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EXISTENCE OF BOUND STATE SOLUTIONS FOR DEGENERATE SINGULAR PERTURBATION PROBLEMS WITH SIGN-CHANGING POTENTIALS

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Abstract. In this article, we study the degenerate singular perturbation problems

$$-\varepsilon^2 \operatorname{div}(|x|^{-2a}\nabla u) + |x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u),$$
$$-\operatorname{div}(|x|^{-2a}\nabla u) + \lambda|x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u),$$

for ε small and λ large positive, where $x \in \mathbb{R}^N$ with $N \geqslant 3$. We search for solutions that decay to zero as $|x| \to +\infty$, when g is superlinear in the potential function changes signs. We prove the existence of bound state solutions for degenerate, singular, semilinear elliptic problems. Additionally, when the nonlinearity $g(x,u)$ is an odd function of u, we obtain infinitely many geometrically distinct solutions.

1. Introduction and main results

This work concerns the study the degenerate singular perturbation problems

$$-\varepsilon^2 \operatorname{div}(|x|^{-2a}\nabla u) + |x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u), \tag{1.1}$$

for small $\varepsilon \in \mathbb{R}^+$, and

$$-\operatorname{div}(|x|^{-2a}\nabla u) + \lambda|x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u), \tag{1.2}$$

for large $\lambda \in \mathbb{R}^+$. We consider the case where $x \in \mathbb{R}^N$ for $N \geqslant 3$ and we search for decaying solutions, i.e., solutions such that $u(x) \to 0$ as $|x| \to +\infty$. The other parameters are such that

$$0 \leqslant a < (N-2)/2, \quad \text{and} \quad a \leqslant b < a+1. \tag{1.3}$$

Additionally, we define

$$2^*(a,b) := \frac{2N}{[N-2(a+1-b)]}, \quad 2^* = 2^*(a,a) := \frac{2N}{N-2}.$$

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We are mainly interested in a superlinear, critical nonlinearity g and in a sign-changing potential V.

To present a brief historical review, we write the two equations as

$$-\varepsilon^2 \operatorname{div}(|x|^{-2a}\nabla u) + |x|^{-2(a+1-c)}V(x)u = |x|^{-b2^*(a,b)}g(x,u), \tag{1.4}$$

$$-\operatorname{div}(|x|^{-2a}\nabla u) + \lambda|x|^{-2(a+1-c)}V(x)u = |x|^{-b2^*(a,b)}g(x,u). \tag{1.5}$$

Clearly, problems (1.1) and (1.2) correspond to the above equations with $c = 0$.

These type of problems come from the study of standing waves in anisotropic Schrödinger equation. The transition from quantum mechanics to classical mechanics can be formally realised by letting $\varepsilon \to 0$; therefore, the existence of solutions for ε small is of physical interest. Aside from being one of the main objects of quantum physics, the Schrödinger equation also appears in problems of nonlinear optics, in plasma physics, and in condensed matter physics, where one simulates the interaction effect among many particles through a nonlinear term. Moreover, several physical phenomena related to equilibrium of anisotropic continuous media that possibly are "perfect" insulators can be modeled by this type of elliptic problem, where it is allowed for the coefficient of the operator to be unbounded. For more details, the reader is referred to the papers [3, 12, 15, 19, 20, 31, 41, 48], to the book [2] and to the excellent article [42].

In the case $a = 0$, $b = 0$, $c = 1$, one expects problem (1.4) to possess nontrivial solutions $u \in H^1(\mathbb{R}^N)$ if $\liminf_{|x|\to+\infty} V(x) > 0$ and imposing a superlinear type condition on g, and this is independent whether or not the potential V changes sign. For example, it is known by variational arguments that the problem with homogeneous Dirichlet boundary condition on a smooth, bounded domain $\Omega \subset \mathbb{R}^N$ with $2 < p < 2^*$ always has solutions $u \in H^1(\Omega)$ and this is independent of the sign of V. See, for example, the papers [4, 23, 24, 30, 32, 33, 34, 43, 44, 46], where the main ideas to prove existence results rely in an essential way on the nondegeneracy of the critical points of the potential V. Regarding problem (1.2) with parameters $a = 0$, $b = 0$, and $c = 1$, see the papers [7, 8, 9, 13, 14, 16, 17, 37, 40].

Still in the case $a = 0$, $b = 0$, and $c = 1$, but with $\min_{x\in\mathbb{R}^N} V(x) = 0$ there are considerably less results. See, for example, the papers [16, 25].

In the case of the Schrödinger problem with a sign-changing potential V, the corresponding energy functional is indefinite, i.e., is neither bounded from above nor bounded from below; consequently, it does not have the geometry of the mountain-pass theorem or some of its variants. This has stimulated the development of new approaches to the problem. See, for example, the papers [26, 27, 28, 29].

In the case $V(x) := 1$, $g(x,u) := |u|^{2^*(a,b)-2}u$, $c = 0$, and where at least one of the parameters a or b is different from zero, problems (1.4) and (1.5) are closely related. In fact, when $\varepsilon^2 = \lambda^{-1}$, then u is a solution to problem (1.5) if and only if $v(x) = \lambda^{-1/(2^*(a,b)-2)}u(x)$ is a solution to problem (1.4). Hence, as far as the existence and the number of solutions are concerned, problems (1.4) and (1.5) are equivalent. However, for more general perturbations g this is no longer true.

The degenerate cases mentioned in the previous paragraph have the following variational structure. Let the space $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ be defined as the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm given by

$$\|u\| := \Big[\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \,\mathrm{d}x\Big]^{1/2}.$$

We define $S(a,b)$ as the best constant of the Sobolev embedding; that is,

$$S(a,b) := \inf_{u\in\mathcal{D}^{1,2}_a(\mathbb{R}^N),\, u\not\equiv 0} \frac{\int_{\mathbb{R}^N}|x|^{-2a}|\nabla u|^2\,\mathrm{d}x + \lambda\int_{\mathbb{R}^N}|x|^{-2(a+1)}|u|^2\,\mathrm{d}x}{\big[\int_{\mathbb{R}^N}|x|^{-b2^*(a,b)}|u|^{2^*(a,b)}\,\mathrm{d}x\big]^{2/2^*(a,b)}}.$$

It is well known that if the value of $S(a,b)$ is attained by a function $u\in\mathcal{D}^{1,2}_a(\mathbb{R}^N)$, then a convenient multiple of u is a weak solution to problem (1.2). For results of existence and non-existence of solutions, see the papers [5, 10, 11, 21, 22, 38, 39, 48].

Inspired by Ding and Szulkin in [27], we prove results of existence of bound state solutions to the case of degenerate, singular, semilinear elliptic problems. We look for solutions to problems (1.1) and (1.2) in the space $\mathcal{D}^{1,2}_a(\mathbb{R}^N)$. Additionaly, in the case where the nonlinearity $g(x,u)$ is an odd function of u, we obtain infinitely many geometrically distinct solutions.

To state our hypotheses we set

$$G(x,u) := \int_0^u g(x,s)\,\mathrm{d}s \quad\text{and}\quad \widetilde{G}(x,u) := \frac{g(x,u)u}{2} - G(x,u);$$

we also assume the following conditions on the potential V and on the perturbation g.

- (V1) $V\in C(\mathbb{R}^N,\mathbb{R})$ and V is bounded from below.
- (V2) There exists $b>0$ such that the set $\{x\in\mathbb{R}^N : V(x)<b\}$ is non-empty and has finite measure.
- (G1) $g\in C(\mathbb{R}^N,\mathbb{R})$, $G(x,u)\geqslant 0$ for all (x,u) and $g(x,u)=o(u)$ uniformly in x as $u\to 0$.
- (G2) $G(x,u)/u^2\to+\infty$ uniformly in x as $|u|\to+\infty$.
- (G3) $\widetilde{G}(x,u)>0$ whenever $u\neq 0$.
- (G4) $|g(x,u)|^\tau \leqslant a_1\widetilde{G}(x,u)|u|^\tau$ for some $a_1>0$, $\tau>\max\{1,N/2\}$, and for all (x,u) with $|u|$ large enough.

The main results of our work can be stated as follows.

Theorem 1.1. *Suppose that assumptions* (V1), (V2), (G1)–(G4) *are satisfied and that $V^{-1}(0)$ has a nonempty and bounded interior Ω. Suppose that conditions* (1.3) *on the parameters a and b are also satisfied.*

1. *If $G(x,u)\geqslant a_0|u|^\delta$ for some $a_0>0$, for some $2<\delta<2^*$ and for all $|u|$ small enough, then there exists $\varepsilon_0>0$ such that problem* (1.1) *has at least one nontrivial solution whenever $\varepsilon\in(0,\varepsilon_0)$. Moreover, if g is odd in u, then for each $k\geqslant 1$ there exists $\varepsilon_k>0$ such that problem* (1.1) *has at least k pairs of nontrivial solutions whenever $\varepsilon\in(0,\varepsilon_k)$.*
2. *There exists $\Lambda_0>0$ such that problem* (1.2) *has at least one nontrivial solution whenever $\lambda>\Lambda_0$. Moreover, if g is odd in u, then for each $k\geqslant 1$ there exists $\Lambda_k>0$ such that problem* (1.2) *has at least k pairs of nontrivial solutions whenever $\lambda>\Lambda_k$.*

Theorem 1.2. *Suppose that assumptions* (V1), (V2), (G1)–(G4) *and the conditions* (1.3) *on the parameters a and b are satisfied.*

1. *If $V(x)<0$ for some x and $G(x,u)\geqslant a_0|u|^\delta$ for some $a_0>0$, for some $2<\delta<2^*$ and for all $|u|$ small enough, then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset\mathbb{R}$ with $\varepsilon_k\to 0$ such that problem* (1.1) *has a nontrivial solution for each $\varepsilon=\varepsilon_k$.*

(2) *If $V(x) < 0$ for some x, then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\lambda_k \to +\infty$ such that problem* (1.2) *has a nontrivial solution for each $\lambda = \lambda_k$.*

The present work has two main goals. The first one is to show the decaying of the weak solutions $u \in \mathcal{D}_a^{1,2}(\Omega)$ to problems (1.1) and (1.2) defined for open, bounded domains $\Omega \subset \mathbb{R}^N$. It is worth mentioning that the ideas used by Ding and Szulkin in [27] to get this result do not apply here. The second one is to prove a concentration-compactness lemma similar to the one by Ackermann in [1]. To the best of our knowledge, there is no similar result in the literature.

