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EXISTENCE OF SOLUTIONS FOR NON-AUTONOMOUS FUNCTIONAL EVOLUTION EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract. In this work, we study the existence of mild solutions and strict solutions of semilinear functional evolution equations with nonlocal conditions, where the linear part is non-autonomous and generates a linear evolution system. The fraction power theory and α-norm are used to discuss the problems so that the obtained results can be applied to the equations in which the nonlinear terms involve spatial derivatives. In particular, the compactness condition or Lipschitz condition for the function g in the nonlocal conditions appearing in various literatures is not required here. An example is presented to show the applications of the obtained results.

1. Introduction

In this article, we study the existence of solutions for semilinear neutral functional evolution equations with nonlocal conditions. More precisely, we consider the nonlocal Cauchy problem

$$\begin{gathered}\frac{d}{dt}[x(t)+F(t,x(t))]+A(t)x(t)=G(t,x(r(t))),\quad 0\le t\le T,\\ x(0)+g(x)=x_0,\end{gathered}\tag{1.1}$$

where the family $\{A(t): 0\le t\le T\}$ of linear operators generates a linear evolution system, and F,G are given functions to be specified later. The nonlocal Cauchy problem was considered by Byszewski [4] and the importance of nonlocal conditions in different fields has been discussed in [4] and the references therein. In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problems with nonlocal conditions have been studied extensively, see, for example, papers [1]-[9] and the references therein.

When $F(\cdot,\cdot)=0$ and A generating a C_0- semigroup in Eq. (1.1), Byszewski and Akca have investigated the existence of mild solutions and classical solutions in paper [5] by using Schauder's fixed point theorem. To take away an unsatisfactory

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condition on solutions and extend the results in [5] to neutral equations, in [14] the authors have studied the existence of mild solutions and strong solutions for the equations with the form

$$\frac{d}{dt}[x(t)+F(t,x(h_1(t)))]+Ax(t)=G(t,x(h_2(t))),\quad 0\le t\le T,$$
$$x(0)+g(x)=x_0\in X,$$

where the operator $-A$ generates a compact analytic semigroup. The main tools and techniques in [14] are the properties of fractional power and Sadovskii fixed point theorem. Papers [3], [8] and [12] have established the corresponding results for the situation in which the linear operator A is non-densely defined. Paper [9] and [13] have investigated the existence topics on impulsive nonlocal problems, and in Papers [1] and [25] the authors have studied the nonlocal Cauchy problems for the case that A generates a nonlinear semigroup. Existence of solutions for quasilinear delay integrodifferential equations with nonlocal conditions has been established in Paper [2]. In order to establish the existence results for the non-autonomous equations, Paper [15] has considered existence of solutions for (1.1) which is a more general situation as $A(t)$ is non-autonomous. However, although the system

$$\begin{aligned}&\frac{\partial}{\partial t}[u(t,x)+f(t,u(b(t),x),\frac{\partial}{\partial x}u(b_1(t),x))]+c(t)\frac{\partial^2}{\partial x^2}u(t,x)\\&=h(t,u(a(t),x),\frac{\partial}{\partial x}u(a_1(t),x)),\\&\qquad u(0)+g(x)=u_0.\end{aligned}\tag{1.2}$$

can also be rewritten as an abstract equation of form (1.1), those results founded in [15] become invalid for it, since the functions f,h in (1.2) involve spatial derivatives.

The purpose of the present note is to extend and develop the work in [15] and [13]. We shall discuss this problem by using fractional power operators theory and $\alpha-$norm; i.e., we will restrict this equation in a Banach space $X_\alpha(t_0)(\subset X)$ and investigate the existence and regularity of mild solutions for (1.1). In Particular, borrowing the idea from [17] we do not require the function g in the nonlocal condition satisfy the compactness condition or Lipschitz condition, instead, it is continuous and is completely determined on $[\tau,T]$ for some small $\tau>0$. The compactness condition or Lipschitz condition for g appear, respectively, in almost all the papers on the topics of nonlocal problem, for example in [3, 5, 7, 13, 18, 25]. Although paper [25] has also discussed the case that the function g is continuous, it assumed additionally some pre-compact condition relative to g. In addition, the obtained results extend also the ones in [20] and [21].

This article is organized as follows: we firstly introduce some preliminaries about the linear evolution operator and fractional power operator theory in Section 2. The main results are arranged in Section 3. In Subsection 3.1 we discuss the existence of mild solutions by Sadovskii fixed point theorem and limit arguments, and in Subsection 3.2 we show the regularity of mild solutions. Finally, an examples is presented in Section 4 to show the applications of our obtained results.

2. Preliminaries

Throughout this paper X will be a Banach space with norm $\|\cdot\|$. For the family $\{A(t):0\le t\le T\}$ of linear operators, we impose the following restrictions:

- (B1) The domain $D(A)$ of $\{A(t) : 0 \leq t \leq T\}$ is dense in X and independent of t, $A(t)$ is closed linear operator;
- (B2) For each $t \in [0, T]$, the resolvent $R(\lambda, A(t))$ exists for all λ with $Re\lambda \leq 0$ and there exists $K > 0$ so that $\|R(\lambda, A(t))\| \leq K/(|\lambda| + 1)$;
- (B3) There exists $0 < \delta \leq 1$ and $K > 0$ such that $\|(A(t) - A(s))A^{-1}(\tau)\| \leq K|t - s|^{\delta}$ for all $t, s, \tau \in [0, T]$;
- (B4) For each $t \in [0, T]$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, the resolvent $R(\lambda, A(t))$, is a compact operator.

