightarrow c, c, s, s\,; N; \\
\begin{cases}\hat{\gamma}I_9 > 0, \hat{\beta}\hat{\gamma} > 0, I_9(4-\hat{\gamma}) \leq 0, \\ \text{or } \hat{\gamma} = 0,\ \hat{\beta} < 0\end{cases} \Leftrightarrow Vul_2 \Rightarrow \begin{cases} c, \$\,; N \text{ if } \hat{\gamma}(\hat{\gamma}-4)(\hat{\gamma}-6) \neq 0; \\ c, \$\,; \overline{\binom{2}{1}}N \text{ if } \hat{\gamma} = 0; \\ c, \$\,; \overline{\binom{0}{3}}N \text{ if } \hat{\gamma} = 4; \\ c, \$\,; N \text{ if } \hat{\gamma} = 6; \end{cases} \\
I_9 < 0,\ 0 \leq \hat{\gamma} \leq 4,\ \hat{\beta} > 0 \Leftrightarrow Vul_{20} \Rightarrow \begin{cases} c, c\,; S \text{ if } \hat{\gamma}(\hat{\gamma}-4) \neq 0; \\ c, c\,; \overline{\binom{2}{1}}S \text{ if } \hat{\gamma} = 0; \\ c, c\,; \overline{\binom{0}{3}}S \text{ if } \hat{\gamma} = 4; \end{cases} \\
I_9 < 0,\ \hat{\gamma} > 4,\ \hat{\beta} > 0 \Leftrightarrow Vul_{21} \Rightarrow c, c\,; S, S, N; \\
I_9 > 0,\ 0 \leq \hat{\gamma} < 4 \Leftrightarrow Vul_{19} \Rightarrow \begin{cases} c, \$\,; S, N, N \text{ if } \hat{\gamma} \neq 0; \\ c, \$\,; \overline{\binom{2}{1}}S, N, N \text{ if } \hat{\gamma} = 0; \end{cases}
\end{gathered} \tag{6.9}$$

$$\hat\beta<0,\ 0<\hat\gamma\le 4 \Leftrightarrow Vul_{25} \Rightarrow \begin{cases} c, \$, n, n\,; S \text{ if } \hat\gamma\neq 4;\\ c, \$, n, n\,; \overline{\binom{0}{3}}S \text{ if } \hat\gamma = 4;\end{cases}$$

and

$$\begin{gathered}
(i)\quad \hat\gamma I_9<0,\ \hat\gamma(\hat\gamma-6)>0 \Leftrightarrow Vul_8,\ Vul_9,\ Vul_{10} \Rightarrow c, \$, s, s\,; N, N, N;\\
(ii)\quad I_9<0,\ \hat\gamma=6 \Leftrightarrow Vul_{16},\ Vul_{17},\ Vul_{18} \Rightarrow c, \$;\ N, \overline{\begin{pmatrix}1\\1\end{pmatrix}}SN, \overline{\begin{pmatrix}1\\1\end{pmatrix}}SN;\\
(iii)\quad \hat\beta<0,\ 4<\hat\gamma<6 \Leftrightarrow Vul_{26},\ Vul_{27},\ Vul_{28} \Rightarrow c, \$, n, n\,; S, S, N.
\end{gathered}\tag{6.10}$$

On the other hand for systems $(S_3^{(c)})$ we calculate

$$\begin{gathered}
\mu_0=4de(e-1)^2,\quad \mathbf{D}=192e(d+2-2e)^3,\quad \eta=4d(2-3e)^3,\\
\widetilde{K}=4(e-1)(dx^2-ey^2),\quad \mu_1=4e(e-1)(d+e-1)y,\\
\mu_2\big|_{e=0}=d(d+2)x^2,\quad \mu_2\big|_{e=1}=d(dx^2+y^2).
\end{gathered}\tag{6.11}$$

So considering (6.8) and (6.11) if $\mathbf{D}\mu_0\eta\neq 0$ we obtain

$$\operatorname{sign}(\mu_0)=\operatorname{sign}(\hat\gamma I_9),\quad \operatorname{sign}(\mathbf{D})=\operatorname{sign}(\hat\beta\hat\gamma),\quad \operatorname{sign}(\eta)=\operatorname{sign}(I_9(4-\hat\gamma)),$$

and due to $\mathcal{T}_2\neq 0$ we have that $\mu_0=0$ (respectively $\mathbf{D}=0$; $\eta=0$) if and only if $\hat\gamma(\hat\gamma-6)=0$ (respectively $\hat\gamma=0$; $\hat\gamma=4$). Therefore it is not too hard to determine that in cases (6.9) (when we have the unique phase portrait) the conditions from Table 2 (the case of statement (c) of Proposition 4.1) are equivalent to the respective conditions from (6.9).

We consider now the remaining cases (6.10). According to [30] the phase portraits Vul_8, Vul_9, Vul_{10} (respectively Vul_{16}, Vul_{17}, Vul_{18}; Vul_{26}, Vul_{27}, Vul_{28}) are distinguished via the GL-invariant $\hat\gamma I_3I_4$. More precisely in the mentioned cases the phase portrait corresponds to Vul_8 (respectively Vul_{16}; Vul_{26}) if $\hat\gamma I_3I_4>0$; Vul_9 (respectively Vul_{18}; Vul_{28}) if $\hat\gamma I_3I_4<0$ and it corresponds to Vul_{10} (respectively Vul_{17}; Vul_{27}) if $I_3=0$.

On the other hand for systems $(S_3^{(c)})$ we calculate

$$B_3=-6(2+d-3e)(d+e)x^3y,\quad B_3B_5=288d(2+d-3e)e(d+e)x^4y^2.$$

We claim that in all three cases (6.10), if $B_3\neq 0$ then we have

$$\operatorname{sign}(B_3B_5)=\operatorname{sign}(\hat\gamma I_3I_4)=\operatorname{sign}\big(e(d+e)\big),\tag{6.12}$$

and $B_3=0$ if and only if $I_3=0$ (i.e. $d+e=0$). To prove this claim we shall consider each one of the cases $(i)-(iii)$ from (6.10).

Case (i). Considering (6.8) we have $de<0$, $e(e-1)>0$ and herein it can easily be detected that $\operatorname{sign}(d+2-3e)=-\operatorname{sign}(e)$ and this leads to (6.12).

Case (ii). As $e=1$ we have $B_3B_5=288d(d-1)(1+d)x^4y^2$ and due to $d<0$ this evidently implies (6.12).

Case (iii). In this case considering (6.8) we have $2/3<e<1$ and $d+2-2e<0$. Therefore $d<0$ and $2-3e<0$ that gives $d+2-3e<0$. As $e>0$ we again obtain (6.12).

It remains to note that due to the conditions discussed above in all three cases we can have $B_3=0$ if and only if $d+e=0$ (i.e. $I_3=0$).

Thus our claim is proved and obviously we arrive to the conditions of Table 3 corresponding to the configurations 8, 9 and 22 respectively.

