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METHOD OF UPPER AND LOWER SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we show the existence and uniqueness of solutions for the boundary-value problems of fractional differential equations, using the upper and lower solutions method and monotone iterative algorithm. An example is also included to illustrate our results.

1. Introduction

In this article, we show the existence and uniqueness of solutions for the boundary value problems of the fractional differential equation

$$\begin{gathered} D^\delta u(t) - Mu(t) = f(t, u(t)), \quad t \in J,\ 0 < \delta < 1, \\ u(0) = ru(T), \end{gathered} \tag{1.1}$$

where $J = [0, T]$, $0 < T < +\infty$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $M \geq 0$, $0 < r < \frac{1}{E_{\delta,1}(MT^\delta)}$. $E_{n_1,n_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn_1+n_2)}$, $n_1, n_2 > 0$ is the Mittag-Leffler function (see [1, 2]). D^δ is the Caputo fractional derivative of order δ (see [1, 2]); that is,

$$D^\delta u(t) = \frac{1}{\Gamma(1-\delta)} \int_0^t (t-s)^{-\delta} u'(s) \mathrm{d}s.$$

The Riemann-Liouville fractional integral operator of order δ is defined by

$$I^\delta u(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} u(s) \mathrm{d}s.$$

Fractional differential equations are thought of an important research branch of fractional calculus, to which much attention has been paid. They arise in the models of many phenomena in various fields of science and engineering as a valuable tool. Indeed, we can find numerous applications in physics, chemistry, biology, etc. (See [3, 4]). Hence, some meaningful results on this kind of problems have been obtained, (see [5, 6, 7, 8, 9, 10, 11, 12, 13]).

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The theory of upper and lower solutions is known to be an effective method to deal with the boundary-value problems of the ordinary differential equations and functional differential equations, (see [14, 15, 16, 17, 18, 19, 20, 21, 22]). Li et al. [15, 16] found a direct method which is quite simple and practical to prove the existence and uniqueness of solutions of the second-order three-point boundary-value problem and gave some examples to illustrate the effectiveness of the result. In [18], using the upper and lower solutions method, the authors considered the periodic boundary-value problems for functional differential equations. In [22], the authors presented the existence of extreme solutions of the boundary-value problem for a class of first-order functional equations with a nonlinear boundary condition

$$\begin{gathered} u'(t) = f(t, u(t), u(\theta(t))), \\ g(u(0)) = ru(T), \end{gathered}$$

by the method of upper and lower solutions and monotone iterative techniques. One interesting thing is that the method is also appropriate for fractional differential equations, (see [24, 23]). In [23], Barrett proved the existence and uniqueness of solutions for the following initial value problems

$$\begin{gathered} (D^{\delta}_{a+}u)(t) - Mu(t) = f(t), \quad (n-1 < \delta < n), \\ (D^{\delta-k}_{a+}u)(a+) = b_k, \quad b_k \in C (k = 1, \dots, n). \end{gathered}$$

Zhang and Su [24] used the method of upper and lower solutions to study the existence for a linear fractional differential equation with nonlinear boundary condition

$$\begin{gathered} D^{\delta}u(t) - du(t) = h(t), \quad t \in J,\ 0 < \delta < 1, \\ g(u(0)) = u(T), \end{gathered} \tag{1.2}$$

where $d \geq 0$, $h \in C^1[0, T]$ is a given function, D^{δ} is a regularized fractional derivative (the Caputo derivative) of order $0 < \delta < 1$, and there exists a constant $\frac{dT^{\delta}}{\Gamma(\delta) - dT^{\delta}} < r_0 < 1$ such that $r_0 < g'(s) < 1$ for $s \in \mathbb{R}$. They presented an existence theorem for the boundary-value problem (1.2).

The boundary-value problem (1.1) which we study is different from the problem (1.2). We study a nonlinear fractional differential equation, and $0 < r < \frac{1}{E_{\delta,1}(MT^{\delta})}$. Since $E_{\delta,1}(MT^{\delta}) = \sum_{j=0}^{\infty} \frac{MT^j}{\Gamma(j\delta+1)} \geq \frac{1}{\Gamma(1)} = 1$, which implies $r \leq 1$. In fact, in the boundary-value problem (1.1), $g(s) = \frac{1}{r}s$, and $g'(s) = \frac{1}{r} \geq 1$, the condition $0 < r_0 < g'(s) < 1$ in (1.2) is not satisfied. The purpose of this paper is to prove the existence and uniqueness of solutions to the boundary-value problem (1.1) by the method of upper and lower solutions and monotone iterative techniques. We not only present the existence and the uniqueness theorem, but also present the iterative sequence for solving the solution and its error estimate formula under the condition of unique solution.

This article is organized as follows. In section 2, we introduce the basic properties, and some comparison principles are studied. In section 3, we consider the existence and uniqueness of the solution of a linear problem associated with (1.1). In section 4, we obtain the extreme solution of (1.1) and prove that there exists a unique solution of (1.1) by using the method of upper and lower solutions and monotone iterative technique. In section 5, we give an example to illustrate the results which have been proved.

2. Preliminaries and comparison principle

In this article, we use the following conditions:

(H0) $0 \leq \frac{MT^\delta}{\Gamma(\delta+1)} < 1$, $c(t) = 1 - \cos\frac{\pi t}{2T}$, $t \in J$.

