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CAUCHY-KOWALEVSKI AND POLYNOMIAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The Cauchy-Kowalevski Theorem is the foremost result guaranteeing existence and uniqueness of local solutions for analytic quasilinear partial differential equations with Cauchy initial data. The techniques of Cauchy-Kowalevski may also be applied to initial-value ordinary differential equations. These techniques, when applied in the polynomial ordinary differential equation setting, lead one naturally to a method in which coefficients of the series solution are easily computed in a recursive manner, and an explicit majorization admits a clear a priori error bound. The error bound depends only on immediately observable quantities of the polynomial system; coefficients, initial conditions, and polynomial degree. The numerous benefits of the polynomial system are shown for a specific example.

1. Introduction

The Cauchy-Kowalevski Theorem is the main tool in showing the existence and uniqueness of local solutions for analytic quasilinear partial differential equations (PDE) with Cauchy initial data. Cauchy developed a proof in a restricted setting by 1842 [3], and in 1875 Kowalevski presented the full result [11]; existence of a unique solution to the general quasilinear system of partial differential equations given initial conditions prescribed on some non-characteristic curve. In [8], a proof in the fully nonlinear setting is presented. The Cauchy-Kowalevski argument is based on the construction of a power series solution, in which the coefficients of the series expansion are reconstructed recursively, and the method of majorants applied to verify that this solution converges locally. Convergence is demonstrated by comparison with the analytic solution of an associated PDE.

Although the Picard-Lindelöf Theorem is the fundamental local existence argument for a large class of initial value ordinary differential equations (IVODE), in 1835 Cauchy demonstrated existence and uniqueness in the ODE setting, applying a majorant based argument similar to that both he and Kowalevski would later use in the PDE setting. That is, Cauchy methods can be used to show that u satisfies the real analytic ODE $d_t u(t) = f(u(t))$, where $u(0) = u_0$ using a constructive approach, provided $f(u)$ is analytic near u_0. A nice treatment may be found in [5].

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Given that the power series solution is directly accessible via the Cauchy-Kowalevski construction but that the method is rarely applied suggests practical difficulties. In fact, the coefficients of the series solution can be tedious to construct as typically posed, as is a key constant in the comparison solution. In this paper, we demonstrate that a subtle recasting of the ODE system meliorates these difficulties: the coefficients of the analytic solution become remarkably easy to recover, and a computable choice of the key constant in the majorization leads to an *attractive a priori error bound.* To make these ideas clear, we consider the simple quasilinear problem

$$d_t u(t) = f(u(t)) := \frac{\mathrm{e}^{2u(t)^2}}{\sin u(t)}, \quad \text{with } u(0) = 1. \tag{1}$$

We first consider (1) using the methods of Cauchy, and identify steps in which the construction of solution becomes tedious. We then recast the problem as a polynomial system, as might be done when using Taylor series based automatic differentiation, and apply the same methods. It will be clear the computations necessary to generate the series solution are basic, and that a simple majorization which depends only on the magnitude of the *initial conditions*, the *degree* of the polynomial system and the magnitude of the *constant coefficients* of the system leads to an explicit bound of the remainder when approximating with the Taylor Polynomial. Although not demonstrated here, the method applied is quite general. The authors view this note as complimentary to [19]. Most importantly, we conjecture that it may be possible to extend the method to analytic IVPDE.

2. Recasting (non)linear ODE(s) as polynomial systems: Why?

Ordinary differential equations, particularly nonlinear and those with singularities, play a fundamental role in understanding the principles that govern the world around us. Left in their original (or *classic*) form, various analytic and numeric methods exist which are problem specific, yet there remains a need for a systematic method to calculate solutions of general problems. The approach presented here, perhaps first introduced by Cauchy and subsequently rediscovered and coupled with power series methods by Fehlberg in 1964 ([7]) and others since, is simple and surprisingly general. A recasting of the original ODE as a system of constant coefficient polynomial ODEs via an introduction of auxiliary variables leads to a straight forward iterative calculation of power series coefficients. This allows a clear and systematic construction of numeric solutions and provides an immediate and explicit *a priori* error bound.