6.3.2. The case $\mathcal{T}_2 = 0$

Then $d(d+2-2e) = 0$ and considering (6.8) for systems $(S_3^{(c)})$ we have $I_9\hat{\beta} = 0$. We recall that by Proposition 4.1 in this case for a system $(S_3^{(c)})$ has to be satisfied the conditions of the statement (e), i.e. besides the center we could not have another weak singularity. So according to [30] (see the proof of Lemma 3) the phase portrait (and hence, the respective configuration of singularities) of a system from the family $(S_3^{(c)})$ with the condition $I_9\hat{\beta} = 0$ is determined by the following GL-invariant conditions

$$\begin{gathered}
I_9 = 0,\ \hat{\gamma}(\hat{\gamma}-6) > 0 \Leftrightarrow \mathit{Vul}_5 \Rightarrow c, s, s\,; N, \widehat{\binom{1}{2}} P\ E\ P - H;\\
I_9 = 0,\ 0 \le \hat{\gamma} \le 3 \Leftrightarrow \mathit{Vul}_{12} \Rightarrow \begin{cases} c\,; S, \widehat{\binom{1}{2}} P\ E\ P - H & \text{if } \hat{\gamma} \neq 0;\\ c\,; \overline{\binom{2}{1}}\ S, \widehat{\binom{1}{2}} P\ E\ P - H & \text{if } \hat{\gamma} = 0; \end{cases}\\
I_9 = 0,\ 3 < \hat{\gamma} < 4 \Leftrightarrow \mathit{Vul}_{14} \Rightarrow c\,; S, \widehat{\binom{1}{2}} P\ E\ P - P\ H\ P;\\
I_9 = 0,\ \hat{\gamma} = 4 \Leftrightarrow \mathit{Vul}_{15} \Rightarrow c\,; [\infty, S];\\
I_9 = 0,\ 4 < \hat{\gamma} \le 6 \Leftrightarrow \mathit{Vul}_{13} \Rightarrow \begin{cases} c\,; N, \widehat{\binom{1}{2}} H - HHH & \text{if } \hat{\gamma} \neq 6;\\ c\,; N, \widehat{\binom{3}{2}} H - HHH & \text{if } \hat{\gamma} = 6; \end{cases}
\end{gathered} \tag{6.13}$$

and

$$\begin{gathered}
\hat{\beta} = 0,\ \hat{\gamma} I_9 > 0, I_9(4-\hat{\gamma}) < 0 \Leftrightarrow \mathit{Vul}_2 \Rightarrow c, \overline{s}_{(3)}\,; N;\\
\hat{\beta} = 0,\ 0 < \hat{\gamma} < 3 \Leftrightarrow \mathit{Vul}_{22} \Rightarrow c, \widehat{es}_{(3)}\,; S;\\
\hat{\beta} = 0,\ 3 \le \hat{\gamma} \le 4 \Leftrightarrow \mathit{Vul}_{23} \Rightarrow \begin{cases} c, \widehat{es}_{(3)}\,; S & \text{if } \hat{\gamma} \neq 4;\\ c, \widehat{es}_{(3)}\,; \overline{\binom{0}{3}} S & \text{if } \hat{\gamma} = 4; \end{cases}\\
\hat{\beta} = 0,\ 4 < \hat{\gamma} < 6 \Leftrightarrow \mathit{Vul}_{24} \Rightarrow c, \widehat{es}_{(3)}\,; S, S, N;\\
\hat{\beta} = 0,\ \hat{\gamma} = 0 \Leftrightarrow \mathit{Vul}_{29} \Rightarrow c, \big(\ominus [|]; \emptyset\big);\ \big(\ominus [|]; \emptyset\big).
\end{gathered} \tag{6.14}$$

We remark that by (6.8) the condition $I_9 = 0$ gives $d = 0$ whereas the condition $\hat{\beta} = 0$ gives $d = 2(e-1)$.

(1) Assume first $I_9 = 0$, i.e. $d = 0$. Then for systems $(S_3^{(c)})$ we obtain

$$\begin{gathered}
\mu_0 = \eta = 0, \quad \mathbf{D} = -1536e(e-1)^3, \quad \widetilde{N} = 4(1-e)(2e-1)y^2,\\
\tilde{L} = 8e(3e-2)y^2, \quad \mu_1 = 4e(e-1)^2 y, \quad \mu_3 = 2(1-e)x^2y + ey^3.
\end{gathered} \tag{6.15}$$

As $\hat{\gamma} = 6e$ (see (6.8)) this implies $\hat{\gamma} - 6 = 6(e-1)$, $\hat{\gamma} - 4 = 2(3e-2)$ and $\hat{\gamma} - 3 = 3(2e-1)$. So if $\hat{\gamma}(\hat{\gamma}-3)(\hat{\gamma}-4)(\hat{\gamma}-6) \neq 0$ then we have

$$\begin{gathered}
\operatorname{sign}(\mathbf{D}) = -\operatorname{sign}\big(\hat{\gamma}(\hat{\gamma}-6)\big), \quad \operatorname{sign}(\widetilde{N})\big|_{\{\mathbf{D}>0\}} = \operatorname{sign}(\hat{\gamma}-3\big),\\
\operatorname{sign}(\tilde{L})\big|_{\{\mathbf{D}>0, \widetilde{N}>0\}} = \operatorname{sign}(\hat{\gamma}-4\big).
\end{gathered} \tag{6.16}$$

Moreover if $\mu_1 \neq 0$ then $\widetilde{N} = 0$ (respectively $\tilde{L} = 0$) if and only if $\hat{\gamma} = 3$ (respectively $\hat{\gamma} = 4$). Therefore to determine the cases (6.13) (when $I_9 = 0$) the conditions of Table 2 (the case of statement (e) of Proposition 4.1) are equivalent to the respective conditions of (6.13).

(2) Suppose now $\hat{\beta} = 0$ and $I_9 \neq 0$. Then $d = 2(e-1) \neq 0$ and for systems $(S_3^{(c)})$ we have

$$\begin{gathered} \mu_0 = 8e(e-1)^3, \quad \eta = 8(e-1)(2-3e)^3, \quad \mu_1 = 12e(e-1)^2 y, \\ \mu_3 = ey^3, \quad \theta_1 = 128(e-1)(2e-1)(3e-2)(5-6e). \end{gathered} \tag{6.17}$$

We observe that if $\mu_0 \neq 0$ then

$$\begin{gathered} \operatorname{sign}(\mu_0) = \operatorname{sign}\big(\hat{\gamma}(\hat{\gamma}-6)\big), \quad \operatorname{sign}(\eta)\big|_{\{\mu_0<0\}} = \operatorname{sign}(\hat{\gamma}-4\big), \\ \operatorname{sign}(\theta_1)\big|_{\{\mu_0<0,\eta<0\}} = \operatorname{sign}(\hat{\gamma}-3\big). \end{gathered} \tag{6.18}$$

We claim that the conditions $\hat{\gamma} I_9 > 0$ and $I_9(4-\hat{\gamma}) < 0$ corresponding to the phase portrait *Vul*$_2$ (see (6.14)) are equivalent to $\mu_0 > 0$. Indeed, as $I_9 = d = 2(e-1) \neq 0$ we obtain $\hat{\gamma} I_9 = 12e(e-1)$ and hence the condition $\hat{\gamma} I_9 > 0$ is equivalent to $\mu_0 > 0$. It remains to note that in the case $e(e-1) > 0$ we have $I_9(4-\hat{\gamma}) = -4(e-1)(3e-2) < 0$. So $3e-2 \neq 0$ (i.e. $\hat{\gamma} \neq 4$) and we arrive to the respective conditions from Table 2.

Considering the remaining cases (6.14) corresponding to the condition $\hat{\beta} = 0$ and (6.18) we arrive to the respective conditions provided by Table 2 (the case of statement (e) of Proposition 4.1).