The followings fundamental properties for fractional differential equations, which are necessary for our analysis.

Lemma 2.1 ([2, Example 4.9]). *The linear initial value problem*

$$\begin{gathered} D^\delta u(t) - Mu(t) = x(t), \quad t \in J, \\ u(0) = u_0, \end{gathered} \tag{2.1}$$

where M is a constant, has a unique solution

$$u(t) = u_0 E_{\delta,1}(Mt^\delta) + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta) x(s) \mathrm{d}s.$$

In particular, when $M = 0$, the initial problem (2.1) has the solution

$$u(t) = u_0 + \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} x(s) \mathrm{d}s.$$

Lemma 2.2 ([2, Lemma 2.22]). *If $u \in C(J)$ and $0 < \alpha < 1$, then $I^\alpha D^\alpha u(t) = u(t) - u(0)$.*

To investigate the boundary-value problem (1.1), for $x \in C(J)$, we consider the boundary-value problem

$$\begin{gathered} D^\delta u(t) - Mu(t) = x(t), \quad t \in J, \\ u(0) = ru(T). \end{gathered} \tag{2.2}$$

Definition 2.3. Let $\alpha \in C^1(J)$. We say that α is a lower solution of the boundary-value problem (2.2), if

$$D^\delta \alpha(t) - M\alpha(t) \leq x(t) - a_\alpha(t), \ t \in J,$$

where

$$a_\alpha(t) = \begin{cases} 0, & r\alpha(T) \geq \alpha(0), \\ \frac{1}{r}(D^\delta c(t) - Mc(t))(\alpha(0) - r\alpha(T)), & r\alpha(T) < \alpha(0). \end{cases} \tag{2.3}$$

Let $\beta \in C^1(J)$. We say that β is an upper solution of the boundary-value problem (2.2), if

$$D^\delta \beta(t) - M\beta(t) \geq x(t) + b_\beta(t), \ t \in J,$$

where

$$b_\beta(t) = \begin{cases} 0, & r\beta(T) \leq \beta(0), \\ \frac{1}{r}(D^\delta c(t) - Mc(t))(r\beta(T) - \beta(0)), & r\beta(T) > \beta(0), \end{cases} \tag{2.4}$$

where $c(t)$ is defined in (H0).

The following comparison principle will play a very important role in our main results.

Lemma 2.4. *Let* (H0) *hold. Assume that $u \in C^1(J)$ and satisfies*

$$\begin{gathered} D^\delta u(t) - Mu(t) \leq 0, \\ u(0) \leq 0. \end{gathered} \tag{2.5}$$

Then $u(t) \leq 0$ for $t \in J$.

Proof. Suppose this is false, then there exist $t_1, t_2 \in J$ such that $u(t) \le 0$ for $0 \le t \le t_1$ and $u(t) > 0$ for $t_1 < t \le t_2$. Let

$$u(t_0) = \max_{t\in[t_1,t_2]} u(t) > 0. \tag{2.6}$$

By (2.5) and Lemma 2.2, we can obtain

$$\begin{gathered} I^\delta D^\delta u(t) - I^\delta Mu(t) \le 0,\\ u(t) - u(0) - I^\delta Mu(t) \le 0,\\ u(t) - I^\delta Mu(t) \le 0. \end{gathered}$$

So

$$u(t_0) - MI^\delta u(t_0) \le 0.$$

Since

$$\begin{aligned} I^\delta u(t_0) &= \frac{1}{\Gamma(\delta)}\int_0^{t_1}(t_0-s)^{\delta-1}u(s)\mathrm{d}s + \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_0}(t_0-s)^{\delta-1}u(s)\mathrm{d}s\\ &\le \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_0}(t_0-s)^{\delta-1}u(s)\mathrm{d}s\\ &\le \frac{1}{\Gamma(\delta)}\int_{t_1}^{t_0}(t_0-s)^{\delta-1}u(t_0)\mathrm{d}s\\ &\le \frac{u(t_0)}{\delta\Gamma(\delta)}(t_0-t_1)^\delta\\ &\le \frac{u(t_0)T^\delta}{\Gamma(\delta+1)}, \end{aligned}$$

we have

$$u(t_0) - \frac{Mu(t_0)T^\delta}{\Gamma(\delta+1)} \le u(t_0) - MI^\delta u(t_0) \le 0;$$

that is,

$$u(t_0)(1 - \frac{MT^\delta}{\Gamma(\delta+1)}) \le 0.$$

It follows from (H0), that $1 - \frac{MT^\delta}{\Gamma(\delta+1)} > 0$, which contradicts $u(t_0) > 0$. This proves that $u(t) \le 0$ on J. The proof is complete. □

Lemma 2.5. *Suppose* (H0) *holds, and* $u \in C^1(J)$ *satisfies*

$$\begin{gathered} D^\delta u(t) - Mu(t) \le -a_\alpha(t), \quad t \in J,\\ u(0) \le 0. \end{gathered} \tag{2.7}$$

Then $u(t) \le 0$ *for* $t \in J$.

Proof. We consider the following two cases.