Under these assumptions, the family $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system, or called linear evolution operator, $U(t, s)$, $0 \leq s \leq t \leq T$, and there exists a family of bounded linear operators $\{R(t, \tau)|0 \leq \tau \leq t \leq T\}$ with $\|R(t, \tau)\| \leq K|t - \tau|^{\delta-1}$ such that $U(t, s)$ has the representation

$$U(t, s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(\tau)} R(\tau, s) d\tau, \tag{2.1}$$

where $\exp(-\tau A(t))$ denotes the analytic semigroup having infinitesimal generator $-A(t)$ (note that Assumption (B2) guarantees that $-A(t)$ generates an analytic semigroup on X). The family of the linear operator$\{U(t, s) : 0 \leq s \leq t \leq T\}$ satisfies the following properties:

- (a) $U(t, s) \in L(X)$, the space of bounded linear transformations on X, whenever $0 \leq s \leq t \leq T$ and for each $x \in X$, the mapping $(t, s) \to U(t, s)x$ is continuous;
- (b) $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t \leq T$;
- (c) $U(t, t) = I$;
- (d) $U(t, s)$ is a compact operator whenever $t - s > 0$;
- (e) $\frac{\partial}{\partial t} U(t, s) = -A(t)U(t, s)$, for $s < t$.

Condition (B4) ensures the generated evolution operator satisfies (d) (see [10, Proposition 2.1]). We have also the following inequalities:

$$\|e^{-tA(s)}\| \leq K, \quad t \geq 0,\ s \in [0, T],$$
$$\|A(s)e^{-tA(s)}\| \leq \frac{K}{t}, \quad t,\ s \in [0, T],$$
$$\|A(t)U(t, s)\| \leq \frac{K}{|t - s|}, \quad 0 \leq s \leq t \leq T.$$

Furthermore, Assumptions (B1)–(B3) imply that for each $t \in [0, T]$, the integral

$$A^{-\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} s^{\alpha-1} e^{-sA(t)} ds$$

exists for each $\alpha \in (0, 1]$. The operator defined by this formula is a bounded linear operator and yields $A^{-\alpha}(t)A^{-\beta}(t) = A^{-(\alpha+\beta)}(t)$. Thus, we can define the fractional power as

$$A^{\alpha}(t) = [A^{-\alpha}(t)]^{-1},$$

which is a closed linear operator with $D(A^{\alpha}(t))$ dense in X and $D(A^{\alpha}(t)) \subset D(A^{\beta}(t))$ for $\alpha \geq \beta$. $D(A^{\alpha}(t))$ becomes a Banach space endowed with the norm $\|x\|_{\alpha,t} = \|A^{\alpha}(t)x\|$, and is denoted by $X_{\alpha}(t)$.

The following estimates and lemmas are from ([11, Part II]):

$$\|A^{\alpha}(t)A^{-\beta}(s)\| \leq C_{(\alpha,\beta)}, \tag{2.2}$$

where $C_{(\alpha,\beta)}$ is a constant related to T and δ, t, $s \in [0,T]$ and $0 \le \alpha < \beta$.

$$\|A^{\beta}(t)e^{-sA(t)}\| \le \frac{C_\beta}{s^\beta} e^{-ws}, \quad t > 0,\ \beta \le 0,\ w > 0, \tag{2.3}$$

$$\|A^{\beta}(t)U(t,s)\| \le \frac{C_\beta}{|t-s|^\beta}, \quad 0 < \beta < \delta + 1, \tag{2.4}$$

$$\|A^{\beta}(t)U(t,s)A^{-\beta}(s)\| \le C'_\beta, \quad 0 < \beta < \delta + 1, \tag{2.5}$$

for some $t > 0$, where $C_{(\alpha,\beta)}$ C_β and C'_β indicate their dependence on the constants α, β.

Lemma 2.1. *Assume that* (B1)–(B3) *hold. If* $0 \le \gamma \le 1$, $0 \le \beta \le \alpha < 1 + \delta$, $0 < \alpha - \gamma \le 1$, *then for any* $0 \le \tau < t + \Delta t \le t_0$, $0 \le \zeta \le T$,

$$\|A^\gamma(\zeta)(U(t+\Delta t,\tau) - U(t,\tau))A^{-\beta}(\tau)\| \le C(\beta,\gamma,\alpha)(\Delta t)^{\alpha-\gamma}|t-\tau|^{\beta-\alpha}. \tag{2.6}$$

Lemma 2.2. *Assume that* (B1)–(B3) *hold and let* $0 \le \gamma < 1$. *Then for any* $0 \le \tau < t + \Delta t \le t_0$ *and for any continuous function* $f(s)$,

$$\begin{aligned} &\|A^\gamma(\zeta)[\int_t^{t+\Delta t} U(t+\Delta t,s)f(s)ds - \int_\tau^t U(t,s)f(s)ds]\| \\ &\le C_\gamma(\Delta t)^{1-\gamma}(|\log(\Delta t)| + 1)\max_{\tau\le s\le t+\Delta t}\|f(s)\|. \end{aligned} \tag{2.7}$$

For more details about the theory of linear evolution system, operator semigroups and fraction powers of operators, we refer the reader to [11, 22, 23].