6.4. **The family of systems** $(S_4^{(c)})$**.** For these systems we have $\sigma = 0$; i.e., this is a class of Hamiltonian systems with center. Moreover any elemental point which is not a center must be an integrable saddle. Calculations yield

$$\begin{gathered} \hat{\alpha} = 8(a-2c)^2(b^2-4ac+8c^2), \quad I_8 = 2a(ab^2+4c^3), \\ I_{10} = -b^2-(a+c)^2, \quad I_{16} = b(3ac^2-a^3+ab^2+2c^3). \end{gathered} \tag{6.19}$$

According to [30] (see the proof of Lemma 2) the phase portrait (and hence, the respective configuration of singularities) of a quadratic system from the family $(S_4^{(c)})$ is determined by the following GL-invariant conditions

$$\begin{gathered} \begin{cases} \hat{\alpha} < 0, \text{ or} \\ \hat{\alpha} = I_{16} = 0 \end{cases} \Leftrightarrow \mathit{Vul}_2 \Rightarrow \begin{cases} c, \$; N & \text{if } \hat{\alpha} \neq 0; \\ c, \overline{s}_{(3)}; N & \text{if } \hat{\alpha} = 0,\ I_8 \neq 0; \\ c, \$; \widehat{\binom{2}{3}} N & \text{if } \hat{\alpha} = 0,\ I_8 = 0; \end{cases} \\ \hat{\alpha} > 0,\ I_8 < 0 \Leftrightarrow \mathit{Vul}_8, \mathit{Vul}_9, \mathit{Vul}_{10}, \mathit{Vul}_{11} \Rightarrow c, \$, \$, \$; N, N, N; \\ \hat{\alpha} > 0,\ I_8 > 0 \Leftrightarrow \mathit{Vul}_3, \mathit{Vul}_4 \Rightarrow c, c, \$, \$; N; \\ \hat{\alpha} > 0,\ I_8 = 0 \Leftrightarrow \mathit{Vul}_5, \mathit{Vul}_6 \Rightarrow c, \$, \$; N, \widehat{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} PEP - H; \\ \hat{\alpha} = 0,\ I_{16} \neq 0 \Leftrightarrow \mathit{Vul}_7 \Rightarrow c, \$, \widehat{cp}_{(2)}; N. \end{gathered} \tag{6.20}$$

On the other hand for systems $(S_4^{(c)})$ calculations yield

$$\begin{gathered} \mu_0 = a(ab^2+4c^3) = I_8/2, \quad \mathbf{D} = -48(a-2c)^2(b^2-4ac+8c^2) = -6\hat{\alpha}, \\ \mu_1 = 2ab(a-c)x - 2(ab^2+2ac^2+2c^3)y. \end{gathered} \tag{6.21}$$

We observe that the condition $I_8 < 0$ implies $c \neq 0$ (then by Proposition 6.1 we have $c = 1/2$) and $a < 0$. In this case we evidently obtain $\hat{\alpha} > 0$. Similarly the condition $\hat{\alpha} < 0$ gives $c \neq 0$ (i.e. $c = 1/2$) and $a > 1 + b^2/2$ and this implies $I_8 > 0$. Herein we conclude that in the case $I_8\hat{\alpha} \neq 0$ (i.e. $\mu_0 \mathbf{D} \neq 0$) the conditions provided by Table 2 (the case of statement (f) of Proposition 4.1) for distinguishing the configurations of the singularities are equivalent to the respective conditions of (6.20).

Assume now $I_8\hat{\alpha} = 0$ (i.e. $\mu_0\mathbf{D} = 0$).

(1) If $I_8 = 0$ then by (6.19) we have $a(ab^2 + 4c^3) = 0$ and then $\mu_0 = 0$. We claim that in the case $\hat{\alpha} \neq 0$ we obtain $\hat{\alpha} > 0$ and this is equivalent to $\mu_1 \neq 0$. Indeed as the condition $\hat{\alpha} \neq 0$ implies $(a^2 + c^2)(b^2 + c^2) \neq 0$, we conclude that the condition $I_8 = 0$ gives $a \leq 0$ and $c \neq 0$. So $c = 1/2$ and we obtain $\hat{\alpha} > 0$. On the other hand since $a - c \neq 0$ we obtain that $\mu_1 = 0$ if and only if $ab = a + c = 0$ but in this case the condition $I_8 = 0$ implies $\hat{\alpha} = 0$. So our claim is proved and this shows the equivalence of the respective conditions of Table 2 and (6.20).

Assume $I_8 = 0 = \hat{\alpha}$. Then considering (6.20) we obtain portrait Vul_2 and the configuration of singularities indicated in the row 41 of Table 2. It remains to observe that the condition above is equivalent to $\mu_0 = \mu_1 = 0$.

(2) Suppose now $\hat{\alpha} = 0$ and $I_8 \neq 0$. This implies $\mu_0 \neq 0$ and $\mathbf{D} = 0$, i.e. $(a - 2c)(b^2 - 4ac + 8c^2)$=0. So we obtain

$$\mathbf{T} = -48b^2c^4y^2(2cx + by)^2(bx - cy)^2, \quad I_{16} = 2b^3c, \quad \mu_0 = 4c^2(b^2 + 2c^2),$$

if $a = 2c$ and

$$\begin{aligned}\mathbf{T} &= -\frac{3b^2}{4096c^6}\,(b^2 + 4c^2)^4y^2(bx - 2cy)^2(b^2x + 8c^2x + 2bcy)^2,\\ I_{16} &= -\frac{1}{64c^3}\,b^3(b^2 + 4c^2)^2, \quad \mu_0 = \frac{1}{16c^2}(b^2 + 4c^2)^2(b^2 + 8c^2),\end{aligned}$$

if $a = (b^2 + 8c^2)/(4c)$ (we note that $c \neq 0$ due to $I_8 \neq 0$).

Thus clearly if $\hat{\alpha} = 0$ and $I_8 \neq 0$ then the condition $I_{16} = 0$ is equivalent to $\mathbf{T} = 0$ and as $\mu_0 > 0$ we obtain that the respective conditions provided by Table 2 (see rows 38 and 39) are equivalent to the corresponding conditions from (6.20).

To finish the proof of Theorem 1 it remains to examine the conditions for distinguishing the different phase portraits which correspond to the same configuration of singularities. We have three groups of such phase portraits: (i) $\mathit{Vul}_8 - \mathit{Vul}_{11}$; (ii) Vul_3, Vul_4 and (iii) Vul_5, Vul_6. According to [31] quadratic systems $(S_4^{(c)})$ possess one of the mentioned phase portraits if and only if the following conditions are fulfilled

$$\begin{gathered}\hat{\alpha} > 0,\ I_8 > 0 \Rightarrow \begin{cases}\mathit{Vul}_3 & \text{if } I_{16} = 0;\\ \mathit{Vul}_4 & \text{if } I_{16} \neq 0;\end{cases}\\ \hat{\alpha} > 0,\ I_8 = 0 \Rightarrow \begin{cases}\mathit{Vul}_5 & \text{if } I_{16} = 0;\\ \mathit{Vul}_6 & \text{if } I_{16} \neq 0;\end{cases}\\ \hat{\alpha} > 0,\ I_8 < 0 \Rightarrow \begin{cases}\mathit{Vul}_8 & \text{if } I_{10} \neq 0, I_{16} = 0, \hat{\xi} > 0;\\ \mathit{Vul}_9 & \text{if } I_{10} \neq 0, I_{16} = 0, \hat{\xi} < 0;\\ \mathit{Vul}_{10} & \text{if } I_{10} = 0;\\ \mathit{Vul}_{11} & \text{if } I_{16} \neq 0.\end{cases}\end{gathered} \tag{6.22}$$

On the other hand for systems $(S_4^{(c)})$ we have

$$\begin{gathered}B_1 = -I_{16}[b^2 + (a - 3c)^2], \quad I_{16} = b(3ac^2 - a^3 + ab^2 + 2c^3),\\ B_3 = -3abx^4 - 6(a - 3c)(a + c)x^3y + 18bcx^2y^2 + 6b^2xy^3 + 3b(a - 2c)y^4,\end{gathered}$$

and hence the condition $I_{16} = 0$ is equivalent to $B_1 = 0$. Herein we arrive to the conditions provided by Table 3 for the portraits $Vul_3 - Vul_6$ respectively.