This article is organized as follows. In section 2 we present an auxiliary variational problem; then we use a result by Vassilev in [47, Theorem 2.9] to prove the decaying of the weak solutions (Proposition 2.1). We also state two well known results that guarantee the existence of critical points for the energy functional, namely, the Linking Theorem by Rabinowitz (Proposition 2.2) and a least bound to the number of pairs of critical points of some functionals by Bartolo, Benci and Fortunato (Proposition 2.3). In section 3 we state and prove several technical lemmas, the two major results being a lemma on the boundedness of the Cerami sequences for the energy functional (Lemma 3.3) and the crucial lemma that can be used to prove the convergence of subsequences of Palais-Smale sequences (Lemma 3.4). In section 4 we concatenate the previous results to prove the theorems.

2. A variational problem and some preliminary results

We begin by making some remarks about the hypotheses on the perturbation g. If assumptions (G1) and (G4) hold, then $|g(x,u)|^\tau \leqslant \frac{1}{2}\, a_1 |g(x,u)||u|^{\tau+1}$ for $|u|$ large enough; hence, g satisfies the growth restriction

$$|g(x,u)| \leqslant a_2(|u| + |u|^{p-1}) \tag{2.1}$$

where $2 < p = 2\tau/(\tau-1) < 2^*$. On the other hand, if g satisfies inequality (2.1) with $2 < p < 2^*$ and the Ambrosetti-Rabinowitz superlinearity condition

$$0 < \mu G(x,u) \leqslant g(x,u)u \tag{2.2}$$

for some $\mu > 2$ and for all (x,u) with $u \neq 0$, then it is easy to see that assumptions (G2) and (G3) hold, and so does (G4). We will also see that assumptions (G2) to (G4) imply that $\widetilde{G}(x,u) \to +\infty$ as $|u| \to +\infty$.

In addition to problems (1.1) and (1.2), we will consider the problem

$$-\operatorname{div}(|x|^{-2a}\nabla u) + |x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u), \tag{2.3}$$

with $V = V^+ - V^-$, where $V^\pm \geqslant 0$, verifying assumptions (V1), (V2), and g verifying (G1) to (G4). Let

$$E := \Big\{u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N) \colon \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^+(x)|u|^2 \,\mathrm{d}x < +\infty\Big\} \tag{2.4}$$

be equipped, for $\lambda > 0$, with the inner product and norm, respectively, given by

$$\langle u,v\rangle_\lambda := \int_{\mathbb{R}^N} |x|^{-2a}\nabla u\cdot\nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \lambda|x|^{-2(a+1)}V^+(x)uv \,\mathrm{d}x,$$
$$\|u\|_\lambda := \langle u,u\rangle_\lambda^{1/2}.$$

We denote $\|\cdot\|_1 = \|\cdot\|$, and $E_\lambda = (E, \|\cdot\|_\lambda)$; in particular, we set $E_1 = E$. Clearly, $\|u\| \leqslant \|u\|_\lambda$ if $\lambda \geqslant 1$.

Let $\Phi\colon E \to \mathbb{R}$ be the energy functional given by

$$\begin{aligned}\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2(a+1)}V(x)u^2 \,\mathrm{d}x \\ - \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)}G(x,u)\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

It is well known that under the previously introduced hypotheses the functional Φ is continuously differentiable, that is, $\Phi \in C^1(E,\mathbb{R})$; see, for example, the paper [5]. Moreover, $\Phi'(u) = 0$ if, and only, if $u \in E$ is a weak solution of (2.3).

In the following proposition we consider the decay of the weak solutions to problems of the type (1.2) defined on open, bounded domains; a similar conclusion holds for weak solutions to problems of the type (1.1).

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain with $N \geqslant 3$ and such that $\partial\Omega \in C^1$ and suppose that inequalities* (1.3) *hold. Suppose also that u is a non-negative solution of the problem*

$$\begin{gathered}-\operatorname{div}(|x|^{-2a}\nabla u) + \lambda|x|^{-2(a+1)}V(x)u = |x|^{-b2^*(a,b)}g(x,u) \quad x \in \Omega \\ u(x) = 0 \quad x \in \partial\Omega.\end{gathered} \tag{2.6}$$

Then there exists a positive constant $C = C(p,\mathbb{R}^N,\|u\|_{\mathcal{D}^{1,2}_a(\Omega)})$ such that for every $q < N-2$ there holds the inequality

$$u(x) \leqslant \frac{C}{|x|^{q-a}}\,\|u\|_{\mathcal{D}^{1,2}_a(\Omega)}. \tag{2.7}$$

Proof. Following an idea by Hsia, Lin and Wadade in [36], we use the change of variables $w(x) := |x|^{-a}u(x)$. Then

$$\int_\Omega |x|^{-2a}|\nabla u|^2 \,\mathrm{d}x = \int_\Omega |\nabla w|^2 \,\mathrm{d}x - \gamma\int_\Omega |x|^{-2}w^2(x)\,\mathrm{d}x, \tag{2.8}$$

where $\gamma := a(N-2-a)$. This implies that $u \in \mathcal{D}^{1,2}_a(\Omega)$ if, and only if, $w \in \mathcal{D}^{1,2}_0(\Omega)$; moreover, u is a solution to problem (2.6) if, and only if, w is a solution to problem

$$\begin{gathered}-\Delta w - \gamma|x|^2 w = |x|^{-(b-a)2^*(a,b)}g(x,|x|^a w) \quad x \in \Omega \\ w(x) = 0 \quad x \in \partial\Omega.\end{gathered}$$

To show equality (2.8) we begin by noting that

$$\begin{aligned}\int_\Omega |x|^{-2a}|\nabla u|^2 \,\mathrm{d}x = \int_\Omega a^2|x|^{-2}w^2(x)\,\mathrm{d}x + \int_\Omega 2a|x|^{-2}w(x)[x\cdot\nabla w(x)]\,\mathrm{d}x \\ + \int_\Omega |\nabla w|^2 \,\mathrm{d}x.\end{aligned} \tag{2.9}$$

To handle the second integral on the right-hand side of the previous equation we evaluate

$$2a|x|^{-2}w(x)[x\cdot\nabla w(x)] = \frac{1}{2}\operatorname{div}(2a|x|^{-2}w^2(x)x) + a(2-N)|x|^{-2}w^2(x).$$

Integrating over Ω and using the divergence theorem, we obtain

$$\int_\Omega 2a|x|^{-2}w(x)[x\cdot\nabla w(x)]\,\mathrm{d}x = \int_\Omega a(2-N)|x|^{-2}w^{-2}(x)\,\mathrm{d}x. \tag{2.10}$$

Substituting equation (2.10) in equation (2.9) we obtain equality (2.8).

Now we evaluate directly the divergence of $|x|^{-2a}\nabla u(x)$ to obtain

$$\operatorname{div}(|x|^{-2a}\nabla u(x)) = \gamma |x|^{-(a+2)} w(x) + |x|^{-a}\Delta w(x). \tag{2.11}$$

Supposing that the function u is a solution to problem (2.6) and using inequality (2.1) we obtain

$$\begin{aligned}
&-\operatorname{div}(|x|^{-2a}\nabla u(x))\\
&= -\lambda |x|^{-(a+2)} V(x) w(x) + |x|^{-b2^*(a,b)} g(x, |x|^a w)\\
&\leqslant -\lambda |x|^{-(a+2)} V(x) w(x) + a_2 |x|^{-b2^*(a,b)} |x|^{a(p-1)} |w(x)|^{p-1}.
\end{aligned} \tag{2.12}$$

Isolating the term $-|x|^{-a}\Delta w(x)$ in equality (2.11) and using (2.12) we obtain

$$\begin{aligned}
&-|x|^{-a}\Delta w(x)\\
&\leqslant -\lambda |x|^{-(a+2)}(V^+(x) - V^-(x)) w(x) + a_2 |x|^{-b2^*(a,b)+a(p-1)} |w(x)|^{p-1}\\
&\quad + \gamma |x|^{-(a+2)} w(x)\\
&\leqslant \lambda |x|^{-(a+2)} V^-(x) w(x) + a_2 |x|^{-b2^*(a,b)+a(p-1)} |w(x)|^{p-1} + \gamma |x|^{-(a+2)} w(x)\\
&= (\lambda V^-(x) + \gamma) |x|^{-(a+2)} w(x) + a_2 |x|^{-b2^*(a,b)+a(p-1)} |w(x)|^{p-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
-\Delta w(x) &\leqslant (\lambda V^-(x) - \gamma)|x|^{-2} w(x) + a_2 |x|^{-b2^*(a,b)+ap} |w(x)|^{p-1}\\
&:= R|x|^{-2} w(x) + V_0
\end{aligned}$$

where $R := \lambda V^-(x) + \gamma$ and $V_0 := a_2 |x|^{-b2^*(a,b)+ap} |w(x)|^{p-1}$. Since we assume that Ω is bounded, then $R \in L^{r_0}(\Omega) \cap L^{t_0}(\Omega)$ for some $t_0 > r_0 := 2^*/[2^*(a,b) - 2]$ and $V_0 \in L^1(\Omega) \cap L^{r_0}(\Omega)$. Applying a result by Vassilev in [47, Theorem 2.9] there exists a positive constant $C = C(p, \mathbb{R}^N, \|w\|_{\mathcal{D}_0^{1,2}(\Omega)})$ such that

$$w(x) \leqslant \frac{C}{|x|^q} \|w\|_{\mathcal{D}_0^{1,2}(\Omega)}$$

for every $q < N - 2$; that is,

$$|x|^{-a} u(x) \leqslant \frac{C}{|x|^q} \||x|^{-a} u\|_{\mathcal{D}_0^{1,2}(\Omega)}. \tag{2.13}$$

Now we observe that

$$\begin{aligned}
&\operatorname{div}(a|x|^{-2(a+1)} u^2(x) x)\\
&= [Na - 2a(a+1)]|x|^{-2(a+1)} u^2(x) + 2a|x|^{-2(a+1)} u(x)[x \cdot \nabla u(x)].
\end{aligned}$$