The considerations of this paper are based on the following result.

Theorem 2.3 ([24])**.** *Let P be a condensing operator on a Banach space X; i.e., P is continuous and takes bounded sets into bounded sets, and $\alpha(P(B)) \le \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$. If $P(H) \subset H$ for a convex, closed and bounded set H of X, then P has a fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of non-compactness).*

3. Main results

The main results of this note are presented in this section. We shall study the existence and regularity of mild solutions for (1.1), and we consider this problem on the Banach subspace $X_\alpha(t_0)$ defined in Section 2 for some $0 < \alpha < 1$ and $t_0 \in [0,T]$.

3.1. **Existence of mild solutions.**

Firstly we consider the existence of mild solutions for (1.1). The mild solutions are defined as follows.

Definition 3.1. A continuous function $x(\cdot) : [0,T] \to X_\alpha(t_0)$ is said to be a mild solution of the nonlocal Cauchy problem (1.1), if the function

$$U(t,s)A(s)F(s,x(s)), \quad s \in [0,t)$$

is integrable on $[0,t)$ and the following integral equation is verified:

$$\begin{aligned} x(t) &= U(t,0)[x_0 + F(0,x(0)) - g(x)] - F(t,x(t)) \\ &\quad + \int_0^t U(t,s)A(s)F(s,x(s))ds \\ &\quad + \int_0^t U(t,s)G(s,x(r(s)))ds, \quad 0 \le t \le T. \end{aligned} \tag{3.1}$$

Now we present the basic assumptions on (1.1).

(H1) $F : [0, T] \times X_\alpha(t_0) \to X$ is a continuous function, $F([0, T] \times X_\alpha(t_0)) \subset D(A)$, and there exist constants $L, L_1 > 0$ such that the function $A(t)F$ satisfies the Lipschitz condition

$$\|A(t)F(s_1, x_1) - A(t)F(s_2, x_2)\| \leq L(|s_1 - s_2| + \|x_1 - x_2\|_\alpha) \tag{3.2}$$

for every $0 \leq s_1,\ s_2 \leq T$, $x_1, x_2 \in X_\alpha(t_0)$; and

$$\|A(t)F(t, x)\| \leq L_1(\|x\|_\alpha + 1)\,. \tag{3.3}$$

(H2) The function $G : [0, T] \times X_\alpha(t_0) \to X$ satisfies the following conditions:
  (i) For each $t \in [0, T]$, the function $G(t, \cdot) : X_\alpha(t_0) \to X$ is continuous, and for each $x \in X_\alpha(t_0)$, the function $G(\cdot, x) : [0, T] \to X$ is strongly measurable;
  (ii) There is a positive function $w(\cdot) \in C([0, T])$ such that

$$\sup_{\|x\|_\alpha \leq k} \|G(t, x)\| \leq w(k), \quad \liminf_{k \to +\infty} \frac{w(k)}{k} = \gamma < \infty.$$

(H3) $r \in C([0, T]; [0, T])$. $g : E \to D(A)$ is a function satisfying that $A(t)g$ is continuous on E and there exists a constant $L_2 > 0$ such that $\|A(t)g(u)\| \leq L_2\|u\|_E$ for each $x \in E$, where $E = C([0, T]; X_\alpha(t_0))$. In addition, there is a $\tau(k) > 0$ such that $g(u) = g(v)$ for any $u,\ v \in B_k$ with $u(s) = v(s)$, $s \in [\tau,\ a]$, where $B_k = \{u \in E : \|u(\cdot)\|_E \leq k\},$.

Theorem 3.2. *If* (B1)–(B4), (H1)–(H3) *are satisfied,* $x_0 \in X_\beta(t_0)$ *for some* β, $0 < \alpha < \beta \leq 1$. *Then the nonlocal Cauchy problem* (1.1) *has a mild solution provided that* L, L_1 *and* γ *are small enough; more precisely,*

$$L_0 := \Big[C'_\beta C_{(\alpha,\beta)} C_{(\beta,1)} + C_{(\alpha,1)} + C_{(\alpha,\beta)} \frac{C_\beta T^{1-\beta}}{1 - \beta}\Big] L < 1 \tag{3.4}$$

and

$$\begin{aligned} &C'_\beta C'_1 C_{(\alpha,\beta)} C_{(\beta,1)} L_2 + \Big[C'_\beta C_{(\alpha,\beta)} C_{(\beta,1)} + C_{(\alpha,1)} + C_{(\alpha,\beta)} \frac{C_\beta T^{1-\beta}}{1 - \beta}\Big] L_1 \\ &+ C_{(\alpha,\beta)} \frac{C_\beta T^{1-\beta}}{1 - \beta} \gamma < 1. \end{aligned} \tag{3.5}$$

We remark that inequalities (3.4) and (3.5) are verified explicitly by the example given in Section 4, which shows that they are applicable.