Next we examine the conditions for the phase portraits $Vul_8 - Vul_{11}$. First we observe that by (6.19) the condition $I_{10} = 0$ yields $b = a + c = 0$ and this implies $I_{16} = B_3 = B_1 = 0$. Therefore in the case $I_{16} \neq 0$ (this is equivalent with $B_1 \neq 0$) we get Vul_{11} and in the case $I_{10} = 0$ (then $B_3 = 0$) we obtain Vul_{10} (see Table 3, configuration number 35). So it remains to consider the phase portraits Vul_8 and Vul_9. We claim that in the case $I_{16} = 0$ we have $\operatorname{sign}(\hat{\xi}) = \operatorname{sign}(B_3 B_4)$. Indeed assuming $I_{16} = 0$ we shall examine two cases, $b = 0$ and $b \neq 0$.

(1) If $b = 0$ (then $I_{16} = 0$) a straightforward calculation for systems $(S_4^{(c)})$ yields

$$\hat{\xi} = -4(a-c)(a+c)^3, \quad B_3 B_4 = -192(a-3c)^3 c^4 (a+c)^3 x^4 y^2, \quad I_8 = 8ac^3,$$

and as $I_8 < 0$ (i.e. $ac < 0$) we get $(a-c)(a-3c) > 0$. So clearly our claim is proved in this case.

(2) Assume now $b \neq 0$. Then $I_8 < 0$ gives $ac < 0$ and hence we can set a new parameter u as follows $u^2 = (a-2c)/a$. Herein we have $c = a(1-u^2)/2$ and calculation gives

$$I_{16} = ab(2b - 3au + au^3)(2b + 3au - au^3)/4.$$

Hence due to $ab \neq 0$ the condition $I_{16} = 0$ gives $b = \pm au(u^2-3)/2$ and then we calculate

$$\hat{\xi} = -a^4(u^2-1)(u^2-3)^3(1+u^2)^2/2, \quad I_8 = a^4(2-u^2)(1+u^2)^2/4,$$
$$B_3 B_4 = -3a^{10}(u^2-3)^3(1+u^2)^5(ux \pm y)^2(uy \mp x)^4/16.$$

As $I_8 < 0$ we have $u^2 - 2 > 0$ and this implies $u^2 - 1 > 0$. Therefore $\operatorname{sign}(\hat{\xi}) = \operatorname{sign}(B_3 B_4)$, i.e. our claim is valid and we arrive to the respective conditions given by Table 3 in the considered case.

As all the cases are examined Theorem 1.1 is proved. □

7. The family of Hamiltonian quadratic systems

In this section to prove Theorem 1.2 we need the following invariant polynomials defined in [16] using the invariant polynomials (5.3) and (5.4) (we keep the respective notations adding only the "hat")

$$
\begin{gathered}
2\widehat{\mu}=I_8, \quad \widehat{H}=-K_{14}, \quad 4\widehat{G}=-5I_2K_7+2I_5K_2+4K_3^2+8K_{31},\\
2\widehat{F}=-I_2K_{11}-4I_{19}K_5+4K_2K_{27}, \quad \widehat{V}=K_{11}K_{22}+K_{23}^2,\\
\widehat{R}=3\widehat{H}^2-2\widehat{G}\widehat{\mu}, \quad \widehat{S}=2\widehat{F}\widehat{H}\widehat{\mu}^2+\widehat{G}^2\widehat{\mu}^2-4\widehat{G}\widehat{H}^2\widehat{\mu}+3\widehat{H}^4-4\widehat{\mu}^3\widehat{V},\\
\widehat{P}=\widehat{G}^2-6\widehat{F}\widehat{H}+12\widehat{\mu}\widehat{V}, \quad \widehat{U}=\widehat{F}^2-4\widehat{G}\widehat{V},\\
\widehat{T}=9\widehat{F}^2\widehat{\mu}^2-14\widehat{F}\widehat{G}\widehat{H}\widehat{\mu}+12\widehat{F}\widehat{H}^3+2\widehat{G}^3\widehat{\mu}-2\widehat{G}^2\widehat{H}^2-8\widehat{G}\widehat{\mu}^2\widehat{V}+12\widehat{H}^2\widehat{\mu}\widehat{V},\\
\widehat{W}_1=K_2^2-4K_5K_{21}, \quad \widehat{W}_2=-K_2K_{12}-2K_3K_{11}+4K_5K_{27}+6K_7K_{23},\\
\widehat{E}_1=4I_{10}-5I_2I_5-24I_{30}, \quad \widehat{E}_2=I_2^2+32I_{23}-16I_{24},\\
\widehat{E}_3=I_2^3+32(12I_5I_{18}-3I_{19}^2+16I_{33}),\\
32\widehat{D}=4I_5^2(3I_2\widehat{E}_2-\widehat{E}_3)-\widehat{E}_2(4I_5\widehat{E}_1+I_8\widehat{E}_2),\\
\widehat{T}_c=(9I_5^3+I_8\widehat{E}_1)^2+9I_5^2(I_8^2\widehat{E}_2-3I_5^4)+I_8^3(3I_2\widehat{E}_2-\widehat{E}_3),\\
\widehat{U}_c=3I_8I_{19}(3I_2I_5+16I_{30})-2I_8(3I_2I_{35}+24I_5I_{28}+16I_8I_{21})\\
+3I_5^2(I_5I_{19}+10I_{35})-2I_{16}(2I_2I_5-I_{10}+12I_{30}).
\end{gathered} \tag{7.1}
$$

The proof of Theorem 1.2 is based on the classifications of quadratic Hamiltonian systems given in [1] and [16]. We use the notations of [1] whether the system has a center as $Vul_{\#}$ or if not as $Ham_{\#}$. In the first paper the global phase portraits of this family were studied. In the second one there are determined the affine invariant criteria for the realization of each one of the 28 possible topologically distinct phase portraits constructed in [1]. That is, the phase portraits of the systems

$$\frac{dx}{dt}=\frac{\partial H}{\partial y}, \quad \frac{dy}{dt}=-\frac{\partial H}{\partial x}, \tag{7.2}$$

where $H(x,y)$ is a polynomial of degree 3 in the variables x and y over $\mathbb{R}$.

According to the paper [4] for the quadratic Hamiltonian systems we have

$$H(x,y)=\sum_{j=0}^{2}\frac{1}{j+1}C_j(x,y), \tag{7.3}$$

where $C_j(x,y)$, $j=0,1,2$ are the polynomials (4.3). So the 3rd degree homogeneous part of the polynomial $H(x,y)$ is the polynomial $H_3(x,y)=C_2(x,y)/3$. As it is shown in [1] via linear transformations the non-zero form $H_3(x,y)$ can be derived to one of the 4 canonical forms

$$a)\ x(x^2-y^2); \quad b)\ (x^3+y^3)/3; \quad c)\ x^2y; \quad d)\ x^3/3.$$

By (4.5) we observe that the invariant polynomials η and $\widetilde{M}$ are respectively the discriminant and the Hessian of the binary form $C_2(x,y)$. So considering also [16] we arrive to the next proposition.

Proposition 7.1. *Assume that for a quadratic system the condition $\frac{\partial}{\partial x}P(x,y)+\frac{\partial}{\partial y}Q(x,y)\equiv 0$ holds, i.e. it is Hamiltonian. Then this system could be brought via an affine transformation and time rescaling to one of the canonical forms below if*

Table 4.