Integrating over Ω and using the divergence theorem we obtain

$$\begin{aligned}
&\int_\Omega -2a|x|^{-2(a+1)} u(x)[x \cdot \nabla u(x)]\, \mathrm{d}x\\
&= [Na - 2a(a+1)] \int_\Omega |x|^{-2(a+1)} u^2(x)\, \mathrm{d}x.
\end{aligned} \tag{2.14}$$

Finally, we evaluate the $\mathcal{D}_0^{1,2}(\Omega)$ norm of the function $|x|^{-a}u$; to do this we use (2.14) to obtain

$$\begin{aligned}
&\int_\Omega |\nabla(|x|^{-a}u(x))|^2\,\mathrm{d}x\\
&= a^2\int_\Omega |x|^{-2(a+1)}u^2(x)\,\mathrm{d}x - 2a\int_\Omega |x|^{-2(a+1)}u(x)[x\cdot\nabla u(x)]\,\mathrm{d}x\\
&\quad + \int_\Omega |x|^{-2a}|\nabla u(x)|^2\,\mathrm{d}x\\
&= a(N-2-a)\int_\Omega |x|^{-2(a+1)}u^2(x)\,\mathrm{d}x + \int_\Omega |x|^{-2a}|\nabla u(x)|^2\,\mathrm{d}x.
\end{aligned} \tag{2.15}$$

Using estimate (2.13) and equality (2.15) it follows that

$$\begin{aligned}
u(x) &\leqslant \frac{C}{|x|^{q-a}}\||x|^{-a}u\|_{\mathcal{D}_0^{1,2}(\Omega)}\\
&= \frac{C}{|x|^{q-a}}\Big(a(N-2-a)\int_\Omega |x|^{-2(a+1)}u^2(x)\,\mathrm{d}x + \int_\Omega |x|^{-2a}|\nabla u(x)|^2\,\mathrm{d}x\Big)^{1/2}\\
&\leqslant \frac{C}{|x|^{q-a}}\|u\|_{\mathcal{D}_a^{1,2}(\Omega)}
\end{aligned}$$

for every $q < N-2$, which is the estimate (2.7). This concludes the proof. □

For an arbitrary set $X \subset \mathbb{R}^N$, and for $d \geqslant 0$ and $r \geqslant 1$, we use the notation

$$\|v\|_{L_d^r(X)} := \Big[\int_X |x|^{-dr}|v|^r\,\mathrm{d}x\Big]^{1/r}.$$

Let $F := \{u \in E\colon \operatorname{supp} u \subset V^{-1}([0,+\infty))\}$; then F is a vector subspace and we denote its orthogonal complement in E by $F^\perp$. If $V \geqslant 0$, then $E = F$; otherwise, $F^\perp \neq \{0\}$. Let $A(u) := -\operatorname{div}(|x|^{-a}\nabla u) + |x|^{-2(a+1)}Vu$; then A is formally self-adjoint in $L_a^2(\mathbb{R}^N)$ and the associated bilinear form $a_V\colon E\times E \to \mathbb{R}$ given by

$$a_V(u,v) := \int_{\mathbb{R}^N} |x|^{-2a}\nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-2(a+1)}V(x)uv\,\mathrm{d}x$$

is such that $a_V \in C(E\times E,\mathbb{R})$. Consider also the eigenvalue problem

$$-\operatorname{div}(|x|^{-2a}\nabla u) + |x|^{-2(a+1)}V^+(x)u = \mu|x|^{-2(a+1)}V^-(x)u$$

for $u \in F^\perp$.

Since from hypothesis (V2) we have that $\operatorname{supp} V^-$ is of finite measure, then the quadratic form

$$u \mapsto \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)u^2\,\mathrm{d}x$$

is weakly continuous; hence, there exists a sequence of positive eigenvalues $(\mu_j)_{j\in\mathbb{N}}$ which may be characterized by

$$\mu_j = \inf_{\dim M\geqslant j,\, M\subset F^\perp}\sup\Big\{\|u\|^2\colon u\in M, \int_{\mathbb{R}^N}|x|^{-2(a+1)}V^-(x)u^2\,\mathrm{d}x = 1\Big\}.$$

To show this, we define the bilinear forms

$$a_{V^+}(u,v) := \int_{\mathbb{R}^N} |x|^{-2a}\nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^+(x)\,uv\,\mathrm{d}x$$

$$b_{V^-}(u,v) := \int_{\mathbb{R}^N} |x|^{-2(a+1)} V^-(x)\, uv \,\mathrm{d}x$$

and apply the results in Willem [49, Theorems 4.45 and 4.46].

Moreover, the sequence of eigenvalues is such that $\mu_j \to +\infty$ as $j \to +\infty$ and the corresponding eigenfunctions e_j can be chosen so that $\langle e_i, e_j\rangle = \delta_{ij}$ and form a basis for $F^\perp$.

Let $\widehat{E} := \operatorname{span}\{e_j : \mu_j \leqslant 1\}$ and $E^+ := \overline{\operatorname{span}\{e_j : \mu_j > 1\}}$. Then $E = \widehat{E} \oplus E^+ \oplus F$ is an orthogonal decomposition, with $\dim \widehat{E} < +\infty$. We set the operator

$$A_\lambda(u) := -\operatorname{div}(|x|^{-2a}\nabla u) + \lambda |x|^{-2(a+1)} V u;$$

then A_λ is formally self-adjoint in $L^2_a(\mathbb{R}^N)$ and the associated bilinear form

$$a_\lambda(u,v) := \int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\mathbb{R}^N} \lambda |x|^{-2(a+1)} V(x) uv \,\mathrm{d}x$$

is such that $a_\lambda \in C(E_\lambda \times E_\lambda, \mathbb{R})$. Let $F_\lambda^\perp$ be defined as the orthogonal complement of F in E_λ and $E_\lambda = F \oplus F_\lambda^\perp$. Now consider the eigenvalue problem

$$-\operatorname{div}(|x|^{-2a}\nabla u) + \lambda|x|^{-2(a+1)} V^+(x) u = \lambda\mu |x|^{-2(a+1)} V^-(x) u$$

where $\lambda > 0$ is fixed and $u \in F_\lambda^\perp$.

Then we have a sequence $(\mu_j(\lambda))_{j\in\mathbb{N}}$ defined by

$$\begin{aligned} \mu_j(\lambda) &:= \inf_{\substack{\dim M \geqslant j \\ M \subset F_\lambda^\perp}} \sup \Big\{ \lambda^{-1} \|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^N} |x|^{-2(a+1)} V^-(x) u^2 \,\mathrm{d}x = 1 \Big\} \\ &= \inf_{\substack{\dim M \geqslant j \\ M \cap F = \{0\}}} \sup \Big\{ \lambda^{-1} \|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^N} |x|^{-2(a+1)} V^-(x) u^2 \,\mathrm{d}x = 1 \Big\}, \end{aligned} \tag{2.16}$$

where the equality (2.16) follows from the fact that $\int_{\mathrm{R}^N} |x|^{-2(a+1)} V^-(x) u^2 \,\mathrm{d}x = 0$ when $u \in F$.

Denote the corresponding eigenfunctions by $e_j(\lambda)$ and let

$$\widehat{E}_\lambda := \operatorname{span}\{e_j(\lambda) : \mu_j(\lambda) \leqslant 1\}, \quad E_\lambda^+ := \overline{\operatorname{span}\{e_j(\lambda) : \mu_j(\lambda) > 1\}}.$$

Then $E_\lambda = \widehat{E}_\lambda \oplus E_\lambda^+ \oplus F$ is an orthogonal decomposition of E_λ and $\dim \widehat{E}_\lambda < +\infty$. Moreover, the quadratic form a_λ is negative semidefinite on $\widehat{E}_\lambda$, positive definite on $E_\lambda^+ \oplus F$ and $a_\lambda(u,v) = 0$ if u, v are in different subspaces of the previous decomposition. See Lemma 3.1 for some more properties of $\mu_j(\lambda)$.

We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset E$ is a Cerami sequence for the functional Φ if $(\Phi(u_n))_{n\in\mathbb{N}}$ is bounded and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ as $n \to +\infty$; also, Φ is said to satisfy the Cerami condition if any such sequence has a convergent subsequence. In particular, a Cerami sequence with $\Phi(u_n) \to c$ is denoted by $(C)_c$-sequence, and we say that Φ satisfies the $(C)_c$-condition if each $(C)_c$-sequence has a convergent subsequence.

To prove Theorem 1.1 and Theorem 1.2 we will make use of the following propositions.

Proposition 2.2 (Linking Theorem). *Suppose that $\Phi \in C^1(E, \mathbb{R})$, $E = E_1 \oplus E_2$ where $\dim E_2 < +\infty$, and that there exist $R > \rho > 0$, $\kappa > 0$ and $e_0 \in E_1 \backslash \{0\}$ such that $\inf \Phi(E_1 \cap S_\rho) \geqslant \kappa$ and $\sup \Phi(\partial Q) \leqslant 0$, where $Q = \{v + te_0 : v \in E_2, t \geqslant 0, \|u\| \leqslant R\}$. If Φ satisfies the $(C)_c$-condition for all $\kappa \leqslant c \leqslant \sup \Phi(Q)$, then Φ has a critical value in the interval $[\kappa, \sup \Phi(Q)]$.*

Proof. See Rabinowitz in [45, Theorem 9.12] or Willem [50, Theorem 2.12] for the details. In their proofs, it is usually assumed that the functional Φ verifies the Palais-Smale condition; however, the Cerami condition is sufficient for the deformation lemma, and therefore it is also sufficient for the Linking Theorem to hold. □

In contrast, if the functional is invariant under the action of a compact group of transformations, then we can get sharper results. Considering the action of the symmetry group, we state the following result.

Proposition 2.3. *Suppose that $\Phi \in C^1(E,\mathbb{R})$ is even, $\Phi(0) = 0$ and that there exist closed subspaces E_1, E_2 such that $\operatorname{codim} E_1 < +\infty$, $\inf \Phi(E_1 \cap S_\rho) \geqslant \kappa$ for some $\kappa > 0$, $\rho > 0$ and $\sup \Phi(E_2) < +\infty$. If Φ verifies the $(C)_c$-condition for all $\kappa \leqslant c \leqslant \sup \Phi(E_2)$, then Φ has at least $\dim E_2 - \operatorname{codim} E_1$ pairs of critical points with corresponding critical values in the interval $[\kappa, \sup \Phi(E_2)]$.*

For a proof of the above proposition, see Bartolo, Benci and Fortunato [6, Theorem 2.4].