Proof of Theorem 3.2. The proof is divided into two steps.

Step 1. We first consider that, for any $\epsilon > 0$ very small, the existence of mild solutions for the equation

$$\begin{gathered} \frac{d}{dt}[x(t) + F(t, x(t))] + A(t)x(t) = G(t, x(r(t))), \quad 0 \leq t \leq T, \\ x(0) + U(\epsilon, 0)g(x) = x_0. \end{gathered} \tag{3.6}$$

Define the operator P on E by the formula

$$\begin{aligned} (Px)(t) &= U(t, 0)[x_0 + F(0, x(0)) - U(\epsilon, 0)g(x)] - F(t, x(t)) \\ &\quad + \int_0^t U(t, s)A(s)F(s, x(s))ds + \int_0^t U(t, s)G(s, x(r(s)))ds, \quad 0 \leq t \leq T. \end{aligned}$$

For each positive number k, B_k is clearly a nonempty bounded closed convex set in E. We claim that there exists a positive number k such that $P(B_k) \sqsubseteq B_k$. If it is not true, then for each positive number k, there is a function $x_k(\cdot) \in B_k$, but $Px_k \notin B_k$, that is $\|Px_k(t)\|_\alpha > k$ for some $t(k) \in [0,T]$. On the other hand, however, we have by conditions (H1)–(H3) and (2.2) (2.4) (2.5) that

$$\begin{aligned}
k &< \|(Px_k)(t)\|_\alpha \\
&= \Big\| A^\alpha(t_0)U(t,0)[x_0 - U(\epsilon,0)g(x_k) + F(0,x_k(0))] - F(t,x_k(t)) \\
&\quad + \int_0^t A^\alpha(t_0)U(t,s)A(s)F(s.x_k(s))ds + \int_0^t A^\alpha(t_0)U(t,s)G(s,x_k(r(s)))ds \Big\| \\
&\le \|A^\alpha(t_0)A^{-\beta}(t)\| \, \|A^\beta(t)U(t,0)A^{-\beta}(0)\| \\
&\quad \times \big[\|A^\beta(0)x_0\| + \|A^\beta(0)A^{-1}(t)\| \, \|A(t)F(0,x_k(0))\|\big] \\
&\quad + \|A^\alpha(t_0)A^{-\beta}(t)\| \, \|A^\beta(t)U(t,0)A^{-\beta}(0)\| \, \|A^\beta(0)A^{-1}(\epsilon)\| \\
&\quad \times \|A(\epsilon)U(\epsilon,0)A^{-1}(0)\| \, \|A(0)g(x_k)\| \\
&\quad + \|A^\alpha(t_0)A^{-1}(t)\| \, \|A(t)F(t,x_k(t))\| \\
&\quad + \int_0^t \|A^\alpha(t_0)A^{-\beta}(t)\| \, \|A^\beta(t)U(t,s)\| \, \|A(s)F(s.x_k(s))\|ds \\
&\quad + \int_0^t \|A^\alpha(t_0)A^{-\beta}(t)\|\|A^\beta(t)U(t,s)\| \, \|G(s,x_k(r(s)))\|ds \\
&\le C'_\beta C_{(\alpha,\beta)}C_{(\beta,1)}\|A^\beta(0)x_0\| + C'_\beta C'_1 C_{(\alpha,\beta)}C_{(\beta,1)}L_2 k \\
&\quad + [C'_\beta C_{(\alpha,\beta)}C_{(\beta,1)} + C_{(\alpha,1)} + C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}]L_1(k+1) + C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}w(k)
\end{aligned}$$

Dividing on both sides by k and taking the lower limit as $k \to +\infty$, we get that

$$\begin{aligned}
&C'_\beta C'_1 C_{(\alpha,\beta)}C_{(\beta,1)}L_2 + \big[C'_\beta C_{(\alpha,\beta)}C_{(\beta,1)} + C_{(\alpha,1)} + C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}\big]L_1 \\
&+ C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}\gamma \ge 1.
\end{aligned}$$

This contradicts (3.5). Hence for some positive number k, $PB_k \sqsubseteq B_k$.

Next we will show that the operator P has a fixed point on B_k, which implies that (3.6) has a mild solution. To this end, we decompose P into $P = P_1 + P_2$, where the operators P_1, P_2 are defined on B_k respectively by

$$(P_1x)(t) = U(t,0)F(0,x(0)) - F(t,x(t)) + \int_0^t U(t,s)A(s)F(s,x(s))ds,$$

$$(P_2x)(t) = U(t,0)[x_0 - U(\epsilon,0)g(x)] + \int_0^t U(t,s)G(s,x(r(s)))ds,$$

for $0 \le t \le T$ and we will verify that P_1 is a contraction while P_2 is a compact operator.