A polynomial system is useful computationally, and methods are available to recast an impressively wide variety of ODEs into an augmented polynomial system. This includes ODEs with right-hand sides involving compositions of exponential, logarithmic, and trigonometric functions, as well as those involving the algebraic operations, including exponentiation of complex power and encompassing general reciprocals and singularities. For example, the second order ODE,

$$y''\Big[1 + \frac{\sqrt{2}}{y'^2}\big(\frac{x}{yy'}\big)^{\sqrt{2}-1}\Big] = \Big[\sqrt{2}\big(\frac{x}{yy'}\big)^{\sqrt{2}} + 1\Big]\Big[\frac{y}{x^2} - \frac{y'}{x}\Big] + \frac{\pi^2}{16}y, \tag{2}$$

with $()' := \frac{d}{dx}$, used to model the torsional deformation of a compressible elastic solid cylinder composed of a generalized Blatz-Ko material, does not appear amiable to classic power series methods, and yet its series solution to arbitrary order is easily computable from its equivalent polynomial system [20]. While such computational advantage is not the focus here, see [2, 9, 10, 16] for examples and discussion in the automatic differentiation setting or [13, 15, 14, 17] in the ODE setting. Recasting an ODE as a polynomial system also makes analysis tractable. We now explore one such application: a proof of existence and uniqueness.

3. Cauchy solution: the classic setting

We begin with the precarious assumption that a locally analytic solution $u(t)$ to (1) exists, and repeatedly differentiate the equation, using the fact that $f(u)$ is analytic in u near the initial condition.

$$\begin{aligned}
d_t^2u(t) &= d_uf(u)d_tu \\
&= -\frac{e^{4\,u^2}\,(4\,u\sin u-\cos u)}{(-1+\cos^2 u)\sin u}, \\
d_t^3u(t) &= d_u^2f(u)[d_tu]^2+d_uf(u)d_t^2u \\
&= -\frac{e^{6\,u^2}\left(32\,u^2\cos^2 u+2\,\cos^2 u+16\,u\sin u\cos u-5-32\,u^2\right)}{(1-2\,\cos^2 u+\cos^4 u)\sin u} \\
d_t^4u(t) &= d_u^3f(u)[d_tu]^3+3d_u^2f(u)d_t^2ud_tu+d_uf(u)d_t^3u \\
&= \frac{e^{8\,u^2}\,(M_1+M_2)}{(-1+3\,\cos^2 u-3\,\cos^4 u+\cos^6 u)\sin u}
\end{aligned}$$

where

$$\begin{gathered}
M_1=-288\,u^2\cos^3 u-22\,\cos^3 u+384\,u^3\sin u\cos^2 u-140\,u\sin u \\
M_2=40\,u\sin u\cos^2 u+37\,\cos u+288\,\cos uu^2+384\,u^3\sin u
\end{gathered}$$

and

$$d_t^nu(t)=p_n(f(u),d_uf(u),d_u^2f(u),\dots,d_u^{n-1}f(u)), \tag{3}$$

where $p_n(\cdot)$ denotes a polynomial in n variables (here taken from the set of derivatives of f with respect to u of order less than n; i.e., $\{d_u^{k-1}f\}, k=1,\dots,n$, and having positive integer coefficients). By this process, all coefficients of the power series representation of $u(t)$ may be built;

$$u(t)=\sum_{k=0}^{\infty}\frac{1}{k!}d_t^ku(0)\,t^k. \tag{4}$$

Note that the form of the polynomial p_n in expression (3) allows the coefficients of the power series to be recovered recursively, although the complexity of calculation may (and usually does) grow exponentially.

By its very construction, this power series (4) yields a unique classical solution to the initial-value ODE if it can be shown to converge. Cauchy demonstrated convergence by comparison with a related analytic initial-value ODE, whose individual coefficients majorize (absolutely bound) those of (4). We briefly illustrate the argument. We begin with the assumption of the theorem that $f(u)$ is analytic in some interval of radius $R\in\mathbb{R}$ about $u=1$, and remark that in practice R might be quite difficult to determine. Then for any positive $r<R$, there exists

$$C_\infty:=\max_k\{|C_k|\}<\infty, \quad \text{where } C_n=\frac{1}{n!}d_u^nf(1)r^n,$$