3. Technical lemmas

Lemma 3.1. *Suppose $V^-(x) \not:= 0$. Then for each fixed $j \in \mathbb{N}$ we have the following claims.*

(1) *$\mu_j(\lambda) \to 0$ as $\lambda \to +\infty$; also, $\dim \widehat{E}_\lambda \to +\infty$ as $\lambda \to +\infty$.*
(2) *$\mu_j(\lambda)$ is a non-increasing continuous function of λ.*

Proof. To prove item (1), let $\phi_k \in C_0^\infty(\mathbb{R}^N)$ be functions such that $\operatorname{supp} \phi_i \subset \operatorname{supp} V^-$ for each $1 \leqslant k \leqslant j$; suppose also that for every $k \neq l$ we have $\operatorname{supp} \phi_k \cap \operatorname{supp} \phi_l = \emptyset$. Define $M := \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_j\}$. Then

$$\begin{aligned} \mu_j(\lambda) &\leqslant \sup_{u \in M,\, u \neq 0} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \lambda |x|^{-2(a+1)} V^+(x) |u|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} \lambda |x|^{-2(a+1)} V^-(x) |u|^2 \,\mathrm{d}x} \\ &= \sup_{u \in M,\, u \neq 0} \frac{\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} \lambda |x|^{-2(a+1)} V^-(x) |u|^2 \,\mathrm{d}x}. \end{aligned}$$

Hence, $\mu_j(\lambda) \to 0$ as $\lambda \to +\infty$. It also follows that $\dim \widehat{E}_\lambda \to +\infty$ as $\lambda \to +\infty$.

To prove item (2), let $u \in M$ be such that $\int_{\mathbb{R}^N} |x|^{-2(a+1)} V^-(x) |u|^2 \,\mathrm{d}x = 1$. Since $M \cap F = \{0\}$, for $\lambda_1, \lambda_2 \in \mathbb{R}^+$, we have

$$\lambda_1^{-1} \|u\|_{\lambda_1}^2 - \lambda_2^{-1} \|u\|_{\lambda_2}^2 = (\lambda_1^{-1} - \lambda_2^{-1}) \|\nabla u\|_{L_a^2}^2. \tag{3.1}$$

It readily follows that $\mu_j(\lambda)$ is a non-increasing function of λ.

To show the continuity of this function, let $\lambda_1, \lambda_2 \in (\lambda_0, \tilde{\lambda})$, where $\lambda_0 > 0$. We only have to consider those subspaces M for which the supremum in equation (2.16) is less than or equal to the number $K := \mu_j(\lambda_0) + 1$. For such subspaces M we have $\mu_j(\tilde{\lambda}) \leqslant \mu_j(\lambda_0) < K$.

Let $u \in M$ be normalized by $\int_{\mathbb{R}^N} |x|^{-2(a+1)} V^-(x) |u|^2 \,\mathrm{d}x = 1$. From the definition, we have

$$\mu_j(\tilde{\lambda}) \leqslant \tilde{\lambda}^{-1} \|u\|_\lambda^2 \leqslant K$$

and this implies that $\tilde{\lambda} K \geqslant \|u\|_\lambda^2$. Since

$$\|\nabla u\|_{L_a^2(\mathbb{R}^N)} \leqslant \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 + \int_{\mathbb{R}^N} |x|^{-2(a+1)} V^+(x) |u|^2 \,\mathrm{d}x = \|u\|_\lambda^2,$$

then $\|\nabla u\|^2_{L^2_a(\mathbb{R}^N)} \leqslant \|u\|^2_\lambda \leqslant \tilde{\lambda} K$ and this implies that $\mu_j(\lambda_2) \to \mu_j(\lambda_1)$ as $\lambda_2 \to \lambda_1$ and the function $\mu_j(\lambda)$ is continuous. This concludes the proof of the lemma. □

Lemma 3.2. (1) *Suppose that g verifies inequalities* (2.1) *and* (2.2) *for some $a_2 > 0$, $2 < p < 2^*$ and $\mu > 2$; then assumption (G4) holds for $(N-a)/2 < \tau < p/(p-2)$ with $\tau > 1$.*
(2) *Suppose that g verifies the assumptions (G2) to (G4); then $\widetilde{G}(x,u) \to +\infty$ uniformly in x as $|u| \to +\infty$.*

The proof of the above lemma is similar to the corresponding proof in Ding and Szulkin [27, Lemma 2.2].

Lemma 3.3. *Suppose that the assumptions* (V1), (V2), (G1)–(G4) *hold. Then any Cerami sequence for the functional Φ is bounded.*

Proof. Consider a Cerami sequence $(u_m)_{m\in\mathbb{N}} \subset E$. Since the functional $\Phi \in C^1(E,\mathbb{R})$, we have

$$\|(1+\|u_m\|)\Phi'(u_m)\| := \sup_{\|v\|=1} |(1+\|u_m\|)\Phi'(u_m)v| \leqslant (1+\|u_m\|)\|\Phi'(u_m)\| \to 0$$

as $m \to +\infty$. So

$$\|\Phi'(u_m)u_m\| \leqslant \|\Phi'(u_m)\|\|u_m\| \leqslant \|\Phi'(u_m)\|(1+\|u_m\|) \to 0$$

as $m \to +\infty$. It follows that for $m \in \mathbb{N}$ large enough and for some positive constant $C > 0$, we have

$$C \geqslant \Phi(u_m) - \frac{1}{2}\Phi'(u_m)u_m = \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} \widetilde{G}(x,u)\,\mathrm{d}x. \tag{3.2}$$

Define $h(r) := \inf_{x\in\mathbb{R}^N}\{\widetilde{G}(x,u) \colon |u| \geqslant r\}$. By (G3) and by Lemma 3.2, there exist $R_0 > 0$ such that $\widetilde{G}(x,u) \geqslant 1$ for $|x| > R_0$. In the annulus $r < |u| < R_0$ we have $\widetilde{G}(x,u) > 0$; therefore, $h(r) > 0$. Moreover, also by Lemma 3.2 we have $h(r) \to +\infty$ as $r \to +\infty$.

For $0 < \alpha < \beta$, we define

$$\Omega_m(\alpha,\beta) := \{x \in \mathbb{R}^N \colon \alpha \leqslant |u_m(x)| < \beta\},$$
$$\Omega_m(\beta,+\infty) := \{x \in \mathbb{R}^N \colon \beta \leqslant |u_m(x)| < +\infty\}.$$

Notice that by assumption (G1) we have that $0 \notin \Omega_m(\beta,+\infty)$. We also define

$$C_\alpha^\beta := \inf_{x\in\mathbb{R}^N} \big\{ \frac{\widetilde{G}(x,u)}{|u|^2} \colon \alpha \leqslant |u| < \beta \big\}.$$

Then $\widetilde{G}(x,u) \geqslant C_\alpha^\beta |u_m|^2$ for all $x \in \Omega_m(\alpha,\beta)$. From inequality (3.2) it follows that

$$\begin{aligned} C &\geqslant \int_{\Omega_m(0,\alpha)} |x|^{-b2^*(a,b)} \widetilde{G}(x,u)\,\mathrm{d}x + \int_{\Omega_m(\alpha,\beta)} |x|^{-b2^*(a,b)} \widetilde{G}(x,u)\,\mathrm{d}x \\ &\quad + \int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)} \widetilde{G}(x,u)\,\mathrm{d}x \\ &\geqslant \int_{\Omega_m(0,\alpha)} |x|^{-b2^*(a,b)} \widetilde{G}(x,u)\,\mathrm{d}x + c_\alpha^\beta \int_{\Omega_m(\alpha,\beta)} |x|^{-b2^*(a,b)} |u|^2\,\mathrm{d}x \\ &\quad + h(\beta) \int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\,\mathrm{d}x. \end{aligned} \tag{3.3}$$

To show that a Cerami sequence is bounded we argue by contradiction and assume that $\|u_m\| \to +\infty$ as $m \to +\infty$. Now we define the sequence $(v_m)_{m\in\mathbb{N}} \subset E$ by $v_m := u_m/\|u_m\|$ which implies that $\|v_m\| = 1$. By the Caffarelli, Kohn, and Nirenberg inequality in [18], for $2 \leqslant s < 2^*(a,b)$ there exists a constant $C_s > 0$ such that

$$\|v_m\|_{L^s_b(\mathbb{R}^N)} \leqslant C_s\Big[\int_{\mathbb{R}^N} |x|^{-2a}|\nabla v_m|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^+(x)|v_m|^2\,\mathrm{d}x\Big]^{1/2} = C_s.$$

Using inequality (3.3), we obtain

$$\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\,\mathrm{d}x \leqslant \frac{C}{h(\beta)} \to 0 \tag{3.4}$$

uniformly in $m \in \mathbb{N}$ as $\beta \to +\infty$. For fixed $0 < \alpha < \beta$, and using inequality (3.3) once more, we have

$$\int_{\Omega_m(\alpha,\beta)} |x|^{-b2^*(a,b)}|v_m|^2\,\mathrm{d}x \leqslant \frac{1}{\|u_m\|^2}\int_{\Omega_m(\alpha,\beta)} |x|^{-b2^*(a,b)}|u_m|^2\,\mathrm{d}x \to 0 \tag{3.5}$$

as $m \to +\infty$ because we assume that $\|u_m\| \to +\infty$.