To prove that P_1 is a contraction, we take $x_1, x_2 \in B_k$, then for each $t \in [0,T]$ and by condition (H1), (2.2), (2.4), (2.5) and (3.4), we have

$$\begin{aligned}
&\|(P_1x_1)(t) - (P_1x_2)(t)\|_\alpha \\
&\le \|A^\alpha(t_0)A^{-\beta}(t)\|\|A^\beta(t)U(t,0)A^{-\beta}(0)\|\|A^\beta(0)A^{-1}(t)\|
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \|A(t)[F(0,x_1(0)) - F(0,x_2(0))]\| \\
&\quad + \|A^\alpha(t_0)A^{-1}(t)\|\|A(t)[F(t,x_1(t)) - F(t,x_2(t))]\| \\
&\quad + \|\int_0^t A^\alpha(t_0)A^{-\beta}(t)\|\|A^\beta(t)U(t,s)\|\|A(s)[F(s.x_1(s)) - F(s.x_2(s))]\|ds \\
&\le [C'_\beta C_{(\alpha,\beta)}C_{(\beta,1)} + C_{(\alpha,1)}]L \sup_{0\le s\le T}\|x_1(s) - x_2(s)\|_\alpha \\
&\quad + C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}L\sup_{0\le s\le T}\|x_1(s) - x_2(s)\|_\alpha \\
&\le L[C'_\beta C_{(\alpha,\beta)}C_{(\beta,1)} + C_{(\alpha,1)} + C_{(\alpha,\beta)}\frac{C_\beta T^{1-\beta}}{1-\beta}]\sup_{0\le s\le T}\|x_1(s) - x_2(s)\|_\alpha \\
&= L_0\sup_{0\le s\le T}\|x_1(s) - x_2(s)\|_\alpha.
\end{aligned}
$$

Thus

$$\|P_1x_1 - P_1x_2\|_\alpha \le L_0\|x_1 - x_2\|_\alpha,$$

which shows P_1 is a contraction.

To prove that P_2 is compact, firstly we prove that P_2 is continuous on B_k. Let $\{x_n\} \sqsubseteq B_k$ with $x_n \to x$ in B_k, then by (H2)(i), we have

$$G(s,x_n(r(s))) \to G(s,x(r(s))), \quad n \to \infty.$$

Since

$$\|G(s,x_n(r(s))) - G(s,x(r(s)))\| \le 2w(k),$$

by the dominated convergence theorem we have

$$
\begin{aligned}
&\|P_2x_n - P_2x\|_\alpha \\
&= \sup_{0\le t\le T}\|A^\alpha(t_0)U(t,0)U(\epsilon,0)[g(x_n) - g(x)] \\
&\quad + \int_0^t A^\alpha(t_0)U(t,s)[G(s,x_n(r(s))) - G(s,x(r(s)))]ds\| \\
&\le \sup_{0\le t\le T}\|A^\alpha(t_0)U(t,0)U(\epsilon,0)[g(x_n) - g(x)] \\
&\quad + \int_0^t \|A^\alpha(t_0)A^{-\beta}(t)\|\|A^\beta(t)U(t,s)\|\|[G(s,x_n(r(s))) - G(s,x(r(s)))]\|ds\} \\
&\to 0, \quad \text{as } n \to +\infty;
\end{aligned}
$$

i.e., P_2 is continuous.

Next we prove that the family $\{P_2x : x \in B_k\}$ is a family of equi-continuous functions. To do this, let $0 < t < T$, $h \ne 0$ with $t + h \in [0,T]$, then

$$
\begin{aligned}
&\|(P_2x)(t+h) - (P_2x)(t)\|_\alpha \\
&= \|A^\alpha(t_0)[U(t+h,0) - U(t,0)](x_0 - U(\epsilon,0)g(x)) \\
&\quad + \int_0^{t+h} A^\alpha(t_0)U(t+h,s)G(s,x(r(s)))ds - \int_0^t A^\alpha(t_0)U(t,s)G(s,x(r(s)))ds\| \\
&\le \|A^\alpha(t_0)[U(t_2,0) - U(t_1,0)](x_0 - U(\epsilon,0)g(x))\| \\
&\quad + \int_0^{t-\varepsilon} \|A^\alpha(t_0)(U(t+h,s) - U(t,s))\|\|G(s,x(r(s)))\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t-\varepsilon}^{t}\|A^{\alpha}(t_0)(U(t+h,s)-U(t,s))\|\|G(s,x(r(s)))\|ds\\
&+\int_{t}^{t+h}\|A^{\alpha}(t_0)U(t+h,s)\|\|G(s,x(r(s)))\|ds.
\end{aligned}
$$

Formula (2.1) gives that

$$
\begin{aligned}
&\|(P_2x)(t+h)-(P_2x)(t)\|_{\alpha}\\
&\le \|A^{\alpha}(t_0)[U(t_2,0)-U(t_1,0)](x_0-U(\epsilon,0)g(x))\|\\
&\quad + w(k))\int_0^{t-\varepsilon}\|A^{\alpha}(t_0)[e^{-(t+h-s)A(t+h)}-e^{-(t-s)A(t)}]\|ds\\
&\quad + w(k)\int_0^{t-\varepsilon}\|A^{\alpha}(t_0)\int_s^{t}[e^{-(t+h-\tau)A(\tau)}-e^{-(t-\tau)A(\tau)}]R(\tau,s)d\tau\|ds\\
&\quad + w(k)\int_0^{t-\varepsilon}\|A^{\alpha}(t_0)\int_s^{t+h}e^{-(t+h-\tau)A(\tau)}R(\tau,s)d\tau\|ds\|\\
&\quad + w(k)\int_{t-\varepsilon}^{t}\|A^{\alpha}(t_0)(U(t+h,s)-U(t,s))\|ds\\
&\quad + w(k)\int_{t}^{t+h}\|A^{\alpha}(t_0)U(t+h,s)\|ds := \sum_{i=1}^{6} I_i.
\end{aligned}
$$