which provides the bound

$$\max_k \Big|\frac{1}{k!} d_u^k f(1)\Big| \le C_\infty r^{-k}$$

on the Taylor coefficients of $f(u)$ about $u(0) = 1$. Next we define g via the geometric series

$$g(v) := \sum_{k=0}^{\infty} C_\infty r^{-k}(v-1)^k = C_\infty \frac{r}{r-(v-1)} \quad \text{when } |v-1| < r,$$

and the comparison initial-value ODE

$$d_t v(t) = g(v(t)) \quad \text{with } v(0) = 1. \tag{5}$$

The form of equation (5) is motivated by the observation that the polynomial p_n generated in this case is identical in form to that of (3), allowing a direct comparison of coefficients of $u(t)$ with those of $v(t)$. Also, $g(v)$ majorizes $f(u)$ near 1 and allows (5) an analytic solution $v(t)$ near 0. When $|v-1| < r$,

$$|d_u^n f(1)| = n!\Big|\frac{1}{n!} d_u^n f(1)\Big| \le n!\, C_\infty r^{-n} = d_v^n g(1)$$

for all n. Noting that the structure of the polynomial in (3) is identical in (1) and (5), it follows that

$$\begin{aligned}
|d_t^n u(0)| &= |p_n(f(1), \dots, d_u^{n-1} f(1))| \\
&\le p_n(|f(1)|, \dots, |d_u^{n-1} f(1)|) \\
&\le p_n(g(1), \dots, d_u^{n-1} g(1)) \\
&= d_t^n v(0),
\end{aligned}$$

demonstrating that $u(t)$ is majorized by $v(t)$ in a neighborhood of $t = 0$. It follows immediately that

$$|u(t)| = \Big|\sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0)\, t^k\Big| \le \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k v(0)\, |t|^k \le v(|t|).$$

The existence of an analytic solution of (5) with radius of convergence $|t| < \frac{r}{2C_\infty}$, given by

$$v(t) = 1 + r - r\sqrt{1 - 2C_\infty t/r}, \tag{6}$$

confirms that $u(t)$ must also be locally analytic about $t = 0$.

This argument relies on C_∞, a constant which in practice is often difficult to ascertain. In our example, it can be shown that $R = 1$, with $r = 0.9$, we have

$$\begin{aligned}
C_\infty &= \max_k \{C_0, C_1, C_2, \dots\} \\
&= \max_k \Big\{\frac{e^2}{\sin 1}, \frac{9e^2(4\sin 1 - \cos 1)}{10\sin^2 1}, \frac{81e^2(2 + 19\sin^2 1 - 8\cos 1 \sin 1)}{200\sin^3 1}, \dots\Big\},
\end{aligned}$$

and it is not immediately clear where the maximum might occur. An explicit computation of the C_k terms, plotted in Figure 1, suggests that the maximum occurs near $k = 6$, and one can easily imagine the complexity of C_k. It is also worth noting that the ODE given by (1) is simple in comparison to the systems often used to model problems of technological importance.

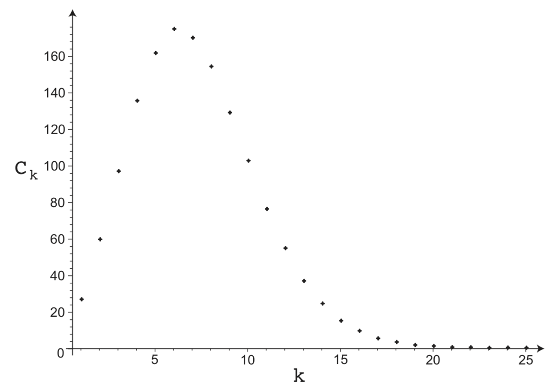

FIGURE 1. C_k coefficient list

4. CAUCHY SOLUTION: THE POLYNOMIAL SETTING

We now apply similar techniques to an equivalent polynomial system. Recall the original problem:

$$d_t u(t) = \frac{e^{2u(t)^2}}{\sin u(t)}, \quad \text{with } u(0) = 1.$$

Now consider the introduction of the auxiliary variables:

$$x(t) := \frac{e^{2u(t)^2}}{\sin u(t)}, \quad y(t) := \frac{\cos u(t)}{\sin u(t)}.$$

These auxiliary variables close the system of successive derivatives that are the foundation of the Cauchy method. This approach is evident in the methods of [9, 16, 10, 2] with an automatic differentiation flavor, or as suggested by examples treated in [14, 13, 17].