From the Hölder inequality and from inequality (3.4), it follows that for $2 \leqslant s < p < 2^*(a,b)$ we have the inequality

$$\begin{aligned}
&\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\frac{g(x,u_m)}{u_m}|v_m||v_m^+|\,\mathrm{d}x\\
&\leqslant \Big[\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\Big|\frac{g(x,u_m)}{u_m}\Big|^{\tau}\,\mathrm{d}x\Big]^{1/\tau}\\
&\quad\times\Big[\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}|v_m|^{2\tau/(\tau-1)}\,\mathrm{d}x\Big]^{(\tau-1)/2\tau}\\
&\quad\times\Big[\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}|v_m^+|^{2\tau/(\tau-1)}\,\mathrm{d}x\Big]^{(\tau-1)/2\tau}
\end{aligned}$$

for all $m \in \mathbb{N}$. In a similar way, by assumption (G4), and using inequality (3.2) together with a suitable constant C_1 we also have

$$\begin{aligned}
&\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\frac{g(x,u_m)}{u_m}|v_m||v_m^+|\,\mathrm{d}x\\
&\leqslant C_1\Big[\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}|v_m|^{2^*(a,b)}\,\mathrm{d}x\Big]^{s/2^*(a,b)}\\
&\quad\times\Big[\int_{\Omega_m(\beta,+\infty)} |x|^{-b2^*(a,b)}\,\mathrm{d}x\Big]^{(2^*(a,b)-s)/2^*(a,b)}.
\end{aligned}$$

Now we write $u = \hat{u} + u^+$ and $v = \hat{v} + v^+$ so that $\hat{u}, \hat{v} \in \widehat{E}$ and $u^+, v^+ \in E^+ \oplus F$; then

$$\|v_m^+\|_{L^s_b(\mathbb{R}^N)} \leqslant C_s\|v_m^+\| \leqslant C_s\|v_m\| = C_s.$$

From the definition of v_m, it follows that $v_m^+ = \frac{u_m^+}{\|u_m\|}$. Dividing the expression of $\Phi'(u_m)u_m^+$ by $\|u_m\|^2$, we obtain

$$\frac{\Phi'(u_m)u_m^+}{\|u_m\|^2} = \int_{\mathbb{R}^N} |x|^{-2a}|\nabla v_m^+|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-2(a+1)}V(x)(v_m^+)^2\,\mathrm{d}x$$

$$- \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{v_m^+} v_m v_m^+ \,\mathrm{d}x$$

But we have $\Phi'(u_m)u_m^+ \to 0$ as $m \to +\infty$; therefore,

$$\frac{\Phi'(u_m)u_m^+}{\|u_m\|^2} = a(v_m^+, v_m^+) - \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{u_m} v_m v_m^+ \,\mathrm{d}x \to 0 \tag{3.6}$$

as $m \to +\infty$.

Let $\varepsilon > 0$. By assumption (G1), there exists $a_\varepsilon > 0$ such that for all $|u| \leqslant a_\varepsilon$ it holds

$$|g(x,u)| \leqslant \frac{\varepsilon}{3C_2^2}|u|.$$

Consequently,

$$\begin{aligned} &\int_{\Omega_m(0,a_\varepsilon)} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{u_m} |v_m||v_m^+| \,\mathrm{d}x \\ &\leqslant \int_{\Omega_m(0,a_\varepsilon)} |x|^{-b2^*(a,b)} \frac{\varepsilon}{3C_2^2} |v_m||v_m^+| \,\mathrm{d}x \\ &\leqslant \frac{\varepsilon}{3C_2^2} \|v_m\|_{L^2_{b2^*(a,b)/2}(\Omega)} \|v_m^+\|_{L^2_{b2^*(a,b)/2}(\Omega)} \leqslant \varepsilon/3 \end{aligned} \tag{3.7}$$

for all $m \in \mathbb{N}$. Using the limit in (3.4), there exist $b_\varepsilon > 0$ large enough such that

$$\int_{\Omega_m(b_\varepsilon,+\infty)} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{u_m} v_m v_m^+ \,\mathrm{d}x < \varepsilon/3 \tag{3.8}$$

uniformly in m.

Following up, we now use the inequality (3.5) to obtain constants $C_3 > 0$ and $m_0 \in \mathbb{N}$ such that

$$\begin{aligned} \int_{\Omega_m(a_\varepsilon,b_\varepsilon)} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{u_m} v_m v_m^+ \,\mathrm{d}x &\leqslant C_3 \int_{\Omega_m(a_\varepsilon,b_\varepsilon)} |x|^{-b2^*(a,b)} v_m v_m^+ \,\mathrm{d}x \\ &\leqslant C_3 \|v_m\|_{L^2(\Omega_m(a_\varepsilon,b_\varepsilon))} \\ &< \varepsilon/3 \end{aligned} \tag{3.9}$$

for all $m \geqslant m_0$.

Finally, using inequalities (3.7), (3.8), and (3.9), for all $m \geqslant m_0$ we have

$$\int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} \frac{g(x,u_m)}{u_m} v_m v_m^+ \,\mathrm{d}x < \varepsilon.$$

It follows from inequality (3.6) that $v_m^+ \to 0$ as $m \to +\infty$, because the quadratic form a is positive definite on $E^+ \oplus F$ and $\varepsilon > 0$ is arbitrary. So, passing to a subsequence if necessary, still denoted in the same way, we obtain $v_m \to v \neq 0$ in E as $m \to +\infty$, because $\dim \widehat{E} < +\infty$. By inequality (3.2), and using assumption (G3), Lemma 3.2, and Fatou's Lemma, we obtain

$$C \geqslant \int_{\mathbb{R}^N} \widetilde{G}(x,u_m) \,\mathrm{d}x \geqslant \int_{\{x \in \mathbb{R}^N :\, v(x) \neq 0\}} \widetilde{G}(x,u_m) \,\mathrm{d}x \to +\infty,$$

which is a contradiction. This concludes the proof of the lemma. □

The following lemma, which is a variant of the Brézis-Lieb lemma, is crucial to prove a result on the convergence of subsequences of a Palais-Smale sequence (Lemma 3.7).

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ be a function such that $|f(x,u)| \leqslant a|u|^s$ for some $a \in \mathbb{R}^+$ and for $1 \leqslant s < +\infty$. Suppose that $s \leqslant p < +\infty$ with $p > 1$, and let $(u_m)_{m\in\mathbb{N}} \subset L^p(\Omega)$ be a bounded sequence such that $u_m \to u$ a.e. in Ω and $u_m \to u$ in $L^p_b(\Omega \cap B_R)$ for all $R \in \mathbb{R}^+$ as $m \to +\infty$. Then, passing to a subsequence, there exists a sequence $(v_m)_{m\in\mathbb{N}} \subset L^p_b(\Omega)$ such that $|x|^{-b}v_m \to |x|^{-b}u$ in $L^p(\Omega)$ as $m \to +\infty$ and*

$$|x|^{-b}f(x,u_m) - |x|^{-b}f(x,u_m - v_m) - |x|^{-b}f(x,u) \to 0 \quad \text{in } L^{p/s}(\Omega).$$

Proof. We adapt the arguments by Ackermann [1, Lemma 3.2] and by Ding and Szulkin [27, Proposition A.1]. Since the sequence $(u_m)_{m\in\mathbb{N}} \subset L^p(\Omega)$ is bounded, and since $u_m \to u$ a.e. in Ω, using a result by Willem [49, Theorem 10.36] we have $u_m \rightharpoonup u$ weakly in $L^p_b(\Omega)$ as $m \to +\infty$.

We claim that there exists a subsequence $(u_{m_j})_{j\in\mathbb{N}} \subset L^p_b(\mathbb{R}^N)$ and that there exists a sequence $(R_{m_j})_{j\in\mathbb{N}} \subset \mathbb{R}^+$ with $R_{m_j} \to +\infty$ as $j \to +\infty$ such that for each $\varepsilon > 0$, for each $R \geqslant R(\varepsilon)$ and for each $j \geqslant j(\varepsilon)$, it holds

$$\int_{\Omega\cap B_{R_{m_j}}\setminus B_R} |x|^{-bp}|u_{m_j}|^p \,\mathrm{d}x \leqslant \varepsilon. \tag{3.10}$$

In fact, by the convergence $u_m \to u$ in $L^p_b(\Omega \cap B_R)$, and by the boundedness of $\Omega \cap B_j$, we can use a result by Hewitt and Stromberg [35, Theorem 13.44] to guarantee that for each fixed $j \geqslant 1$ and almost all $m \in \mathbb{N}$, we have

$$\int_{\Omega\cap B_j} \left[|x|^{-bp}|u_m|^{-p} - |x|^{-bp}|u|^{-p}\right] \mathrm{d}x \leqslant \frac{1}{j}\,. \tag{3.11}$$

Choosing $m_j \geqslant j$ for which the inequality (3.11) holds, we may assume without loss of generality that $m_j < m_{j+1}$ for all $j \in \mathbb{N}$. Now we choose $R \in \mathbb{R}^+$ large enough so that

$$\int_{\Omega\setminus B_R} |x|^{-bp}|u|^p \,\mathrm{d}x \leqslant \frac{\varepsilon}{2}, \tag{3.12}$$

which is possible because $u \in L^p_b(\Omega)$. Assigning the value $R_{m_j} = j$ we have

$$\begin{aligned}\int_{\Omega\cap B_{R_{m_j}}\setminus B_R} |x|^{-bp}|u_{m_j}|^p \,\mathrm{d}x &= \int_{\Omega\cap B_{R_{m_j}}} \left[|x|^{-bp}|u_{m_j}|^p - |x|^{-bp}|u|^p\right] \mathrm{d}x \\ &\quad + \int_{\Omega\cap B_{R_{m_j}}\setminus B_R} |x|^{-bp}|u|^p \,\mathrm{d}x \\ &\quad + \int_{\Omega\cap B_R} \left[|x|^{-bp}|u|^p - |x|^{-bp}|u_{m_j}|^p\right] \mathrm{d}x\end{aligned}$$

for every $R_{m_j} > R$. From the inequalities (3.11) and (3.12), it follows that the last term on the right-hand side of the previous equation tends to 0 as $j \to +\infty$; more precisely,

$$\int_{\Omega\cap B_{R_{m_j}}} \left[|x|^{-bp}|u|^p - |x|^{-bp}|u_{m_j}|^p\right] \mathrm{d}x \leqslant \frac{1}{R_{m_j}} + \frac{\varepsilon}{2} + \frac{1}{j}$$

as $j \to +\infty$. Now we apply Cantor's diagonal argument to obtain inequality (3.10).

Let $\eta \in C_0^\infty(\mathbb{R}, [0,1])$ be a cut-off function such that $\eta(t) = 1$ if $t \leqslant 1$ and $\eta(t) = 0$ if $t \geqslant 2$; set $v_{m_j}(x) := \eta(2|x|/R_{m_j})u(x)$. Then we have

$$\int_\Omega |x|^{-bp}|v_{m_j}|^p \,\mathrm{d}x \to \int_\Omega |x|^{-bp}|u|^p \,\mathrm{d}x \tag{3.13}$$

as $j \to +\infty$.