Affine invariant conditions for configurations					*Configuration of singularities*	No.
$\mu_0 < 0$	$\mathbf{D} < 0$				$\$, \$, \$, c;\ N, N, N$	1
	$\mathbf{D} > 0$				$\$, \$;\ N, N, N$	2
	$\mathbf{D} = 0$	$\mathbf{T} \neq 0$			$\$, \$, \widehat{cp}_{(2)};\ N, N, N$	3
		$\mathbf{T} = 0$	$\mathbf{R} \neq 0$		$\$, \widehat{s}_{(3)};\ N, N, N$	4
			$\mathbf{R} = 0$		$(hhhhhh)_{(4)};\ N, N, N$	5
$\mu_0 > 0$	$\mathbf{D} < 0$	$(\mathbf{R} > 0)\&(\mathbf{S} > 0)$			$\$, \$, c, c;\ N$	6
		$(\mathbf{R} \leq 0) \vee (\mathbf{S} \leq 0)$			$\emptyset; N$	7
	$\mathbf{D} > 0$				$\$, c;\ N$	8
	$\mathbf{D} = 0$	$\mathbf{T} < 0$			$\$, c, \widehat{cp}_{(2)};\ N$	9
		$\mathbf{T} > 0$			$\widehat{cp}_{(2)};\ N$	10
		$\mathbf{T} = 0$	$\mathbf{PR} < 0$		$\emptyset; N$	11
			$\mathbf{PR} > 0$		$\widehat{cp}_{(2)}, \widehat{cp}_{(2)};\ N$	12
			$\mathbf{PR} = 0$	$\mathbf{R} \neq 0$	$c, \widehat{s}_{(3)};\ N$	13
				$\mathbf{R} = 0$	$(hh)_{(4)};\ N$	14
$\mu_0 = 0$	$\mu_1 \neq 0$	$\mathbf{D} < 0$			$\$, \$, c;\ N, \widehat{\binom{1}{2}} PEP-H$	15
		$\mathbf{D} > 0$			$\$;\ N, \widehat{\binom{1}{2}}\ PEP-H$	16
		$\mathbf{D} = 0$	$\mathbf{P} \neq 0$		$\$, \widehat{cp}_{(2)};\ N, \widehat{\binom{1}{2}}\ P\ E\ P-H$	17
			$\mathbf{P} = 0$		$\widehat{s}_{(3)};\ N, \widehat{\binom{1}{2}}\ PEP-H$	18
	$\mu_1 = 0$	$\mu_2 \neq 0$	$\mathbf{U} < 0$	$\widetilde{M} \neq 0$	$\emptyset; N, \binom{2}{2} H-H$	19
				$\widetilde{M} = 0$	$\emptyset; \widehat{\binom{2}{3}}\ N$	20
			$\mathbf{U} > 0$	$\widetilde{M} \neq 0$	$\$, \$;\ N, \binom{2}{2} P\ E\ P-P\ E\ P$	21
				$\widetilde{M} = 0$	$\$, c;\ \widehat{\binom{2}{3}}\ N$	22
			$\mathbf{U} = 0$		$\widehat{cp}_{(2)};\ \widehat{\binom{2}{3}}\ N$	23
		$\mu_2 = 0$	$\mu_3 \neq 0$		$\$;\ \binom{3}{3}\ PEPEP-P$	24
			$\mu_3 = 0$, $\mu_4 \neq 0$	$\widetilde{M} \neq 0$	$\emptyset; N, \binom{4}{2}\ PHP-PHP$	25
				$\widetilde{M} = 0$	$\emptyset; \binom{4}{3}\ N$	26
			$\mu_3 = \mu_4 = 0, \widetilde{M} \neq 0$		$(\ominus[\vert]; s); N, (\ominus[\vert]; N)$	27
			$\mu_3 = 0$, $\mu_4 = 0$, $\widetilde{M} = 0$	$B_6 < 0$	$(\ominus[\vert\vert^c]; \emptyset); (\ominus[\vert\vert^c]; N)$	28
				$B_6 > 0$	$(\ominus[\vert\vert\vert]; \emptyset); (\ominus[\vert\vert\vert]; N)$	29
				$B_6 = 0$	$(\ominus[\vert 2]; \emptyset); (\ominus[\vert 2]; N)$	30

and only if the respective conditions hold

$$(S_1^{(h)}) \quad \begin{cases} \dot{x} = \alpha + bx + cy - 2xy, \\ \dot{y} = \beta - ax - by - 3x^2 + y^2, \end{cases} \quad \Leftrightarrow \eta > 0;$$

$$(S_2^{(h)}) \quad \begin{cases} \dot{x} = \alpha + bx + cy + y^2, \\ \dot{y} = \beta - ax - by - x^2, \end{cases} \quad \Leftrightarrow \eta < 0;$$

TABLE 5.

Configuration	*Phase portrait*	*Configuration*	*Phase portrait*	*Configuration*	*Phase portrait*
1	Vul_{11} if $B_1 \neq 0$	8	Vul_2	20	Ham_{11}
	Vul_9 if $B_1 = 0,\ B_3B_4 < 0$	9	Vul_7	21	Ham_{20} if $B_3 \neq 0$
	Vul_8 if $B_1 = 0,\ B_3B_4 > 0$	10	Ham_{17}		Ham_{21} if $B_3 = 0$
	Vul_{10} if $B_1 = B_3 = 0$	11	Ham_{11}	22	Vul_2
2	Ham_{26} if $B_1 = 0, B_3B_4 < 0$	12	Ham_{28}	23	Ham_{17}
	Ham_{25} if $\begin{cases} B_1 \neq 0, \ or \\ B_1 = 0, B_3B_4 > 0 \end{cases}$	13	Vul_2	24	Ham_{18}
3	Ham_{27} if $B_1 \neq 0$	14	Ham_{17}	25	Ham_{13}
	Ham_{24} if $B_1 = 0$	15	Vul_6 if $B_1 \neq 0$	26	Ham_{11}
4	Ham_{25}		Vul_5 if $B_1 = 0$	27	Ham_{14}
5	Ham_{23}	16	Ham_{19}	28	Ham_{11}
6	Vul_4 if $B_1 \neq 0$	17	Ham_{22}	29	Ham_{15}
	Vul_3 if $B_1 = 0$	18	Ham_{19}	30	Ham_{16}
7	Ham_{11}	19	Ham_{12}		

$$(S_3^{(h)}) \quad \begin{cases} \dot{x} = \alpha + bx + cy + x^2, \\ \dot{y} = \beta - ax - by - 2xy, \end{cases} \quad \Leftrightarrow \eta = 0,\ \widetilde{M} \neq 0;$$

$$(S_4^{(h)}) \quad \begin{cases} \dot{x} = \alpha + bx + cy, \\ \dot{y} = \beta - ax - by - x^2, \end{cases} \quad \Leftrightarrow \eta = 0,\ \widetilde{M} = 0.$$

Remark 7.2. For a quadratic Hamiltonian system the relation $\eta = -27\mu_0$ holds. This could be easy established via the frontal evaluations of the invariant polynomials η and μ_0 for systems $(S_1^{(h)}) - (S_4^{(h)})$.

Proof of Theorem 1.2. In what follows we shall consider each one of the systems $(S_1^{(h)}) - (S_4^{(h)})$ and will show that the conditions given in [16] for distinguishing the phase portraits of the respective systems, are equivalent with the affine invariant provided in Tables 4 and 5.