Case 1. $r\alpha(T) \ge \alpha(0)$. We have $a_\alpha(t) = 0$ if $r\alpha(T) \ge \alpha(0)$. By Lemma 2.4, we can obtain $u(t) \le 0$ for $t \in J$.

Case 2. $r\alpha(T) < \alpha(0)$. When $r\alpha(T) < \alpha(0)$, $a_\alpha(t) = \frac{1}{r}(D^\delta c(t) - Mc(t))(\alpha(0) - r\alpha(T))$.

Let $v(t) = u(t) + \frac{1}{r}c(t)(\alpha(0) - r\alpha(T))$. Obviously, $v(t) \geq u(t)$. Since $c(t) = 1 - \cos\frac{\pi t}{2T} \geq 0$ for $t \in J$, we can get that $v(t) \geq u(t)$ for all $t \in J$. Follows from (2.7) we have

$$\begin{aligned} D^\delta v(t) - Mv(t) &= D^\delta u(t) - Mu(t) + \frac{1}{r}(D^\delta c(t) - Mc(t))(\alpha(0) - r\alpha(T)) \\ &= D^\delta u(t) - Mu(t) + a_\alpha(t) \leq 0, \end{aligned}$$

and

$$v(0) = u(0) + c(0)(\alpha(0) - r\alpha(T)) = u(0) \leq 0.$$

In view of Lemma 2.4, $v(t) \leq 0$ for $t \in J$, which implies that $u(t) \leq 0$. This completes the proof. □

In a similar way, we can get the following lemma.

Lemma 2.6. *Suppose that* (H0) *holds,* $u \in C^1(J)$ *and satisfies*

$$\begin{gathered} D^\delta u(t) - Mu(t) \leq -b_\beta(t), \quad t \in J, \\ u(0) \leq 0. \end{gathered} \tag{2.8}$$

Then $u(t) \leq 0$*, for* $t \in J$*.*

Lemma 2.7 ([1, Theorem 4.1])**.** *Consider the two-parameter Mittag-Leffler function* E_{n_1,n_2} *for some* $n_1, n_2 > 0$*. The power series defining* $E_{n_1,n_2}(z)$ *is convergent for all* $z \in \mathbb{R}$*.*

Lemma 2.8 ([15, Lemma 2.5])**.** *Let* E *be a partially ordered Banach space,* $\{x_n\} \subset E$ *a monotone sequence and relatively compact set, then* $\{x_n\}$ *is convergent.*

Lemma 2.9 ([15, Lemma 2.6])**.** *Let* E *be a partially ordered Banach space,* $x_n \leq y_n$ $(n = 1, 2, 3 \dots)_0$*, if* $x_n \to x^*$*,* $y_n \to y_0^*$*, we have* $x^* \leq y^*$*.*

3. Existence and uniqueness of solutions for the Linear problems

In this section, we present the existence and uniqueness theorems of solutions of the boundary-value problem (2.2) based on the method of the upper and the lower solutions.

Lemma 3.1. *Let* (H0) *hold. Assume that there exist upper and lower solutions* $\beta, \alpha \in C^1(J)$ *of the boundary-value problem* (2.2) *such that* $\alpha(t) \leq \beta(t)$ *on* J*. Then the boundary-value problem* (2.2) *has a unique solution* u*. Moreover,* $\alpha \leq u \leq \beta$ *on* J*, respectively.*

Proof. (1) We can prove that the boundary-value problem (2.2) has a unique solution. Let

$$p(t) = \begin{cases} r\alpha(t), & r\alpha(T) \geq \alpha(0), \\ r\alpha(t) + c(t)(\alpha(0) - r\alpha(T)), & r\alpha(T) < \alpha(0), \end{cases}$$

and

$$q(t) = \begin{cases} r\beta(t), & r\beta(T) \leq \beta(0), \\ r\beta(t) - c(t)(r\beta(T) - \beta(0)), & r\beta(T) > \beta(0). \end{cases}$$

It is obvious that $p(0) = r\alpha(0)$ and $q(0) = r\beta(0)$.

If $r\alpha(T) \geq \alpha(0)$, then

$$p(T) = r\alpha(T) \geq \alpha(0) = \frac{p(0)}{r}.$$

If $r\alpha(T) < \alpha(0)$, then

$$p(T) = r\alpha(T) - r\alpha(T) + \alpha(0) = \alpha(0) = \frac{p(0)}{r}.$$

Thus, $rp(T) \geq p(0)$.

Analogously, we can get $rq(T) \leq q(0)$. Therefore,

$$p(0) = r\alpha(0), \quad rp(T) \geq p(0), \tag{3.1}$$

$$q(0) = r\beta(0), \quad rq(T) \leq q(0). \tag{3.2}$$

If $r\alpha(T) \geq \alpha(0)$, we obtain

$$D^\delta p(t) - Mp(t) = r(D^\delta \alpha(t) - M\alpha(t)) \leq rx(t), \quad t \in J,$$

and if $r\alpha(T) < \alpha(0)$, for $t \in J$, we have

$$\begin{aligned} D^\delta p(t) - Mp(t) &= r(D^\delta \alpha(t) - M\alpha(t)) + (D^\delta c(t) - Mc(t))(\alpha(0) - r\alpha(T)) \\ &\leq rx(t) - ra_\alpha(t) + ra_\alpha(t) = rx(t). \end{aligned}$$