By Lemma 2.1 we deduce easily that $I_1 \to 0$ as $h \to 0$. Since $A(t)e^{-\tau A(s)}$ is uniformly continuous in (t,τ,s) for $0 \le t \le T$, $m \le \tau \le T$ and $0 \le s \le T$, where m is any positive number (cf. [11] and [22]), we see that I_2 also tends to 0 as $h \to 0$. And

$$
\begin{aligned}
I_3 &= w(k)\int_0^{t-\varepsilon}\|\int_s^{t-\varepsilon}A^{\alpha}(t_0)[e^{-(t+h-\tau)A(\tau)}-e^{-(t-\tau)A(\tau)}]R(\tau,s)d\tau\|ds\\
&\quad + w(k)\int_0^{t-\varepsilon}\|\int_{t-\varepsilon}^{t}A^{\alpha}(t_0)[e^{-(t+h-\tau)A(\tau)}-e^{-(t-\tau)A(\tau)}]R(\tau,s)d\tau\|ds\\
&:= I_{3i}+I_{32}.
\end{aligned}
$$

Again from the uniform continuity of $A(t)e^{-\tau A(s)}$ and the estimate of $R(\tau,s)$ it is easy to infer that $I_{31} \to 0$ as $h \to 0$. For I_{32}, there yields by (2.3) that

$$
\begin{aligned}
I_{32} &\le w(k)\int_0^{t-\varepsilon}\int_{t-\varepsilon}^{t}C_{(\alpha,\beta)}[\frac{C_\beta}{t+h-\tau}+\frac{C_\beta}{t-\tau}]\frac{K}{|\tau-s|^{1-\delta}}d\tau ds\\
&\le w(k)KC_\beta C_{(\alpha,\beta)}\int_0^{t-\varepsilon}\frac{1}{|t-\varepsilon-s|^{1-\delta}}ds\int_{t-\varepsilon}^{t}[\frac{1}{t+h-\tau}+\frac{1}{t-\tau}]d\tau\\
&= w(k)KC_\beta C_{(\alpha,\beta)}\frac{1}{\delta}\frac{1}{1-\beta}(t-\varepsilon)^{\delta}[(h+\varepsilon)^{1-\beta}-h^{1-\beta}+\varepsilon^{1-\beta}] \to 0.
\end{aligned}
$$

Similarly, one can verify by the estimate of $R(\tau,s)$ and (2.2)-(2.4) that I_4, I_5 and I_6 all tend to 0 as $h \to 0$. Therefore, $\|(P_2x)(t+h)-(P_2x)(t)\|_{\alpha}$ tends to zero independently of $x \in B_k$ as $h \to 0$ with ε sufficiently small. Observe that $U(\epsilon,0)g$ is compact in X, by the similar method as above we also get that $\|(P_2x)(t)-(P_2x)(0)\|_{\alpha} \to 0$ as $t \to 0^+$. Hence, P_2 maps B_k into a family of equi-continuous functions..

It remains to prove that $V(t) = \{(P_2x)(t) : x \in B_k\}$ is relatively compact in $X_\alpha(t_0)$. It is easy to verify that $V(0)$ is relatively compact in $X_\alpha(t_0)$. Now, for any β, $0 \le \alpha < \beta < 1$, and $t \in (0,T]$,

$$\begin{aligned}
\|(A^\beta(t_0)P_xx)(t)\| &\le \int_0^t \|A^\beta(t_0)U(t,s)G(s,x(r(s)))\|ds \\
&\le w(k)\int_0^t \|A^\beta(t_0)U(t,s)\|ds \\
&\le w(k)C_{(\beta,\beta')}\frac{C_{\beta'}}{1-\beta'}a^{\beta'},
\end{aligned}$$

where $\beta < \beta' < 1$. This shows that $A^\beta(t_0)V(t)$ is bounded in X. On the other hand, $A^{-\beta}(t_0)$ is compact since $A^{-1}(t_0)$ is compact by Assumption (B4), thus $A^{-\beta}(t_0) : X \to X_\alpha(t_0)$ is compact for each $\beta > \alpha$ (note that the imbedding $X_\beta(t_0) \hookrightarrow X_\alpha(t_0)$ is compact). Therefore, we infer that $V(t)$ is relatively compact in $X_\alpha(t_0)$. Thus, by Arzela-Ascoli theorem P_2 is a compact operator. These arguments above enable us to conclude that $P = P_1 + P_2$ is a condense mapping on B_k, and by Theorem 2.3 there exists a fixed point $x_\epsilon(\cdot)$ for P on B_k, which is a mild solution for the problem (3.6).