7.1. **The family of systems $(S_1^{(h)})$.** In this case according to [16, Theorem 3] the phase portrait (and this yields the respective configuration of singularities) of a system of the family $(S_1^{(h)})$ is determined by the following affine invariant conditions

$$\begin{gathered}
\widehat{D} < 0,\ \widehat{U}_c = 0,\ \widehat{T}_c > 0 \Leftrightarrow Vul_8 \Rightarrow \$, \$, \$, c\,; N, N, N; \\
\widehat{D} < 0,\ \widehat{U}_c = 0,\ \widehat{T}_c < 0 \Leftrightarrow Vul_9 \Rightarrow \$, \$, \$, c\,; N, N, N; \\
\widehat{D} < 0,\ \widehat{U}_c = 0,\ \widehat{T}_c = 0 \Leftrightarrow Vul_{10} \Rightarrow \$, \$, \$, c\,; N, N, N; \\
\widehat{D} < 0,\ \widehat{U}_c \neq 0 \Leftrightarrow Vul_{11} \Rightarrow \$, \$, \$, c\,; N, N, N; \\
\widehat{D} = 0,\ \widehat{U}_c = 0,\ \widehat{T}_c = 0 \Leftrightarrow Ham_{23} \Rightarrow (hhhhhh)_{(4)}\,; N, N, N; \\
\widehat{D} = 0,\ \widehat{U}_c = 0,\ \widehat{T}_c < 0 \Leftrightarrow Ham_{24} \Rightarrow \$, \$, \widehat{cp}_{(2)}\,; N, N, N; \\
\begin{cases} \widehat{D} > 0,\ \widehat{U}_c \neq 0,\ \text{or} \\ \widehat{D} \geq 0,\ \widehat{U}_c = 0,\ \widehat{T}_c > 0 \end{cases} \Leftrightarrow Ham_{25} \Rightarrow \begin{cases} \$, \$; N, N, N & \text{if } \widehat{D} \neq 0; \\ \$, \hat{s}_{(3)}\,; N, N, N & \text{if } \widehat{D} = 0; \end{cases} \\
\widehat{D} > 0,\ \widehat{U}_c = 0,\ \widehat{T}_c < 0 \Leftrightarrow Ham_{26} \Rightarrow \$, \$; N, N, N; \\
\widehat{D} = 0,\ \widehat{U}_c \neq 0 \Leftrightarrow Ham_{27} \Rightarrow \$, \$, \widehat{cp}_{(2)}\,; N, N, N.
\end{gathered} \tag{7.4}$$

According to [1] systems $(S_1^{(h)})$ have at least one real singular point. So we may assume $\alpha = \beta = 0$ due to a translation and then calculations yield

$$\begin{gathered}
\widehat{D} = 4(b^2 - ac)^2[-8a^3c - (36c^2 + b^2)a^2 + 6c(4b^2 - 9c^2)a + 3(b^4 + 33b^2c^2 - 9c^4)],\\
\widehat{U}_c = 108bc(a - 2b - 3c)(a + 2b - 3c)(a - 2b + c)(a + 2b + c),\\
\widehat{T}_c = 3456\big[(a + 2b + c)^3(a - 2b - 3c)^3 + 24bc(a + 2b + c)^2(a - 2b - 3c)^2\\
- 384b^2c^2(a + 2b + c)(a - 2b - 3c) - 4096b^3c^3\big],\\
\mu_0 = -12 < 0, \quad \mathbf{D} = 48\widehat{D}, \quad B_1 = \widehat{U}_c/2, \qquad (7.5)\\
B_3 = 18(a^2 - 4b^2 - 2ac - 3c^2)x^3y - 18bc(3x^4 + 6x^2y^2 - y^4),\\
B_4 = 288[(a^2 - 4b^2 - 2ac - 3c^2)x - 8bcy][24bcx + (a^2 - 4b^2 - 2ac - 3c^2)y].
\end{gathered}$$

Remark 7.3. We claim that in the case $\widehat{U}_c = 0$ we have $\operatorname{sign}(\widehat{T}_c) = \operatorname{sign}(B_3B_4)$ and $\widehat{T}_c = 0$ if and only if $B_3 = 0$.

Indeed assume $\widehat{U}_c = 0$. We observe that the change of variables $(x, y, t) \mapsto (x, -y, -t)$ keeps the systems $(S_1^{(h)})$ with $\alpha = \beta = 0$ except the sign of the parameter b, which will be changed. Hence without loss of generality we could consider only the equality $bc(a+2b+c)(a-2b-3c) = 0$. If $bc \neq 0$ then by (7.5) evidently we get $\widehat{T}_c = -2^{19}3^3b^3c^3$. On the other hand for $a = -2b-c$ as well as for $a = 2b+3c$, we obtain $B_3B_4 = -2^{12}3^4b^3c^3(x-y)^4(3x+y)^2$ and $B_3 = -18bc(x-y)^3(3x+y)$. So our claim is proved in the considered case.

Assume now $bc = 0$. Then calculations yield

$$\widehat{T}_c = 3456(a - 3c)^3(a + c)^3, \quad B_3B_4 = 3\,\widehat{T}_c x^4y^2/2, \quad B_3 = 18(a - 3c)(a + c)x^3y$$

if $b = 0$ and

$$\widehat{T}_c = 3456(a - 2b)^3(a + 2b)^3, \quad B_3B_4 = 3\widehat{T}_c x^4y^2/2, \quad B_3 = 18(a - 2b)(a + 2b)x^3y$$

if $c = 0$. Now evidently the proof of the claim is completed.

7.1.1. *The case* $\widehat{D} < 0$. Then $\mathbf{D} < 0$ and considering (7.5) and Remark 7.3 we conclude that the conditions for phase portraits $Vul_8 -\ Vul_{11}$ given by Tables 4 and 5 are equivalent to the respective conditions in (7.4).

7.1.2. *The case* $\widehat{D} > 0$. According to (7.4) in this case we could have only the phase portraits Ham_{25} and Ham_{26}. So considering (7.5) and Remark 7.3 we again arrive to the equivalence of the respective conditions from (7.4) and from Tables 4 and 5.

7.1.3. *The case* $\widehat{D} = 0$. By (7.4) we have: *(i)* the phase portraits Ham_{24} (if $\widehat{U}_c = 0$, $\widehat{T}_c < 0$) and Ham_{27}(if $\widehat{U}_c \neq 0$) with the same configuration having three finite singularities (one double); *(ii)* the phase portrait Ham_{25} (if $\widehat{U}_c = 0$, $\widehat{T}_c > 0$) possessing only two real finite singularities (one triple); *(iii)* the phase portrait Ham_{23} (if $\widehat{U}_c = \widehat{T}_c = 0$) possessing only one finite singularity of multiplicity four. Considering the diagram in Figure 1 and (7.5) we conclude that the conditions for determining the mentioned phase portraits from Tables 4 and 5 are equivalent to the respective conditions from (7.4).

7.2. **The family of systems** $(S_2^{(h)})$**.** In this case according to [16, Theorem 4] the phase portrait (and this yields the respective configuration of singularities) of a system from the family $(S_2^{(h)})$ is determined by the following affine invariant conditions

$$\begin{gathered}
\begin{cases} \widehat{D} > 0, \text{ or} \\ \widehat{D} = \widehat{T} = \widehat{P} = 0, \ \widehat{R} \neq 0 \end{cases} \Leftrightarrow Vul_2 \Rightarrow \begin{cases} \$, c\,; N & \text{if } \widehat{D} \neq 0; \\ c, \hat{s}_{(3)}\,; N & \text{if } \widehat{D} = 0; \end{cases} \\
\widehat{D} < 0, \ \widehat{R} > 0, \widehat{S} > 0, \widehat{U}_c = 0 \Leftrightarrow Vul_3 \Rightarrow \$, \$, c, c\,; N; \\
\widehat{D} < 0, \ \widehat{R} > 0, \widehat{S} > 0, \widehat{U}_c \neq 0 \Leftrightarrow Vul_4 \Rightarrow \$, \$, c, c\,; N; \\
\widehat{D} = 0, \ \widehat{T} < 0 \Leftrightarrow Vul_7 \Rightarrow \$, c, \widehat{cp}_{(2)}\,; N; \\
\begin{cases} \widehat{D} < 0, (\widehat{R} \leq 0) \vee (\widehat{S} \leq 0), \text{ or} \\ \widehat{D} = \widehat{T} = 0, \ \widehat{P}\widehat{R} < 0 \end{cases} \Leftrightarrow Ham_{11} \Rightarrow \emptyset; N; \\
\begin{cases} \widehat{D} = 0, \ \widehat{T} > 0, \text{ or} \\ \widehat{D} = \widehat{T} = \widehat{P} = \widehat{R} = 0 \end{cases} \Leftrightarrow Ham_{17} \Rightarrow \begin{cases} \widehat{cp}_{(2)}\,; N & \text{if } \widehat{T} \neq 0; \\ (hh)_{(4)}\,; N & \text{if } \widehat{T} = 0; \end{cases} \\
\widehat{D} = \widehat{T} = 0, \ \widehat{P}\widehat{R} > 0 \Leftrightarrow Ham_{28} \Rightarrow \widehat{cp}_{(2)}, \widehat{cp}_{(2)}\,; N.
\end{gathered} \tag{7.6}$$

Taking into account (7.1) and (5.2) a straightforward calculation gives for systems $(S_2^{(h)})$

$$\mu_0 = 1, \ \widehat{D} = \mathbf{D}/48, \ \widehat{T} = \mathbf{T}/6, \ \widehat{R} = \mathbf{R}/4, \ \widehat{S} \quad = \mathbf{S}/48, \ \widehat{P} = \mathbf{P}, \ \widehat{U}_c \quad = 2B_1. \tag{7.7}$$

Herein considering the diagram from Figure 1, it is easy to observe that the conditions for determining the phase portraits from Tables 4 and 5 corresponding to the case $\mu_0 > 0$ are equivalent to the respective conditions from (7.6).