Hence, we obtain

$$D^\delta p(t) - Mp(t) \leq rx(t), \quad t \in J. \tag{3.3}$$

Similarly, we can show that

$$D^\delta q(t) - Mq(t) \geq rx(t), \quad t \in J. \tag{3.4}$$

Let $y(t) = p(t) - q(t),\ t \in J$. It follows that

$$\begin{gathered} D^\delta y(t) - My(t) \leq 0, \\ y(0) = p(0) - q(0) \leq 0. \end{gathered}$$

From (3.1), (3.2), (3.3) and (3.4). By Lemma 2.4, we have

$$p(t) \leq q(t) \quad \text{for } t \in J. \tag{3.5}$$

For each $\lambda \in \mathbb{R}$, we consider the initial problem

$$\begin{gathered} D^\delta u(t) - Mu(t) = x(t), \\ u(0) = \lambda. \end{gathered} \tag{3.6}$$

According to Lemma 2.1, the initial problem (3.6) has a unique solution

$$u(t,\lambda) = \lambda E_{\delta,1}(Mt^\delta) + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta) x(s) \mathrm{d}s,\ t \in J. \tag{3.7}$$

It is easy to see $u(t,\lambda)$ is continuous on $\lambda \in \mathbb{R}$.

Let $z(t) = p(t) - ru(t,\lambda)$, where $u(t,\lambda)$ is the solution of (3.6). If $p(T) \leq \lambda \leq q(T)$, then

$$\begin{gathered} D^\delta z(t) - Mz(t) \leq rx(t) - rx(t) = 0, \\ z(0) = p(0) - ru(0,\lambda) \leq rp(T) - r\lambda \leq 0. \end{gathered}$$

From Lemma 2.4, we obtain that $z(t) \leq 0$ for $t \in J$, so $p(T) \leq ru(T,\lambda)$.

Using the same method, we can get $ru(T,\lambda) \leq q(T)$. Hence,

$$p(T) \leq ru(T,\lambda) \leq q(T) \quad \text{for } \lambda \in [p(T), q(T)].$$

Let $g(\lambda) = ru(T,\lambda) - \lambda$ for $\lambda \in \mathbb{R}$, then $g'(\lambda) = rE_{\delta,1}(MT^\delta) - 1 < 0$, and g is strictly decreasing. We see that the equation $g(\lambda) = 0$ has at most one solution on $\mathbb{R}$.

Since $g(q(T))g(p(T)) = (ru(T, q(T)) - q(T))(ru(T, p(T)) - p(T)) \leq 0$ and $g(\lambda)$ is continuous for $\lambda \in \mathbb{R}$, we can show that the equation $g(\lambda) = 0$ has a unique solution $\lambda_0 \in \mathbb{R}$ and $p(T) \leq \lambda_0 \leq q(T)$ with $ru(T, \lambda_0) = \lambda_0 = u(0)$. Hence,

$$u(t, \lambda_0) = \lambda_0 E_{\delta,1}(Mt^\delta) + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta)x(s)\mathrm{d}s, \quad t \in J$$

is the unique solution of the boundary-value problem (2.2).

(2) We can prove that the solution $u(t, \lambda_0)$ satisfies $\alpha(t) \leq u(t, \lambda_0) \leq \beta(t)$ for $t \in J$. Let $h(t) = \alpha(t) - u(t, \lambda_0)$. If $r\alpha(T) \geq \alpha(0)$, we know $a_\alpha(t) = 0$. By (3.2), we have

$$\begin{gathered} D^\delta h(t) - Mh(t) = D^\delta \alpha(t) - M\alpha(t) - x(t) \leq 0, \\ h(0) = \alpha(0) - u(0, \lambda_0) \leq p(T) - \lambda_0 \leq 0. \end{gathered}$$

In view of Lemma 2.4, we have $h(t) = \alpha(t) - u(t, \lambda_0) \leq 0$ for $t \in J$.

If $r\alpha(T) < \alpha(0)$, then $a_\alpha(t) = \frac{1}{r}(D^\delta c(t) - Mc(t))(\alpha(0) - r\alpha(T))$. By (3.2), it is easy to see that

$$\begin{gathered} D^\delta h(t) - Mh(t) = D^\delta \alpha(t) - M\alpha(t) - x(t) \leq -a_\alpha(t), \\ h(0) = \alpha(0) - u(0, \lambda_0) \leq p(T) - \lambda_0 \leq 0. \end{gathered}$$

As a consequence of Lemma 2.5, we obtain that $h(t) \leq 0$, for $t \in J$, this is $u(t, \lambda_0) \geq \alpha(t)$.

In a similar way, we can obtain that $\beta(t) \geq u(t, \lambda_0)$ for $t \in J$. Therefore, the unique solution $u(t, \lambda_0)$ of the boundary-value problem satisfies $\alpha(t) \leq u(t, \lambda_0) \leq \beta(t)$ for $t \in J$. The proof is complete. □

4. Main results

Definition 4.1. Let $\beta_0, \alpha_0 \in C^1(J)$. We say that β_0, α_0 are the upper solution and the lower solution of the boundary-value problem (1.1), respectively, if

$$D^\delta \alpha_0(t) - M\alpha_0(t) \leq f(t, \alpha_0(t)) - a_{\alpha_0}(t),\ t \in J,$$

and

$$D^\delta \beta_0(t) - M\beta_0(t) \geq f(t, \beta_0(t)) + b_{\beta_0}(t),\ t \in J,$$

where $a_{\alpha_0}(t), b_{\beta_0}(t)$ are defined in (2.3), (2.4), respectively.