We now generate the polynomial system

$$\begin{gathered} d_t u = x \quad u(0) = 1 \\ d_t x = (4xu - xy)\, d_t u = 4x^2 u - x^2 y \quad x(0) = \frac{e^2}{\sin 1} \\ d_t y = -\left(1 + y^2\right) d_t u = -x - y^2 x \quad y(0) = \cot 1. \end{gathered}$$

The first equation is our original ODE; the additional equations serve a purely computational purpose.

As earlier, we assume the existence of an analytic solution u. We continue by assuming a formal power series for x and y, which can be shown (along with u) to be convergent via a majorant argument. Now,

$$u(t) = \sum_{k=0}^{\infty} u_k t^k, \quad x(t) = \sum_{k=0}^{\infty} x_k t^k, \quad y(t) = \sum_{k=0}^{\infty} y_k t^k.$$

The constant on which the previous argument relies is C_∞, which is difficult in general to construct. The constants related to the polynomial argument are easy to construct. In this new setting, consider the companion problem

$$d_t z = \mathcal{C} z^m \quad z(0) = c. \tag{7}$$

The analytic solution to (7) is given explicitly by

$$z(t) = \left(\mathcal{C}t - \mathcal{C}tm + c^{1-m}\right)^{-(m-1)^{-1}}. \tag{8}$$

In general, (8) will bound solutions to all autonomous polynomial systems of degree m, with suitable choice of $\mathcal{C}$ and initial condition c. If $\mathcal{C} = 5, m = 3$ and $c = \mathrm{e}^2/\sin 1$, we claim that $z(t)$ majorizes $u(t), x(t)$ and $y(t)$. These parameters arise naturally when considering the majorization; $\mathcal{C}$ from the largest row sum of the absolute value of coefficients in the system, m from the largest degree of the polynomial system, and c from the largest of the absolute value of the initial conditions and 1. As a brief exercise, we demonstrate this by applying an inductive argument to verify that the coefficients of the power series representation of $z(t) = \sum_{k=0}^{\infty} z_k t^k$ bound those of $x(t)$. Clearly $z_0 = c \geq \{|u_0|, |x_0|, |y_0|\}$, since $c \geq \{1, \mathrm{e}^2/\sin 1, \cot 1\}$ the absolute value of the initial conditions. Obviously,

$$z_1 = 5z_0^3 \geq |4x_0^2 u_0 - x_0^2 y_0| = |x_1|.$$

Assuming $z_k \geq \{|u_k|, |x_k|, |y_k|\}$ for $k = 0, \dots, n$, it follows that

$$\begin{aligned} z_{n+1} &= \frac{1}{n+1} \cdot 5 \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} z_i z_{k-i} \Big) z_{n-k} \\ &\geq \frac{1}{n+1} \cdot \Big| 4 \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} x_i x_{k-i} \Big) u_{n-k} - \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} x_i x_{k-i} \Big) y_{n-k} \Big| \\ &= |x_{n+1}| \end{aligned} \tag{9}$$

where a Cauchy product of two series has been applied twice. An important (and obvious) observation used in (9) is that

$$x_{n+1} = \frac{1}{n+1} \cdot \Big[4 \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} x_i x_{k-i} \Big) u_{n-k} - \sum_{k=0}^{n} \Big(\sum_{i=0}^{k} x_i x_{k-i} \Big) y_{n-k} \Big],$$

which can easily be implemented to construct the coefficient x_{n+1} using only coefficients of order n or less. The software tools `ATOMFT`, `Taylor`, and most recently `TIDES` are three such packages that exploit this recursive feature [4, 10, 1], although only the last appears to still be supported. The polynomial used to construct coefficients in the classic setting, p_n, has now been replaced by an algebraic expression whose complexity is only $\mathcal{O}(n^3)$. (In fact, augmenting the system allows reduction to $\mathcal{O}(n^2)$ [18].) Since $z(t)$ converges on some open interval containing $t = 0$ and majorizes $x(t)$ for $|t| < 1$, $x(t)$ must also converge on the intersection of these intervals. The demonstration is now complete; an explicit verification that $x(t)$ converges via a term-by-term comparison with the convergent series representation of $z(t)$. It is easy to see that a similar argument may be used for $u(t)$ and $y(t)$.