7.3. **The family of systems** $(S_3^{(h)})$**.** For this family of systems we have $\eta = \mu_0 = 0$ and $\widetilde{M} \neq 0$. According to [28, Table 2] at infinity we have two real singularities of total multiplicity at least four. So in order to determine exactly the configuration of the singularities at infinity we shall use the classification of the behavior of the trajectories in the neighbourhood at infinity of quadratic differential systems, given in [28].

In this order of ideas we need the following additional invariant polynomials, defined in [28]

$$\begin{gathered}
\kappa(\tilde{a}) = (\widetilde{M}, \widetilde{K})^{(2)}, \quad \kappa_1(\tilde{a}) = (\widetilde{M}, C_1)^{(2)}, \\
\widetilde{K}_1(\tilde{a}, x, y) = p_1(x, y)q_2(x, y) - p_2(x, y)q_1(x, y), \\
\widetilde{K}_2(\tilde{a}, x, y) = 4(T_2, \omega)^{(1)} + 3D_1(C_1, \omega)^{(1)} - \omega\big(16T_3 + 3T_4/2 + 3D_1^2\big), \\
\widetilde{K}_3(\tilde{a}, x, y) = C_2^2(4T_3 + 3T_4) + C_2(3C_0\widetilde{K} - 2C_1T_7) + \widetilde{K}_1(3\widetilde{K}_1 - C_1D_2),
\end{gathered} \tag{7.8}$$

where $\omega = \widetilde{M} - 8\widetilde{K}$. According to [16, Theorem 2] the phase portrait of a system from the family $(S_3^{(h)})$ is determined by the following affine invariant conditions

$$
\begin{gathered}
Vul_5 \Leftrightarrow \widehat{D} < 0,\ \widehat{U}_c = 0;\\
Vul_6 \Leftrightarrow \widehat{D} < 0,\ \widehat{U}_c \neq 0;\\
Ham_{12} \Leftrightarrow \widehat{R} = 0,\ \widehat{U} < 0;\\
Ham_{13} \Leftrightarrow \widehat{R} = \widehat{U} = 0,\ \widehat{V} \neq 0;\\
Ham_{19} \Leftrightarrow \begin{cases} \widehat{D} > 0,\ \text{or}\\ \widehat{D} = 0,\ \widehat{R} \neq 0,\ \widehat{U}_c = 0; \end{cases}\\
Ham_{20} \Leftrightarrow \widehat{R} = 0,\ \widehat{U} > 0,\ \widehat{W}_2 \neq 0;\\
Ham_{21} \Leftrightarrow \widehat{R} = 0,\ \widehat{U} > 0,\ \widehat{W}_2 = 0;\\
Ham_{22} \Leftrightarrow \widehat{D} = 0,\ \widehat{U}_c \neq 0.
\end{gathered}
\tag{7.9}
$$

For systems $(S_3^{(h)})$ calculations yield

$$
\begin{gathered}
\widehat{D} = 108c^4(ab + 2\beta)^2 - 4c^2(b^2 + 2ac - 4\alpha)^3,\\
\widehat{U}_c = -108c^4(ab + 2\beta), \quad \widehat{R} = 12c^2x^2,\\
\mu_0 = 0, \quad \widetilde{M} = -72x^2, \quad \mathbf{D} = 48\widehat{D}, \quad B_1 = \widehat{U}_c/2,\\
\mu_1 = 4cx, \quad \tilde{L} = 24x^2 > 0, \quad \widetilde{K} = -4x^2 < 0,
\end{gathered}
\tag{7.10}
$$

7.3.1. *The case* $\widehat{R} \neq 0$. In this case the condition $c \neq 0$ (i.e., $\mu_1 \neq 0$) holds and as $\tilde{L} > 0$ and $\widetilde{K} < 0$ according to [28, Table 4] at infinity the behavior of the trajectories corresponds to Figure 9; i.e., we have the following configuration of singularities $N, \widehat{\binom{1}{2}} PEP - H$.

We observe that in the case $\widehat{D} = 0$ and (i.e., $\mathbf{D} = 0$) by (7.9) we have the phase portrait Ham_{19} with one finite singularity (if $\widehat{U}_c = 0$) and Ham_{22} with two finite singularities (if $\widehat{U}_c \neq 0$). Considering the diagram (Figure 1), (7.10) and the fact, that the condition $\widehat{D}^2 + \widehat{U}_c^2 \neq 0$ implies $\widehat{R} \neq 0$, we conclude that the conditions for determining the phase portraits from Tables 4 and 5 corresponding to the case $\mu_0 = 0$ and $\mu_1 \neq 0$ are equivalent to the respective conditions of (7.9) (i.e. the conditions for the phase portraits Vul_5, Vul_6, Ham_{19} and Ham_{22}).

7.3.2. The case $\widehat{R} = 0$

Then $c = 0$ and for systems $(S_3^{(h)})$ we calculate

$$
\begin{gathered}
\widehat{U} = (b^2 - 4\alpha)\big[(ab + 2\beta)x - (b^2 - 4\alpha)y\big]^2 x^4, \quad \widehat{W}_2 = -6x^4(ab + 2\beta),\\
\mu_0 = \mu_1 = 0,\ \mu_2 = (4\alpha - b^2)x^2, \quad \mathbf{U} = \widehat{U}, \quad B_3 = 3\widehat{W}_2,\\
\kappa = \kappa_1 = 0, \quad \widetilde{K}_2 = 768(b^2 - 4\alpha)x^2.
\end{gathered}
\tag{7.11}
$$

We observe that $\operatorname{sign}(\widehat{U}) = \operatorname{sign}(\mathbf{U}) = -\operatorname{sign}(\mu_2) = \operatorname{sign}(\widetilde{K}_2)$.

Assume first $\widehat{U} < 0$. According to (7.9) the phase portrait of a system $(S_3^{(h)})$ corresponds to Ham_{12} (without real finite singularities). On the other hand as $\mu_0 = \mu_1 = \kappa = \kappa_1 = 0$, $\mu_2 > 0$, $\tilde{L} > 0$ and $\widetilde{K}_2 < 0$, according to [28, Table 4] at infinity the behavior of the trajectories corresponds to Figure 8, i.e. we arrive to the configuration $N, \binom{2}{2} H - H$.

Admit now $\widehat{U} > 0$. Then by (7.9) we obtain the phase portrait Ham_{20} if $\widehat{W_2} \neq 0$ and Ham_{21} if $\widehat{W_2} = 0$ in both cases having two finite integrable saddles. As regarding the configuration of infinite singularities we observe that $\mu_2 < 0$, $\tilde{L} > 0$ and $\widetilde{K} < 0$. So following [28, Table 4] we obtain Figure 9, i.e. we get the configuration $N, \binom{2}{2} P\ E\ P - P\ E\ P$. This leads to the total configuration 21 of Table 4. It remains to note that by (7.11) the condition $\widehat{W_2} = 0$ is equivalent to $B_3 = 0$.