Step 2. We prove that there is a subsequence $x_\epsilon(\cdot)$ converging to a mild solution of (1.1). We denote by Σ the set of all the fixed points $x_\epsilon(\cdot)$ of operator P on B_k obtained above for $\epsilon > 0$, that is,

$$\Sigma = \{x_\epsilon(\cdot) \in E : x_\epsilon(\cdot) = (Px_\epsilon)(\cdot)\}.$$

We shall prove that Σ is relatively compact in E.

For $\epsilon > 0$, each $x_\epsilon(\cdot) \in \Sigma$ satisfies

$$\begin{aligned}
x_\epsilon(t) &= U(t,0)[x_0 - U(\epsilon,0)g(x_\epsilon) - F(0,x_\epsilon(0))] + F(t,x_\epsilon(t)) \\
&\quad + \int_0^t U(t,s)A(s)F(s,x_\epsilon(s))ds + \int_0^t U(t,s)G(s,x_\epsilon(r(s)))ds.
\end{aligned}$$

Let $0 < t < T$, $h > 0$ very small, then

$$\begin{aligned}
&\|x_\epsilon(t+h) - x_\epsilon(t)\|_\alpha \\
&= \|A^\alpha(t_0)\left(U(t+h,0) - U(t,0)\right)[x_0 - U(\epsilon,0)g(x_\epsilon) - F(0,x_\epsilon(0))]\| \\
&\quad + \|A^\alpha(t_0)A^{-1}(t)A(t)[F(t+h,x_\epsilon(t+h)) - F(t,x_\epsilon(t))]\| \\
&\quad + \|A^\alpha(t_0)[\int_0^{t+h} U(t+h,s)A(s)F(s,x_\epsilon(s))ds - \int_0^t U(t,s)A(s)F(s,x_\epsilon(s))ds]\| \\
&\quad + \|A^\alpha(t_0)[\int_0^{t+h} U(t+h,s)G(s,x_\epsilon(r(s)))ds - \int_0^t U(t,s)G(s,x_\epsilon(r(s)))ds]\|.
\end{aligned}$$

From (3.2) and (3.4) it follows that

$$\begin{aligned}
&\|(1 - C_{(\alpha,1)}L)x_\epsilon(t+h) - x_\epsilon(t)\|_\alpha \\
&\le \|A^\alpha(t_0)U(t+h,0) - U(t,0)[x_0 - U(\epsilon,0)g(x_\epsilon) - F(0,x_\epsilon(0))]\| + C_{(\alpha,1)}Lh \\
&\quad + \|A^\alpha(t_0)[\int_0^{t+h} U(t+h,s)A(s)F(s,x_\epsilon(s))ds - \int_0^t U(t,s)A(s)F(s,x_\epsilon(s))ds]\| \\
&\quad + \|A^\alpha(t_0)[\int_0^{t+h} U(t+h,s)G(s,x_\epsilon(r(s)))ds - \int_0^t U(t,s)G(s,x_\epsilon(r(s)))ds]\|.
\end{aligned}$$

Thus, using the similar arguments as proving the equi-continuity for the family $\{P_2 x : x \in B_k\}$ in Step 1, one can easily prove that Σ is an equi-continuous family on $C([\tau, T], X_\alpha(t_0))$ for $\tau(k) > 0$.

Next we show that, for each fixed $t \in [\tau, T]$, $\Sigma(t)$ is relatively compact in $X_\alpha(t_0)$. From

$$\|A^\beta(t_0)F(t, x_\epsilon(t))\| = \|A^\beta(t_0)A^{-1}A(t)F(t, x_\epsilon(t))\| \le C_{(\beta,1)}L_1(\|x_\epsilon\| + 1)$$

and the compactness of $A^{-\beta}(t_0) : X \to X_\beta(t_0)$ $(\subset X_\alpha(t_0))$ it follows that, for each $t \in [\tau, T]$, $\{F(t, x_\epsilon(t)) : x_\epsilon \in \Sigma\}$ is relatively compact in $X_\alpha(t_0)$. Hence, we can also prove that $\Sigma(t)$ is relatively compact in $X_\alpha(t_0)$ by the same techniques as in Step 1.

Hence, again by Arzela-Ascoli theorem we deduce that $\Sigma|_{[\tau,T]}$ is relatively compact in the space $C([\tau, T], X_\alpha(t_0))$. Set

$$\tilde{x}_\epsilon(t) = \begin{cases} x_\epsilon(t), & t \in [\tau, a], \\ x_\epsilon(\tau), & t \in [0, \tau], \end{cases}$$

then $g(\tilde{x}_\epsilon) = g(x_\epsilon)$ by (H_3') and we may assume without loss of generality that $\tilde{x}_\epsilon(\cdot) \to x(\cdot)$ on interval $[\tau, T]$.