Let us recall that by hypothesis we have a sequence $(u_m)_{m\in\mathbb{N}}$ such that $u_m \to u$ a. e. in Ω and $u_m \to u$ in $L^p(\Omega \cap B_R)$. Then, arguing as the authors in [5, Lemma 2.2], we can conclude that

$$|x|^{-bp}u_m \to |x|^{-bp}u \quad \text{in } L^p_{\rm loc}(\Omega \cap B_R). \tag{3.14}$$

Using the limits (3.13) and (3.14) and the continuity of the Nemytskii operator, we obtain

$$|x|^{-b}f(x,u_{m_j}) - |x|^{-b}f(x,u_{m_j} - v_{m_j}) - |x|^{-b}f(x,u) \to 0 \tag{3.15}$$

in $L^{p/s}_b(\Omega \cap B_R)$ as $j \to +\infty$. Moreover,

$$\begin{aligned}
&\big\||x|^{-b}f(x,u_{m_j}) - |x|^{-b}f(x,u_{m_j} - v_{m_j}) - |x|^{-b}f(x,u)\big\|_{L^{p/s}(\Omega\setminus B_R)}\\
&\leqslant \big\||x|^{-b}f(x,u_{m_j}) - |x|^{-b}f(x,u_{m_j} - v_{m_j}) - |x|^{-b}f(x,v_{m_j})\big\|_{L^{p/s}(\Omega\setminus B_R)}\\
&\quad + \big\||x|^{-b}f(x,v_{m_j}) - |x|^{-b}f(x,u)\big\|_{L^{p/s}(\Omega\setminus B_R)}.
\end{aligned}$$

Using the limit (3.13), once again by the continuity of the Nemytskii operator we have

$$\big\||x|^{-b}f(x,v_{m_j}) - |x|^{-b}f(x,u)\big\|_{L^{p/s}(\Omega\setminus B_R)} \to 0$$

as $j \to +\infty$. And since $|x|^{-b}v_{m_j} \leqslant |x|^{-b}|u|$ and $\operatorname{supp}\{v_{m_j}\} \subset \overline{B}_{R_{m_j}}$, we have

$$\begin{aligned}
&\big||x|^{-b}f(x,u_{m_j}) - |x|^{-b}f(x,u_{m_j} - v_{m_j}) - |x|^{-b}f(x,v_{m_j})\big|^{p/s}\\
&\leqslant C\Big[|x|^{-b}|u_{m_j}|^s + |x|^{-b}|u_{m_j} - v_{m_j}|^s + |x|^{-b}|v_{m_j}|^s\Big]^{p/s}\\
&\leqslant C\Big[|x|^{-b}|u_{m_j}|^p + |x|^{-b}|u|^p\Big],
\end{aligned}$$

where we used twice the well known inequality $|a+b|^p \leqslant C\big[|a|^p + |b|^p\big]$ for $a, b \in \mathbb{R}^+$. By the definitions of η and v_{m_j}, in $\Omega\setminus B_{R_{m_j}}$ we have $v_{m_j}(x) = 0$; hence, the left-hand side of the previous inequality is zero. Finally, by inequality (3.10) and by the limits (3.12) and (3.15), we obtain the conclusion of the lemma. $\square$

Before stating the next result, let us recall that given $\Omega \subset \mathbb{R}^N$, the set $L^p_b(\Omega) + L^q_b(\Omega)$ is the space of functions u defined in Ω such that $u = u_1 + u_2$ where $u_1 \in L^p_b(\Omega)$ and $u_2 \in L^q_b(\Omega)$. This set is equipped with the norm $\|u\|_{p\vee q} := \inf\{\|u_1\|_{L^p_b(\Omega)} + \|u_2\|_{L^q_b(\Omega)}\}$, where the infimum is taken with respect to all decompositions $u = u_1 + u_2$ as above.

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in C(\Omega\times\mathbb{R},\mathbb{R})$ be a function such that $|f(x,u)| \leqslant a\big[|u|^r + |u|^s\big]$ for some $a \in \mathbb{R}^+$ and for $1 \leqslant r < s < +\infty$. Suppose $s \leqslant p < +\infty$, $r \leqslant q < +\infty$ with $q > 1$, and let $(u_m)_{m\in\mathbb{N}} \subset L^p_b(\Omega) \cap L^q_b(\Omega)$ be a bounded sequence such that $u_m \to u$ a.e. in Ω and in $L^p_b(\Omega\cap B_R) \cap L^q_b(\Omega\cap B_R)$ for all $R \in \mathbb{R}^+$. Then, passing to a subsequence, there exists a sequence $(v_m)_{m\in\mathbb{N}} \subset L^p_b(\Omega\cap B_R) \cap L^q_b(\Omega\cap B_R)$ such that $|x|^{-b}v_m \to |x|^{-b}u$ in $L^p_b(\Omega) \cap L^q_b(\Omega)$ as $m \to +\infty$ and*

$$f(x,u_m) - f(x,u_m - v_m) - f(x,u) \to 0 \quad \textit{in } L^{p/s}_b(\Omega) \cap L^{q/r}_b(\Omega).$$

The proof of the above proposition is similar to the corresponding one by Ding and Szulkin in [27, Theorem A.2]. We omit it.

Let E be the space defined in (2.4). Suppose that $L\colon E \to E$ is a bounded linear self-adjoint operator and let $\Phi\colon E \to \mathbb{R}$ be given by

$$\Phi(u) := \frac{1}{2}\langle Lu, u\rangle - \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} G(x,u)\,\mathrm{d}x. \tag{3.16}$$

Using this notation, we have the following result.

Lemma 3.6. *Let $(u_m)_{m\in\mathbb{N}} \subset E$ be a sequence such that $u_m \rightharpoonup u$ weakly in E. Then after passing to a subsequence, there exists a sequence $(v_m)_{m\in\mathbb{N}} \subset E$ such that*

$$\begin{gathered}\Phi(u_m) = \Phi(u_m - v_m) + \Phi(u) + o(1),\\ \Phi'(u_m) = \Phi'(u_m - v_m) + \Phi'(u) + o(1)\end{gathered}$$

as $m \to +\infty$. In particular, if $(u_m)_{m\in\mathbb{N}} \subset E$ is a Palais-Smale $(PS)_c$-sequence, then after passing to a subsequence we have $\Phi(u_m - v_m) \to c - \Phi(u)$ and $\Phi'(u_m - v_m) \to \Phi'(u)$ as $m \to +\infty$.

Proof. Without loss of generality we can suppose that $|x|^{-b}u_m \to |x|^{-b}u$ a. e. in $\mathbb{R}^N$, $|x|^{-b} \rightharpoonup |x|^{-b}u$ weakly in $L^t(\mathbb{R}^N)$ and $|x|^{-b}u_m \to |x|^{-b}u$ in $L^t_{\rm loc}(\mathbb{R}^N)$, for $2 \leqslant t < 2^*$. Applying Proposition 3.5 to the function $|x|^{-b2^*(a,b)}G(x,u)$ with $r = q = 2$ and $s = p$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} G(x,u_m)\,\mathrm{d}x &= \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} G(x,u_m - v_m)\,\mathrm{d}x\\ &\quad + \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} G(x,u)\,\mathrm{d}x + o(1).\end{aligned}$$

Similarly, applying Proposition 3.5 to the function $|x|^{-b2^*(a,b)}g(x,u)$ with $r = 1$, $s = p-1$, and $q = 2$, we obtain

$$|x|^{-b2^*(a,b)}g(x,u_m) - |x|^{-b2^*(a,b)}g(x,u_m - v_m) - |x|^{-b2^*(a,b)}g(x,u) \to 0$$

as $m \to +\infty$. Therefore, it holds

$$\begin{aligned}&\sup_{\|\phi\|\leqslant 1} \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} g(x,u_m)\phi\,\mathrm{d}x\\ &= \sup_{\|\phi\|\leqslant 1} \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)} \big[g(x,u_m - v_m) - g(x,u)\big]\phi\,\mathrm{d}x + o(1)\end{aligned}$$

Lemma 3.4 guarantees the existence of a sequence $(v_m)_{m\in\mathbb{N}} \subset E$ such that $v_m \to u$ in E and, therefore, $Lu_m = L(u_m - v_m) - Lu + o(1)$ as $m \to +\infty$. The conclusion of the proof of the lemma follows. □

The following lemma follows immediately from Lemma 3.4. We omit its proof.

Lemma 3.7. *Suppose that the assumptions* (V1), (V2) *are satisfied and that inequality* (2.1) *holds; let $(u_m)_{m\in\mathbb{N}} \subset E$ be a Palais-Smale sequence for the functional Φ defined by equality* (3.16) *such that $u_m \rightharpoonup u$ weakly in E and $\Phi(u_m) \to c$ as $m \to +\infty$. Passing to a subsequence, there exists a sequence $(v_m)_{m\in\mathbb{N}} \subset E$ such that $v_m \to u$ as $m \to +\infty$ and*

$$\Phi(u_m - v_m) \to c - \Phi(u), \quad \Phi'(u_m - v_m) \to 0. \tag{3.17}$$

Let $\Phi_\lambda\colon E \to \mathbb{R}$ be the functional defined by

$$\begin{aligned}\Phi_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \lambda|x|^{-2(a+1)}V(x)u^2\,\mathrm{d}x \\ - \int_{\mathbb{R}^N} |x|^{-b2^*(a,b)}G(x,u)\,\mathrm{d}x;\end{aligned} \tag{3.18}$$

and for $\lambda = \varepsilon^{-2}$, let $\Psi_\lambda\colon E \to \mathbb{R}$ be the functional defined by

$$\begin{aligned}\Psi_\lambda(u) :=\frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \lambda|x|^{-2(a+1)}V(x)u^2\,\mathrm{d}x \\ - \int_{\mathbb{R}^N} \lambda|x|^{-b2^*(a,b)}G(x,u)\,\mathrm{d}x.\end{aligned} \tag{3.19}$$

We can easily infer that the conclusion of the Lemma 3.7 still holds for the functionals Φ_λ and Ψ_λ defined by (3.18) and (3.19), respectively.