Assume finally $\widehat{U} = 0$. Then $\alpha = b^2/4$ and for systems $(S_3^{(h)})$ we have

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = (ab + 2\beta)^2 x^4/4 = \widehat{V}. \tag{7.12}$$

According to (7.9) the phase portrait of a system $(S_3^{(h)})$ corresponds to Ham_{13} if $\widehat{V} \neq 0$ and we get degenerate systems (with the phase portrait Ham_{14}) if $\widehat{V} = 0$. We claim that in the first case at infinity we have the configuration $N, \binom{4}{2}\, PHP{-}PHP$. Indeed following [28] for systems $(S_3^{(h)})$ we have

$$\kappa = \kappa_1 = 0, \quad \tilde{L} = 24x^2 > 0, \quad \widetilde{K} = -4x^2 < 0, \quad \widetilde{R} = -8x^2 < 0$$

and according to [28, Table 4] at infinity the behavior of the trajectories corresponds to Figure 28; i.e., we arrive to the mentioned above configuration and our claim is proved.

It remains to observe that in the case $\widehat{V} = 0$ (then $\mu_4 = 0$) and the degenerate systems have the phase portrait Ham_{14}. Hence the singular invariant line coincides with the invariant line of the respective linear systems, and using our notations (see page 7) we get the configuration $\big(\ominus [|]; s\big); N, \big(\ominus [|]; N\big)$.

Considering that the condition $\widetilde{M} \neq 0$ for the family $(S_3^{(h)})$ holds and (7.11), we conclude that the conditions for determining the phase portraits from Tables 4 and 5 corresponding to the case $\mu_0 = \mu_1 = 0$ are equivalent to the respective conditions from (7.9) (i.e. the conditions for the phase portraits Ham_{12}, Ham_{13}, Ham_{20} and Ham_{21}).

7.4. **The family of systems** $(S_4^{(h)})$**.** For this family of systems we have $\eta = \mu_0 = \mu_1 = 0$ and $\widetilde{M} = 0$. According to [28, Table 2] at infinity we have one real singularity of total multiplicity at least five. So in order to determine exactly the configuration of the singularities at infinity we shall use again Table 4 from [28].

According to [16, Theorem 1] the phase portrait of a system from the family $(S_4^{(h)})$ is determined by the following affine invariant conditions

$$\begin{gathered} Vul_2 \Leftrightarrow \widehat{P} \neq 0,\ \widehat{U} > 0; \\ Ham_{11} \Leftrightarrow \begin{cases} \widehat{P} \neq 0,\ \widehat{U} < 0, \text{ or} \\ \widehat{P} = \widehat{U} = 0,\ \widehat{V} \neq 0, \text{ or} \\ \widehat{P} = \widehat{U} = \widehat{V} = 0, \widehat{W}_1 < 0; \end{cases} \\ Ham_{15} \Leftrightarrow \widehat{P} = \widehat{U} = \widehat{V} = 0, \widehat{W}_1 > 0; \\ Ham_{16} \Leftrightarrow \widehat{P} = \widehat{U} = \widehat{V} = \widehat{W}_1 = 0; \\ Ham_{17} \Leftrightarrow \widehat{P} \neq 0,\ \widehat{U} = 0; \\ Ham_{18} \Leftrightarrow \widehat{P} = 0,\ \widehat{U} \neq 0. \end{gathered} \tag{7.13}$$

For systems $(S_4^{(h)})$ calculations yield

$$\begin{gathered} \widehat{P} = c^4x^4, \quad \widehat{U} = \big[(ac-b^2)^2 + 4c(b\alpha + c\beta)\big](bx+cy)^2x^4, \\ \eta = \widetilde{M} = \mu_0 = \mu_1 = 0, \quad \mu_2 = c^2x^2, \quad \widetilde{K} = 0, \quad \mathbf{P} = \widehat{P}, \quad \mathbf{U} = \widehat{U}. \end{gathered} \tag{7.14}$$

7.4.1. *The case* $\widehat{P} \neq 0$. In this case according to (7.13) a system from the family $(S_4^{(h)})$ has one of the following phase portraits: Vul_2 (if $\widehat{U} > 0$), Ham_{11} (if $\widehat{U} < 0$) or Ham_{17} (if $\widehat{U} = 0$). In all the cases at infinity we have a multiple node, which according to [28, Table 4] is of multiplicity five, as the conditions $\mu_0 = \mu_1 = \widetilde{M} = \widetilde{K} = 0$ and $\mu_2 > 0$ (as $c \neq 0$) are verified. More precisely at infinity we have Figure 30, i.e. the nilpotent singular point $\widehat{\binom{2}{3}}\, N$.

7.4.2. *The case* $\widehat{P} = 0$. Then $c = 0$ and for systems $(S_4^{(h)})$ we have

$$\begin{gathered} \widehat{U} = b^6x^6, \quad \widehat{V} = \big[(\alpha^2 - ab\alpha - b^2\beta)x - b^2\alpha y\big]x^3, \\ \mu_2 = 0, \quad \mu_3 = b^3x^3, \quad \mathbf{U} = \widehat{U}, \quad \mu_4 = \widehat{V}, \quad \widetilde{K} = 0, \quad \widetilde{K}_1 = -bx^3. \end{gathered} \tag{7.15}$$

So if $\widehat{U} \neq 0$ (i.e., $b \neq 0$) according to (7.13) we get the phase portrait Ham_{18} with a finite integrable saddle. On the other hand as $\mu_3\widetilde{K}_1 = -b^4x^4 < 0$ according to [28, Table 4] the configuration of infinite singularities corresponds to Figure 33; i.e., $\binom{3}{3}\, PEPEP - P$.

Assume $\widehat{U} = 0$ (i.e., $b = 0$). In this case we have

$$\mu_3 = 0, \quad \mu_4 = \widehat{V} = \alpha^2x^4, \quad \widehat{W}_1 = (a^2 + 4\beta)x^4 - 4\alpha x^3 y = B_6, \quad \widetilde{K}_3 = 0 \tag{7.16}$$

and we shall consider two subcases: $\mu_4 \neq 0$ and $\mu_4 = 0$.

(1) If $\mu_4 \neq 0$ systems $(S_4^{(h)})$ are non-degenerate having the phase portrait Ham_{11} (see (7.13), as $\widehat{V} \neq 0$). Clearly at infinity we have a singular point of multiplicity seven. As $\mu_4 > 0$ and $\widetilde{K}_3 = 0$ according to [28, Table 4] at infinity we get Figure 30. More exactly we have the configuration $\binom{4}{3}\, P - P$ as the infinite singular point is intricate.

(2) Assume finally $\mu_4 = 0$, i.e. $\alpha = 0$. Then systems $(S_4^{(h)})$ with $c = b = \alpha = 0$ become degenerate possessing the phase portraits indicated respectively in (7.13). We observe that in the case of all the three phase portraits Ham_{11}, Ham_{15} and

Ham_{16} the systems $(S_4^{(h)})$ have an invariant singular conic which is reducible. More exactly, it splits into two parallel invariant lines which are real and distinct if $B_6 > 0$ (portrait Ham_{15}); they are complex if $B_6 < 0$ (portrait Ham_{11}), and they coincide if $B_6 = 0$ (portrait Ham_{16}). This leads to the respective configurations of singularities described in Table 4 (see rows No. 28–30).

It remains to note taking into account (7.14), (7.15) and (7.16) that for the family of systems $(S_4^{(h)})$ the respective conditions for determining the phase portraits from Tables 4 and 5 in all the cases considered above are equivalent to the respective conditions from (7.13).

As all the cases are considered Theorem 1.2 is proved.

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