Let $E = C(J)$ with $\|x\| = \max_{t\in J} |x(t)|$ for $x \in E$. Then E is a Banach space.

Theorem 4.2. *Suppose* (H0) *holds and there exist $\beta_0, \alpha_0 \in C^1(J)$ such that β_0, α_0 are upper and lower solutions of the boundary-value problem* (1.1) *with $\alpha_0(t) \leq \beta_0(t)$ for $t \in J$, respectively, and f satisfies*

(H1) *$f(t, x_1) \leq f(t, x_2)$ for any $x_1,\ x_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $t \in J$.*

Then the boundary-value problem (1.1) *has a minimal solution α^* and a maximal solution β^* on $[\alpha_0, \beta_0] = \{u \in C(J)|\ \alpha_0(t) \leq u(t) \leq \beta_0(t),\ t \in (J)\}$. Moreover, the monotone iterative sequences defined by*

$$\begin{aligned} \alpha_n(t) = \frac{rE_{\delta,1}(Mt^\delta)}{1 - rE_{\delta,1}(MT^\delta)} \int_0^T (T-s)^{\delta-1} E_{\delta,\delta}(M(T-s)^\delta) f(s, \alpha_{n-1}(s))\mathrm{d}s \\ + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta) f(s, \alpha_{n-1}(s))\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned}\beta_n(t) &= \frac{rE_{\delta,1}(Mt^\delta)}{1-rE_{\delta,1}(MT^\delta)}\int_0^T (T-s)^{\delta-1}E_{\delta,\delta}(M(T-s)^\delta)f(s,\beta_{n-1}(s))\mathrm{d}s\\ &\quad+\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}(M(t-s)^\delta)f(s,\beta_{n-1}(s))\mathrm{d}s,\end{aligned}$$

converge uniformly on J to α^* and β^*, respectively. Namely, for $t\in J$, $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ with

$$\lim_{n\to\infty}\alpha_n(t)=\alpha^*(t),\quad \lim_{n\to\infty}\beta_n(t)=\beta^*(t).$$

Proof. We divide the proof into five parts.

(1) We denote $D=[\alpha_0,\beta_0]$. For any $\varphi\in D$, we consider first the boundary-value problem

$$\begin{gathered}D^\delta u(t)-Mu(t)=f(t,\varphi(t)),\\ u(0)=ru(T).\end{gathered}\tag{4.1}$$

Since $\beta_0(t),\alpha_0(t)$ are upper and lower solutions of the boundary-value problem (1.1), by (H1), for $t\in J$, we have

$$D^\delta\alpha_0(t)-M\alpha_0(t)\le f(t,\alpha_0(t))-a_{\alpha_0}(t)\le f(t,\varphi(t))-a_{\alpha_0}(t),$$

and

$$D^\delta\beta_0(t)-M\beta_0(t)\ge f(t,\beta_0(t))+b_{\beta_0}(t)\ge f(t,\varphi(t))+b_{\beta_0}(t).$$

Therefore, $\beta_0(t),\alpha_0(t)$ are also the upper and lower solutions of the boundary-value problem (4.1). In view of Lemma 3.1, the boundary-value problem (4.1) has the following unique solution u with $u\in D$.

$$u(t)=u(0)E_{\delta,1}(Mt^\delta)+\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}(M(t-s)^\delta)f(s,\varphi(s))\mathrm{d}s,\quad t\in J.$$

Because $u(0)=ru(T)$, we can easily obtain that

$$u(0)=\frac{r}{1-rE_{\delta,1}(MT^\delta)}\int_0^T (T-s)^{\delta-1}E_{\delta,\delta}(M(T-s)^\delta)f(s,\varphi(s))\mathrm{d}s.$$

We define an operator A: $D\to E$ by

$$\begin{aligned}A\varphi(t) &= \frac{rE_{\delta,1}(Mt^\delta)}{1-rE_{\delta,1}(MT^\delta)}\int_0^T (T-s)^{\delta-1}E_{\delta,\delta}(M(T-s)^\delta)f(s,\varphi(s))\mathrm{d}s\\ &\quad+\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}(M(t-s)^\delta)f(s,\varphi(s))\mathrm{d}s.\end{aligned}$$

By Lemma 3.1 we can show that $\alpha_0(t)\le A\varphi(t)\le\beta_0(t)$, for $t\in J$. Hence, we obtain

$$A\alpha_0\ge\alpha_0,\quad A\beta_0\le\beta_0.\tag{4.2}$$

(2) We prove that A is completely continuous. Since D is bounded, there exists a constant $M_0>0$ such that $\|\varphi\|\le M_0$ for $\varphi\in D$. Because f is continuous, there exists a constant $M_1>0$ such that $\max_{s\in J}|f(s,\varphi(s))|\le M_1$ for $s\in J$. We have that

$$\begin{aligned}|A\varphi(t)| &\le \frac{rE_{\delta,1}(MT^\delta)}{1-rE_{\delta,1}(MT^\delta)}M_1E_{\delta,\delta}(MT^\delta)\int_0^T (T-s)^{\delta-1}\mathrm{d}s\\ &\quad+M_1E_{\delta,\delta}(MT^\delta)\int_0^t (t-s)^{\delta-1}\mathrm{d}s\end{aligned}$$

$$\le \frac{M_1 T^\delta E_{\delta,\delta}(MT^\delta)}{\delta(1 - rE_{\delta,1}(MT^\delta))}.$$

So $A(D)$ is uniformly bounded.