Lemma 3.8. *Suppose that the assumptions* (V1), (V2), (G1)–(G4) *hold. Then for any $M > 0$ there exists $\Lambda = \Lambda(M)$ such that the following claims hold.*

1. *Φ_λ verifies the $(C)_c$-condition for all $c \leqslant M$ and for all $\lambda \geqslant \Lambda$.*
2. *Ψ_λ verifies the $(C)_c$-condition for all $c \leqslant M\lambda^{-\alpha}$ and for all $\lambda \geqslant \Lambda$, where $\alpha > 2/(2^* - 2)$ if $N \geqslant 3$ and $\alpha > 0$ if $N = 1$ or $N = 2$.*

Proof. We begin by considering the functional Φ_λ. Let $(u_m)_{m\in\mathbb{N}} \subset E_\lambda$ be a Cerami $(C)_c$-sequence with $c \leqslant M$. Lemma 3.3 implies that $(u_m)_{m\in\mathbb{N}}$ is a bounded sequence. Passing to a subsequence if necessary, still denoted in the same way, we can assume that $u_m \rightharpoonup u$ weakly in E_λ as $m \to +\infty$ and that it verifies the conclusion of Lemma 3.6.

Now we define $w_m := u_m - v_m$. Since $V(x) < b$ on a set of finite measure and since $w_m \rightharpoonup 0$ weakly in E_λ as $m \to +\infty$, we have $w_m > 0$ and $w_m \to 0$ in $L^2(K)$ for compact subsets $K \subset \mathbb{R}^N$. Therefore,

$$\begin{aligned}\||x|^{-(a+1)}w_m\|_2^2 &\leqslant \frac{1}{\lambda b}\int_{V(x)<b} V(x)|x|^{-2(a+1)}w_m^2\,\mathrm{d}x + o(1) \\ &\leqslant \frac{1}{\lambda b}\|w_m\|^2 + o(1) \leqslant \frac{1}{\lambda b}\|w_m\|_\lambda^2 + o(1).\end{aligned} \tag{3.20}$$

Moreover, if $2 < s < p < 2^*$, the previous inequality together with Hölder, and Caffarelli, Kohn and Nirenberg inequalities, imply that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-s(a+1)}|w_m|^s\,\mathrm{d}x &\leqslant \Big[\int_{\mathbb{R}^N} |x|^{-2(a+1)}|w_m|^2\,\mathrm{d}x\Big]^{(p-s)/(p-2)} \\ &\quad\times \Big[\int_{\mathbb{R}^N} |x|^{-p(a+1)}|w_m|^p\,\mathrm{d}x\Big]^{(s-2)/(p-2)} \\ &\leqslant d_1(\lambda b)^{-(p-s)/(p-2)}\|w_m\|_\lambda^{2(p-s)/(p-2)}\|w_m\|_\lambda^{p(s-2)/(p-2)} \\ &\leqslant d_1(\lambda b)^{-(p-s)/(p-2)}\|w_m\|_\lambda^s + o(1).\end{aligned} \tag{3.21}$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|g(x,u)| \leqslant \varepsilon|u|$ for all $x \in \mathbb{R}^N$ and for all $|u| \leqslant \delta$; moreover, assumption (G4) is verified for all $x \in \mathbb{R}^N$ and for $|u| \geqslant \delta$ with the same value of τ but possibly for a larger value of a_1. It follows from inequality

(3.20) that

$$\begin{aligned}&\int_{|w_m|\leqslant\delta}|x|^{-b2^*(a,b)}g(x,w_m)w_m\,\mathrm{d}x\\&\leqslant\varepsilon\int_{|w_m|\leqslant\delta}|x|^{-b2^*(a,b)}|w_m|^2\,\mathrm{d}x\leqslant\frac{\varepsilon}{\lambda b}\|w_m\|_\lambda^2+o(1).\end{aligned}\tag{3.22}$$

By the limits (3.17), we obtain

$$\Phi_\lambda(w_m)-\frac{1}{2}\Phi'_\lambda(w_m)w_m=\int_{\mathbb{R}^N}|x|^{-b2^*(a,b)}\widetilde{G}(x,w_m)\,\mathrm{d}x\to c-\Phi_\lambda(u)\tag{3.23}$$

as $m\to+\infty$. Using assumption (G4), inequality (3.21) with $s=2\tau/(\tau-1)$ and inequalities (3.8) and (3.23), we have

$$\begin{aligned}&\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}g(x,w_m)w_m\,\mathrm{d}x\\&\leqslant\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}\Big|\frac{g(x,w_m)}{w_m}\Big|^\tau\,\mathrm{d}x\Big]^{1/\tau}\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}|w_m|^{2\tau'}\,\mathrm{d}x\Big]^{1/\tau'}\\&\leqslant\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}a_1\widetilde{G}(x,w_m)\,\mathrm{d}x\Big]^{1/\tau}\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}|w_m|^s\,\mathrm{d}x\Big]^{2/s}\\&\leqslant a_1^{1/\tau}\big[c-\Phi_\lambda(u)\big]^{1/\tau}\|w_m\|^2_{L^s_{b2^*(a,b)/s}(\mathbb{R}^N)}+o(1)\\&\leqslant a_1^{1/\tau}M^{1/\tau}\big[d_1(\lambda b)^{-(p-s)/(p-2)}\|w_m\|_\lambda^s\big]^{2/s}+o(1)\\&\quad=d_2M^{1/\tau}(\lambda b)^\theta\|w_m\|_\lambda^2+o(1)\end{aligned}\tag{3.24}$$

where $\theta=2(p-s)/(s(p-2))$ and $d_2=a_1^{1/\tau}d_1^{s/2}$. Set $Z:=\{x\in\mathbb{R}^N: V(x)<0\}$; then $\Phi_\lambda(w_m)w_m\to0$ and $w_m\to0$ in $L^2(Z)$ as $m\to+\infty$. By inequalities (3.22) and (3.24), we have

$$o(1)=\|w_m\|_\lambda^2-\int_{\mathbb{R}^N}|x|^{-b2^*(a,b)}g(x,w_m)w_m\,\mathrm{d}x\leqslant\Big[1-\frac{\varepsilon}{\lambda b}-\frac{d_2M^{1/\tau}}{(\lambda b)^\theta}\Big]\|w_m\|_\lambda^2+o(1).$$

Finally, we take $\Lambda=\Lambda(M)$ large enough so that $1-\varepsilon/(\lambda b)-d_2M^{1/\tau}/(\lambda b)^\theta>0$ for $\lambda\geqslant\Lambda$, and we obtain $w_m\to0$ in E_λ as $m\to+\infty$. But since $w_m:=u_m-v_m$ and $v_m\to u$ as $m\to+\infty$, it follows that $u_m\to u$ as $m\to+\infty$.

Now we consider the functional Ψ_λ. In this case the limits (3.17) and the inequality (3.22) are still valid. However, we have to substitute $\widetilde{G}$ by $\lambda\widetilde{G}$ in inequality (3.23) and g by λg in inequality (3.24); hence, using assumption (G4) once more and Hölder inequality, we obtain

$$\begin{aligned}&\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}g(x,w_m)w_m\,\mathrm{d}x\\&\leqslant\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}\Big|\frac{g(x,w_m)}{w_m}\Big|^\tau\,\mathrm{d}x\Big]^{1/\tau}\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}|w_m|^{2\tau'}\,\mathrm{d}x\Big]^{1/\tau'}\\&\leqslant\Big[\int_{|w_m|>\delta}\lambda^{-1}|x|^{-b2^*(a,b)}a_1\widetilde{G}(x,w_m)\,\mathrm{d}x\Big]^{1/\tau}\Big[\int_{|w_m|>\delta}|x|^{-b2^*(a,b)}|w_m|^s\,\mathrm{d}x\Big]^{2/s}\\&\leqslant a_1^{1/\tau}\lambda^{-1/\tau}\big[\tilde{c}-\Psi_\lambda(u)\big]^{1/\tau}\|w_m\|^2_{L^s_{b2^*(a,b)}(\mathbb{R}^N)}+o(1)\\&\leqslant a_1^{1/\tau}\lambda^{-1/\tau}\big[\tilde{c}-\Psi_\lambda(u)\big]^{1/\tau}\big[d_1(\lambda b)^{-(p-s)/(p-2)}\|w_m\|_\lambda^s\big]^{2/s}+o(1)\end{aligned}$$

$$= d_2\lambda^{-1/\tau}(M\lambda^{-\alpha})^{1/\tau}(\lambda b)^{-\theta}\|w_m\|_\lambda^2 + o(1).$$

Using the same arguments as before, we have

$$\begin{aligned} o(1) &= \|w_m\|_\lambda^2 - \int_{\mathbb{R}^N} \lambda|x|^{-b2^*(a,b)}g(x,w_m)w_m\,\mathrm{d}x + o(1) \\ &\leqslant \Big[1 - \frac{\varepsilon}{\lambda b} - d_3\lambda^\beta\Big]\|w_m\|_\lambda^2 + o(1), \end{aligned} \tag{3.25}$$

where $d_3 := d_2 M^{1/\tau}/b^\theta$. For $N \geqslant 3$, we take $\alpha > 2/(2^* - 2)$ and $\beta := [2 - \alpha(p - 2)]/(\tau(p-2))$; in this way, we can choose $s < p < 2^*$ so that $\beta < 0$. For $N = 1$ or $N = 2$, we take $p > s$ large enough and we still have $\alpha > 2/(p-2)$ and $\beta < 0$. Finally, the limit (3.25) is valid for all λ large enough, provided we choose $\varepsilon < b$; therefore, we have $w_m \to 0$ in E_λ and $u_m \to u$ in E_λ as $m \to +\infty$. This concludes the proof of the lemma. □

Lemma 3.9. *There exist $\kappa, \rho > 0$ such that* $\inf \Phi((E^+ \oplus F) \cap S_\rho) \geqslant \kappa$.

Proof. Let $2 < p < 2^*(a,b)$ and consider the function $h(x) = Ax^2 + Bx^p$, whose behavior near the origin is determined by the quadratic term. Since the first and second terms of the functional (2.5) define a quadratic form which is positive definite on $E^+ \oplus F$, and has a behavior like that of the function h just defined, then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $0 \leqslant G(x,u) \leqslant \varepsilon|u|^2 - C_\varepsilon|u|^p$. Finally, applying a standard argument we obtain the conclusion of the lemma. □

4. Proofs of the theorems

Proof of Theorem 1.1. First we consider the functional Φ_λ. Suppose that $\phi_j \in C_0^\infty(\Omega)$ have disjoint supports and let $W := \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_k\}$. Then $W \subset F$ and, by Lemma 3.9, it holds $\Phi_\lambda((E_\lambda^+ \oplus F) \cap S_\rho) \geqslant \kappa$ where κ and ρ may depend on λ.