In addition to a simple coefficient recursion and explicit majorization, the polynomial comparison solution gives rise to an easily computable local *a priori* error bound. To accomplish this, the comparison solution $z(t)$ is bounded by $w(t)$, a function with a geometric series representation. We begin with the recurrence relation (see [19] for a detailed development) for the coefficients of z,

$$z_{n+1} = \frac{(1 + (m-1)n)c^{m-1}\mathcal{C}}{n+1} z_n \quad z_0 = c, \quad \text{for } n \geq 1. \tag{10}$$

For $m \geq 2$,

$$\frac{(1+(m-1)n)c^{m-1}\mathcal{C}}{n+1} \leq (m-1)c^{m-1}\mathcal{C} := \mathcal{C}_\infty. \tag{11}$$

Combining (10) and (11) yields $z_{n+1} \leq \mathcal{C}_\infty z_n$. If

$$w_{n+1} = \mathcal{C}_\infty w_n, \quad \text{with } w_0 = c, \tag{12}$$

then the coefficients of w majorize those of z (and therefore u), and $w(t)$ majorizes $z(t)$ (and $u(t)$). The recurrence relation (12) leads directly to the geometric series,

$$w(t) = \frac{c}{1-\mathcal{C}_\infty t} = c\sum_{k=0}(\mathcal{C}_\infty t)^k, \quad \text{when } |t| < \frac{1}{\mathcal{C}_\infty}.$$

The function w may be interpreted as a solution to the IVODE

$$d_t w(t) = \mathcal{C}_\infty w, \quad w(0) = c \tag{13}$$

where $\mathcal{C}_\infty$ bounds the coefficient growth of terms of z, playing much the same role as C_∞. Here, however, $\mathcal{C}_\infty$ is trivial to compute from (11).

Finally, a simple bound on the remainder term $\mathcal{R}_n$, given by

$$\mathcal{R}_n(t) := \left|u(t) - \sum_{k=0}^{n} u_k t^k\right| \leq c \sum_{k=n+1}^{\infty} \mathcal{C}_\infty |t|^k \leq c|\mathcal{C}_\infty t|^{n+1}\frac{1}{1-|\mathcal{C}_\infty t|}, \tag{14}$$

provides a concise and computable error bound. For (1),

$$\mathcal{R}_n(t) \leq \frac{10\mathrm{e}^4}{\sin^2 1}|t|^{n+1}\frac{1}{1-\frac{10\mathrm{e}^4}{\sin^2 1}|t|}.$$

For a detailed discussion, and an example for which this bound is tight, see [19]. See [12] for a detailed discussion of Interval Analysis, an alternative approach.

In practice, this error bound may be used in a variety of ways. Here, it suggests a small interval of convergence, with $t < 1/\mathcal{C}_\infty = \sin^2 1/10\mathrm{e}^4 \approx 1.3E-3$, and points to possible singularities in the solution. It can be shown that the solution of (1) becomes singular quickly beyond $t \approx 2.6E-2$. It may used to construct a robust marching method, and provide solutions with known error in regions of particular interest.

Conclusion. We have demonstrated that recasting the original ODE as a polynomial system has several surprising benefits. While the simple differential equation studied here is not tied to a particular modeling scenario, it demonstrates how the conversion makes typically abstract analysis very concrete. The techniques of Cauchy-Kowalevski, when applied to a polynomial system, lead one naturally to a method in which; *(i)* coefficients are easily computed in a recursive manner; i.e., u_{n+1}, x_{n+1}, and y_{n+1} only depend on products and sums of $\{u_k, x_k, y_k\}_{k=1..n}$, *(ii)* the majorization is explicit, and *(iii)* there is a clear *a priori* error bound. The majorization and error bound depend only on immediately observable quantities of the recast system; coefficient sums, initial conditions, and degree.

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