Let $F(t) = E_{\delta,1}(Mt^\delta)$ for $t \in J$, $G(t,s) = E_{\delta,\delta}(M(t-s)^\delta)$ for $(t,s) \in \Omega = \{(t,s)| \ t \in J, \ 0 \le s \le t\}$. As a consequence of Lemma 2.7, we have F and G are uniformly continuous on J and Ω, respectively.

For any $\varepsilon > 0$, there exists $\delta_1 > 0$, for any $t_1, t_2 \in J$ with $t_2 > t_1$, whenever $|t_2 - t_1| < \delta_1$, we obtain

$$|F(t_2) - F(t_1)| < \frac{\varepsilon\delta(1 - rE_{\delta,1}(MT^\delta))}{3rM_1 T^\delta E_{\delta,\delta}(MT^\delta)},$$

moreover, if (t_1, s), $(t_2, s) \in \Omega$, then

$$|G(t_2, s) - G(t_1, s)| < \frac{\varepsilon\delta}{6M_1 T^\delta}.$$

We take $0 < \delta_0 \le \min\{\delta_1, \delta_2, (\frac{\varepsilon\delta}{6M_1 E_{\delta,\delta}(MT^\delta)})^{\frac{1}{\delta}}\}$. Thus, as $t_1, t_2 \in J$ with $t_2 > t_1$, whenever $|t_2 - t_1| < \delta_0$ and $\varphi \in D$, we have

$$\begin{aligned}
&|A\varphi(t_2) - A\varphi(t_1)| \\
&\le \Big| \frac{r\int_0^T (T-s)^{\delta-1} E_{\delta,\delta}(M(T-s)^\delta) f(s,\varphi(s)) \mathrm{d}s}{1 - rE_{\delta,1}(MT^\delta)} \cdot \big(F(t_2) - F(t_1)\big) \Big| \\
&\quad + \Big| \int_{t_1}^{t_2} (t_2 - s)^{\delta-1} E_{\delta,\delta}(M(t_2 - s)^\delta) f(s, \varphi(s)) \mathrm{d}s \Big| \\
&\quad + \Big| \int_0^{t_1} \big((t_2 - s)^{\delta-1} E_{\delta,\delta}(M(t_2 - s)^\delta) - (t_1 - s)^{\delta-1} E_{\delta,\delta}(M(t_1 - s)^\delta)\big) f(s, \varphi(s)) \mathrm{d}s \Big| \\
&\le \frac{\varepsilon}{3} + \frac{M_1 E_{\delta,\delta}(MT^\delta)}{\delta} \cdot |t_2 - t_1|^\delta \\
&\quad + M_1 \int_0^{t_1} (t_2 - s)^{\delta-1} |E_{\delta,\delta}(M(t_2 - s)^\delta) - E_{\delta,\delta}(M(t_1 - s)^\delta)| \mathrm{d}s \\
&\quad + M_1 \int_0^{t_1} \big((t_1 - s)^{\delta-1} - (t_2 - s)^{\delta-1}\big) E_{\delta,\delta}(M(t_1 - s)^\delta) \mathrm{d}s.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^{t_1} (t_2 - s)^{\delta-1} |E_{\delta,\delta}(M(t_2 - s)^\delta) - E_{\delta,\delta}(M(t_1 - s)^\delta)| \mathrm{d}s \\
&= \int_0^{t_1} (t_2 - s)^{\delta-1} |G(t_2, s) - G(t_1, s)| \mathrm{d}s \\
&\le \frac{T^\delta}{\delta} \cdot \frac{\varepsilon\delta}{6M_1 T^\delta} = \frac{\varepsilon}{6M_1},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{t_1} \big((t_1 - s)^{\delta-1} - (t_2 - s)^{\delta-1}\big) E_{\delta,\delta}(M(t_1 - s)^\delta) \mathrm{d}s \\
&\le E_{\delta,\delta}(MT^\delta) \int_0^{t_1} \big((t_1 - s)^{\delta-1} - (t_2 - s)^{\delta-1}\big) \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{E_{\delta,\delta}(MT^\delta)}{\delta}\big((t_2-t_1)^\delta-(t_2^\delta-t_1^\delta)\big)\\
&\le \frac{E_{\delta,\delta}(MT^\delta)}{\delta}(t_2-t_1)^\delta \le \frac{\varepsilon}{6M_1}.
\end{aligned}
$$

Therefore, as $t_1, t_2 \in J$ with $t_2 > t_1$, whenever $|t_2 - t_1| < \delta_0$ and $\varphi \in D$, we can show that

$$|A\varphi(t_2) - A\varphi(t_1)| < \varepsilon.$$

We prove A is equi-continuous. By Arzela-Ascoli theorem, we know that $A(D)$ is relatively compact. We can easily show that A is continuous since f is continuous. Hence, A is completely continuous.