Since $G \geqslant 0$, according to Propositions 2.2 and 2.3, and Lemma 3.8, we only have to show that $\sup \Phi_\lambda(\widehat{E}_\lambda \oplus W)$ is bounded from above by a constant independent of λ and that there exists $R > 0$, possibly depending on λ, such that $\Phi_\lambda \leqslant 0$ whenever $u \in \widehat{E}_\lambda \oplus W$ and $\|u\|_\lambda \geqslant R$.

By assumption (G2), given $\eta > 0$ there exists $r_\eta > 0$ such that $G(x,u) \geqslant \frac{1}{2}\eta|u|^2$ if $|u| \geqslant r_\eta$. Let $u = v + w \in \widehat{E}_\lambda \oplus W$; then, noting that $\operatorname{supp} w \subset \operatorname{int} V^{-1}(0)$ and using the properties of the quadratic form a_λ, we have

$$\begin{aligned} \Phi_\lambda(u) &\leqslant \frac{1}{2}a_\lambda(w,w) - \int_\Omega |x|^{-b2^*(a,b)}G(x,u)\,\mathrm{d}x \\ &= \frac{1}{2}\|\nabla w\|_{L_a^2(\Omega)}^2 - \frac{1}{2}\eta\|u\|_{L_{b2^*(a,b)/2}^2(\Omega)}^2 \\ &\quad + \int_\Omega \Big[\frac{1}{2}\eta|x|^{-b2^*(a,b)}u^2 - |x|^{-b2^*(a,b)}G(x,u)\Big]\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\|\nabla w\|_{L_a^2(\Omega)}^2 - \frac{1}{2}\eta\|u\|_{L_{b2^*(a,b)/2}^2(\Omega)}^2 + C_\eta, \end{aligned} \tag{4.1}$$

where the constant C_η depends on η but does not depend on λ.

Since $w \in C_0^\infty(\Omega)$ and $a_\lambda(v,w)=0$, we have

$$\begin{aligned}
&\|\nabla w\|^2_{L^2_a(\Omega)} \\
&= \int_\Omega |x|^{-2a}\nabla w\cdot\nabla u\,\mathrm{d}x = \int_\Omega -\operatorname{div}\big[|x|^{-a}\nabla w\big]u\,\mathrm{d}x \\
&= \int_\Omega -|x|^{b2^*(a,b)/2}\operatorname{div}\big[|x|^{-a}\nabla w\big]|x|^{-b2^*(a,b)/2}u\,\mathrm{d}x \\
&\leqslant \Big(\int_\Omega |x|^{b2^*(a,b)/2}|\operatorname{div}\big[|x|^{-a}\nabla w\big]|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega |x|^{-b2^*(a,b)}u^2\,\mathrm{d}x\Big)^{1/2} \\
&\leqslant C\Big(\int_\Omega |\operatorname{div}\big[|x|^{-a}\nabla w\big]|^2\,\mathrm{d}x\Big)^{1/2}\|u\|_{L^2_{b2^*(a,b)/2}(\Omega)} \\
&\leqslant b_0\|\nabla w\|_{L^2_a(\mathbb{R}^N)}\|u\|_{L^2_{b2^*(a,b)/2}(\Omega)} \\
&\leqslant \frac{b_0^2}{2\eta}\|\nabla w\|^2_{L^2_a(\mathbb{R}^N)} + \frac{\eta}{2}\|u\|^2_{L^2_{b2^*(a,b)/2}(\Omega)},
\end{aligned} \tag{4.2}$$

where b_0 is a constant depending on the finite-dimensional subspace W. Choosing $\eta \geqslant b_0^2$, we obtain $\|\nabla w\|^2_{L^2} \leqslant \eta\|u\|^2_{L^2(\Omega)}$ and it follows from inequalities (4.2) that $\Phi_\lambda \leqslant C_\eta$. Using assumption (G2) again, and since $\widehat{E}_\lambda \oplus W$ has finite dimension, it follows that $\Phi_\lambda(u) \leqslant 0$ for $\|u\|_\lambda$ large enough.

If λ is large enough, then the functional Φ_λ satisfies the Cerami $(C)_c$-condition for every $c \leqslant C_\eta$. Moreover, we have $\Phi_\lambda \leqslant 0$ outside a certain ball. Therefore, by Proposition 2.2, the functional Φ_λ has a nontrivial critical point u.

Now we consider the functional Ψ_λ. We define W as in the previous case and we have $\Psi_\lambda((E_\lambda^+ \oplus F)\cap S_\rho) \geqslant \kappa$, where κ and ρ may depend on λ and are given by Lemma 3.9.

Inequality (4.1) is still valid for Ψ_λ if we replace G by λG; therefore, denoting by $C_{\eta,\lambda}$ a constant such that

$$\int_\Omega \Big(\frac12\,\eta|x|^{-2(a+1)}|u|^2 - \lambda|x|^{-b2^*(a,b)}G(x,u)\Big)\,\mathrm{d}x \leqslant C_{\eta,\lambda},$$

we have $\Psi_\lambda \leqslant C_{\eta,\lambda}$. To obtain the conclusion, we have to show that $C_{\eta,\lambda} \leqslant M\lambda^{-\alpha}$, where $\alpha > 2/(p-2)$ was identified in Lemma 3.8. Since $G \geqslant 0$, setting $U_{\lambda,\alpha} := \{x \in \mathbb{R}^N : |u(x)| \leqslant \lambda^{-\alpha/2}\}$, we have

$$\int_{\Omega\cap U_{\lambda,\alpha}} \Big[\frac12\eta|x|^{-2(a+1)}|u|^2 - \lambda|x|^{-b2^*(a,b)}G(x,u)\Big]\,\mathrm{d}x \leqslant C_1\lambda^{-\alpha}$$

for some fixed constant C_1, where $\eta \geqslant b_0^2$ is fixed.

By assumptions (G1) and (G3), there exists a constant $C_2 > 0$ such that $G(x,u) \geqslant C_2|u|^\delta$ whenever $|u(x)| \leqslant 1$. In this way, if $\lambda^{-\alpha/2} \leqslant |u| \leqslant 1$, then

$$\begin{aligned}
&\frac12\eta|x|^{-2(a+1)}|u|^2 - \lambda|x|^{-b2^*(a,b)}G(x,u) \\
&\leqslant \frac12\eta|x|^{-2(a+1)}|u|^2 - \lambda C_2|x|^{-b2^*(a,b)}|u|^\delta \\
&\leqslant \Big[\frac12\,\eta|x|^{-2(a+1)} - C_2|x|^{-b2^*(a,b)}\lambda^{1-\alpha(\delta-2)/2}\Big]|u|^2.
\end{aligned}$$

Since $\delta < 2^*$, we may choose $\alpha > 2/(2^*-2)$ so that $1-\alpha(\delta-2)/2 > 0$. Hence, for large λ the right-hand side of the previous inequality is non-positive. It is clear that if $|u| \geqslant 1$ and λ is large, then $\frac{1}{2}\eta|x|^{-2(a+1)}|u|^2 - \lambda|x|^{-b2^*(a,b)}G(x,u) \leqslant 0$. It follows that $\Psi_\lambda(u) \leqslant C_1\lambda^{-\alpha}$. Therefore, by Proposition 2.3, the functional Ψ_λ has k pairs of critical points. This concludes the proof of the theorem. $\square$

Proof of Theorem 1.2. First we consider the functional Φ_λ. Given $\overline{\lambda} > 0$, there exists $\lambda \geqslant \overline{\lambda}$ such that $1 < \mu_k(\lambda) < 1 + 1/\lambda$ for some $k \in \mathbb{N}$. We set $W := \operatorname{span}\{e_k(\lambda)\}$ and write $u = v + w \in \widehat{E}_\lambda \oplus W$. By the orthogonality of $\widehat{E}_\lambda$ and W, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|u|^2\,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|w|^2\,\mathrm{d}x.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Phi_\lambda(u) &\leqslant \frac{1}{2}a_\lambda(w,w) - \int_{\operatorname{supp} V^-} |x|^{-b2^*(a,b)}G(x,u)\,\mathrm{d}x \\
&= \frac{1}{2}\lambda(\mu_k - 1)\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|w|^2\,\mathrm{d}x - \frac{1}{2}\,\eta\|u\|^2_{L^2_{a+1}(\operatorname{supp} V^-)} \\
&\quad + \int_{\operatorname{supp} V^-} \Big[\frac{1}{2}\,\eta|x|^{-2(a+1)}|u|^2 - |x|^{-b2^*(a,b)}G(x,u)\Big]\,\mathrm{d}x \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|w|^2\,\mathrm{d}x \\
&\quad - \frac{1}{2}\,\frac{\eta}{\|V^-\|_{L^\infty(\mathbb{R}^N)}}\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|u|^2\,\mathrm{d}x + C_\eta \\
&\leqslant \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|w|^2\,\mathrm{d}x \\
&\quad - \frac{1}{2}\,\frac{\eta}{\|V^-\|_{L^\infty(\mathbb{R}^N)}}\int_{\mathbb{R}^N} |x|^{-2(a+1)}V^-(x)|w|^2\,\mathrm{d}x + C_\eta,
\end{aligned}$$

where the constant $C_\eta > 0$ is independent of λ and can be chosen following the same steps as in the proof of Theorem 1.1. Choosing $\eta \geqslant \|V^-\|_{L^\infty(\mathbb{R}^N)}$, we obtain $\Phi_\lambda \leqslant C_\eta$. If λ is large enough, then the functional Φ_λ verifies the Cerami $(C)_c$-condition for all values $c \leqslant C_\eta$. Following the proof of Theorem 1.1, we can prove that outside a certain ball we have $\Phi_\lambda \leqslant 0$. Therefore, by Proposition 2.2 the functional Φ_λ has a nontrivial critical point u.

The study of the case for the functional Ψ_λ follows closely the ideas for the corresponding case in the proof of Theorem 1.1; we just have to prove that $C_{\eta,\lambda} \geqslant C_1\lambda^{-\alpha}$ whenever $\lambda \geqslant \overline{\lambda}$ for $\overline{\lambda}$ large enough. Therefore, by Proposition 2.3, the functional Ψ_λ has k pairs of critical points. This concludes the proof of the theorem. $\square$

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