(3) A is an increasing operator on J. For $\alpha_0 \le \omega_1 \le \omega_2 \le \beta_0$, $t \in J$, let

$$B(s) = f(s, \omega_2(s)) - f(s, \omega_1(s)),\ s \in J.$$

From (H1), we have $B(s) \ge 0$ and

$$
\begin{aligned}
A\omega_2(t) - A\omega_1(t) &= \frac{rE_{\delta,1}(Mt^\delta)}{1-rE_{\delta,1}(MT^\delta)}\int_0^T (T-s)^{\delta-1}E_{\delta,\delta}(M(T-s)^\delta)B(s)\mathrm{d}s\\
&\quad+\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}(M(t-s)^\delta)B(s)\mathrm{d}s \ge 0.
\end{aligned}
$$

Thus, A is an increasing operator.

(4) Let $\alpha_n = A\alpha_{n-1}$, $\beta_n = A\beta_{n-1}$, for $n = 1, 2, \dots$. By (4.2), we get monotone iterative sequences

$$\alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \dots \le \beta_n \le \dots \le \beta_2 \le \beta_1.$$

As $\{\alpha_n\}$, $\{\beta_n\} \subset A(D)$, we get that $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone sequences and relatively compact set respectively. In view of Lemma 2.8, we can obtain that there exist α^*, $\beta^* \in C(J)$ such that

$$\lim_{n\to\infty}\alpha_n(t) = \alpha^*(t), \quad \lim_{n\to\infty}\beta_n(t) = \beta^*(t).$$

By the continuity of A, we have $\alpha^* = A\alpha^*$, $\beta^* = A\beta^*$. So α^*, β^* are the fixed points of A.

It is clear that u is a solution of the boundary-value problem (1.1) if and only if u is a fixed point of A. Hence, $\alpha^*(t), \beta^*(t)$ are solutions of the boundary-value problem (1.1).

(5) We prove that α^*, β^* are the minimal solution and the maximal solution of the boundary-value problem (1.1), respectively.

Assume $u \in [\alpha_0, \beta_0]$ is a solution of the boundary-value problem (1.1). We can easily obtain that $A\alpha_0(t) \le Au(t) \le A\beta_0(t)$ by the fact that A is increasing in $[\alpha_0, \beta_0]$. That is $\alpha_1(t) \le u(t) \le \beta_1(t)$. Doing this repeatedly, we have $\alpha_n(t) \le u(t) \le \beta_n(t)$, for $n = 1, 2, \dots$. From Lemma 2.9, we obtain that $\alpha^*(t) \le u(t) \le \beta^*(t)$, as $n \to \infty$.

Therefore, α^*, β^* are the minimal solution and the maximal solution of the boundary-value problem (1.1), respectively. The proof is complete. □

Theorem 4.3. *Suppose the conditions of Theorem 4.2 hold. There exists a constant* $\gamma \in \big[0, \frac{\delta(1-rE_{\delta,1}(MT^\delta))}{T^\delta E_{\delta,\delta}(MT^\delta)}\big)$ *and* f *satisfies*

(H2) $f(t, x_2) - f(t, x_1) \le \gamma(x_2 - x_1)$ *for any* $x_1, x_2 \in \mathbb{R}$ *and* $x_1 \le x_2$.

Then the boundary-value problem (1.1) *has a unique solution* u^* *on* $[\alpha_0, \beta_0]$. *Moreover, for each* $u_0 \in [\alpha_0, \beta_0]$, *the iterative sequence*

$$\begin{aligned} u_n(t) = & \frac{rE_{\delta,1}(Mt^\delta)}{1 - rE_{\delta,1}(MT^\delta)} \int_0^T (T-s)^{\delta-1} E_{\delta,\delta}(M(T-s)^\delta) f(s, u_{n-1}(s)) \mathrm{d}s \\ & + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta) f(s, u_{n-1}(s)) \mathrm{d}s, \ n = 1, 2, \dots \end{aligned}$$

converges uniformly to u^* *on* J. *Its error estimate is*

$$\|u_n - u^*\| \le \rho^n \|\beta_0 - \alpha_0\|, \ n = 1, 2, 3 \dots,$$

where

$$\rho = \frac{\gamma T^\delta E_{\delta,\delta}(MT^\delta)}{\delta(1 - rE_{\delta,1}(MT^\delta))}.$$

Proof. For $\alpha_0 \le \omega_1 \le \omega_2 \le \beta_0$, $t \in J$, from (H2) we have

$$\begin{aligned} & 0 \le A\omega_2(t) - A\omega_1(t) \\ & \le \frac{rE_{\delta,1}(MT^\delta)}{1 - rE_{\delta,1}(MT^\delta)} \int_0^T (T-s)^{\delta-1} E_{\delta,\delta}(M(T-s)^\delta)(f(s,\omega_2(s)) - f(s,\omega_1(s))) \mathrm{d}s \\ & \quad + \int_0^t (t-s)^{\delta-1} E_{\delta,\delta}(M(t-s)^\delta)(f(s,\omega_2(s)) - f(s,\omega_1(s))) \mathrm{d}s \\ & \le \Big(\frac{T^\delta E_{\delta,1}(MT^\delta) E_{\delta,\delta}(MT^\delta)}{\delta(\frac{1}{r} - E_{\delta,1}(MT^\delta))} + \frac{T^\delta}{\delta} E_{\delta,\delta}(MT^\delta) \Big) \gamma \|\omega_2(t) - \omega_1(t)\| \\ & = \frac{\gamma T^\delta E_{\delta,\delta}(MT^\delta)}{\delta(1 - rE_{\delta,1}(MT^\delta))} \|\omega_2(t) - \omega_1(t)\|. \end{aligned}$$

Hence,

$$\|A\omega_2 - A\omega_1\| \le \rho \|\omega_2 - \omega_1\|.$$

It is easy to obtain that

$$\|\beta_n - \alpha_n\| = \|A\beta_{n-1} - A\alpha_{n-1}\| \le \rho \|\beta_{n-1} - \alpha_{n-1}\| \le \dots \le \rho^n \|\beta_0 - \alpha_0\|.$$

From $0 \le \gamma < \frac{\delta(1 - rE_{\delta,1}(MT^\delta))}{T^\delta E_{\delta,\delta}(MT^\delta)}$, we have $0 \le \rho < 1$, and

$$\|\beta_n - \alpha_n\| \to 0, \quad n \to \infty.$$

It follows from Theorem 4.2, there exists a unique $u^* \in [\alpha_0, \beta_0]$ such that $\alpha_n \to u^*$, $\beta_n \to u^*$, when $n \to \infty$. As $\alpha_n \le u^* \le \beta_n$, by the monotonicity of A we have $A\alpha_n \le u^* \le A\beta_n$. When $n \to \infty$, we have that $u^* = Au^*$, u^* is a fixed point of A. Hence, u^* is the unique solution of the boundary-value problem (1.1).

For each $u_0 \in [\alpha_0, \beta_0]$, about the iterative sequence $u_n = Au_{n-1}$, we have $\alpha_n \le u_n \le \beta_n$, and

$$\|u_n - u^*\| \le \|\beta_n - \alpha_n\| \le \rho^n \|\beta_0 - \alpha_0\|.$$

This completes the proof. □

5. Example

Consider the boundary-value problem

$$\begin{aligned} D^{1/2}u(t) &= \frac{1}{2} + \frac{\cos t \arctan u(t)}{\pi(t+6)^2}, \quad 0 \le t \le 1, \\ 3u(0) &= u(1). \end{aligned} \tag{5.1}$$

Let $f(t,x) = \frac{1}{2} + \frac{\cos t \arctan x}{\pi(t+6)^2}$. Clearly, $T = 1$, $\delta = \frac{1}{2}$, $r = \frac{1}{3}$, $M = 0$ and $0 \le f(t,x) \le 1$. We take $\gamma = \frac{1}{36\sqrt{\pi}}$, then $E_{\delta,\delta}(MT^\delta) = \frac{1}{\sqrt{\pi}}$, $E_{\delta,1}(MT^\delta) = 1$, and

$$\rho = \frac{\gamma T^\delta E_{\delta,\delta}(MT^\delta)}{\delta(1 - rE_{\delta,1}(MT^\delta))} = \frac{1}{12} < 1.$$

Obviously, $\alpha_0(t) = 0$ is a lower solution of the boundary-value problem (5.1). Let $\beta_0(t) = \frac{5}{8\sqrt{\pi}}(1 + 2\sqrt{t})$, we have $\frac{1}{3}\beta_0(1) = \beta_0(0)$, then $b_{\beta_0}(t) = 0$.

$$D^{1/2}\beta_0(t) = \frac{5}{8} \ge f(t, \beta_0(t)). \tag{5.2}$$

So β_0 is an upper solution of the boundary-value problem (5.1).

It is easy to verify that the assumptions of Theorem 4.3 are satisfied. Hence, the boundary-value problem (5.1) has a unique solution $u^*(t)$ with $0 \le u^*(t) \le \frac{5}{8\sqrt{\pi}}(1 + 2\sqrt{t})$ on $[0,1]$.

For each $0 \le u_0(t) \le \frac{5}{8\sqrt{\pi}}(1 + 2\sqrt{t})$, let the iterative sequence $u_n = Au_{n-1}$, we have

$$u_n(t) = \frac{1}{2\sqrt{\pi}} \int_0^1 (1-s)^{-\frac{1}{2}} f(s, u_{n-1}(s))\mathrm{d}s + \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-\frac{1}{2}} f(s, u_{n-1}(s))\mathrm{d}s$$

converges uniformly to u^* on J. Its error estimate is

$$\|u_n - u^*\| \le \frac{15}{8\sqrt{\pi}} \cdot \big(\frac{1}{12}\big)^n.$$

We take $u_0 = 0$. If $n = 2$, its error is not more than 0.00734622; if $n = 3$, its error is not more than 0.000612185; if $n = 4$, its error is not more than 0.0000510154.

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