Next we need to prove that $\Sigma(0) = \{x_0 - U(\epsilon, 0)g(x_\epsilon)\}$ is relatively compact in $X_\alpha(t_0)$. In fact, by (2.2), (2.5), (2.6) and condition (H3), we obtain

$$\begin{aligned}
&\|A^\alpha(t_0)U(\epsilon, 0)g(x_\epsilon) - A^\alpha(t_0)g(x)\| \\
&\le \|A^\alpha(t_0)U(\epsilon, 0)g(x_\epsilon) - A^\alpha(t_0)U(\epsilon, 0)g(x)\| \\
&\quad + \|A^\alpha(t_0)U(\epsilon, 0)A^\alpha(t_0)A^\alpha(t_0)g(x) - A^\alpha(t_0)g(x)\| \\
&\le \big\|A^\alpha(t_0)A^{-1}(\epsilon)\big\| \big\|A(\epsilon)U(\epsilon, 0)A^{-1}(0)\big\| \|A(0)g(\tilde{x}_\epsilon) - A(0)g(x)\| \\
&\quad + \big\|A^\alpha(t_0)\,[U(\epsilon, 0) - I]\,A^{-1}(0)\big\| \|A(0)g(x) - A(0)g(x)\| \to 0, \quad \text{as } \epsilon \to 0^+.
\end{aligned}$$

To complete the proof for the relative compactness of Σ in E it remains to verify that Σ is equi-continuous at $t = 0$, while this can be reached readily by the relative compactness of $\{U(\epsilon, 0)g(x_\epsilon) : \epsilon > 0\}$.

Therefore, Σ is relatively compact in E and we may assume that $x_\epsilon(\cdot) \to x(\cdot)$ in E for some $x(\cdot)$. Then, by taking the limit as $\epsilon \to 0^+$ in $x_\epsilon(\cdot) = Px_\epsilon(\cdot)$ and using the Lebesgue dominated convergence theorem, we deduce without difficulty that $x(\cdot)$ is a mild solution to System (1.1). The proof is complete. □

3.2. **Existence of strict solutions.** In this subsection, we provide conditions which allow the differentiability of the mild solutions obtained in Section 3.1.

Definition 3.3. A function $x(\cdot) : [0, T] \to X_\alpha(t_0)$ is said to be a strict solution of the nonlocal Cauchy problem (1.1), if

1. x belongs to $C([0, T]; X_\alpha(t_0)) \cap C^1((0, T]; X)$;
2. x satisfies
$$\frac{d}{dt}[x(t) + F(t, x(t))] + A(t)x(t) = G(t, x(r(t)))$$
on $(0, T]$, and $x(0) + g(x) = x_0$.

For the next theorem, we define the following assumptions:

(H1') For any function $y \in E$, the mapping $t \to F(t, y(t))$ is Hölder continuous on $[0, T]$;

(H4) $G(\cdot,\cdot)$ is Hölder continuous; i.e. for each $(t^0,x^0)\in[0,T]\times X_\alpha(t_0)$, there exist a neighborhood W of (t^0,x^0) in $[0,T]\times X_\alpha(t_0)$ and constants $L_3>0$, $0<\theta\le 1$ such that
$$\|G(s,x)-G(\bar{s},\bar{x})\|\le L_3[|s-\bar{s}|^\theta+\|x-\bar{x}\|_\alpha^\theta]$$
for $(s,x),\ (\bar{s},\bar{x})\in W$;

(H5) There is a constant $l>0$, such that for all $s,\bar{s}\in[0,T]$,
$$\|r(s)-r(\bar{s})\|\le l|s-\bar{s}|\,;$$

(H6) $x_0\in D(A)$.

We remark that (H1') is also verified by the example presented in Section 4.

Theorem 3.4. *Suppose that* (B1)–(B4), (H1), (H2)(ii), (H3), (H1'), (H4)–(H6) *are satisfied. Then the nonlocal Cauchy problem* (1.1) *has a strict solution on* $[0,T]$ *provided that* (3.4) *and* (3.5) *hold.*

Proof. By Theorem 3.2, we see that(1.1) has a mild solution $x(\cdot)$ on $[0,T]$. We now consider the differentiability of $x(t)$. Let
$$\begin{gathered}
f(t)=F(t,x(t)),\\
o(t)=U(t,0)[x_0+F(0,x(0))-g(x)]=U(t,0)[x(0)+F(0,x(0))],\\
p(t)=\int_0^t U(t,s)A(s)F(s,x(s))ds,\\
q(t)=\int_0^t U(t,s)G(s,x(r(s)))ds.
\end{gathered}$$
It follows from Lemma 2.1, Lemma 2.2, (2.6) and (2.7) that
$$\|o(t+h)-o(t)\|_\alpha\le C(\alpha,1)\|A((0)[x(0)+F(0,x(0))]\|h^{1-\alpha},$$
and
$$\begin{aligned}
\|p(t+h)-p(t)\|_\alpha &\le C(\alpha)(|\log h|+1))\max_{0\le s\le t+h}\|A(s)F(s,x(s))\|h^{1-\alpha}\\
&\le C(\alpha)h^\beta(|\log h|+1))\max_{0\le s\le t+h}\|A(s)F(s,x(s))\|h^{1-\alpha-\beta},
\end{aligned}$$
where we have chosen $0<\beta<1$ such that $1-\alpha-\beta>0$. Observing $h^\beta(|\log h|+1))$ is bounded we see that $o(t)$ and $p(t)$ are both Hölder continuous on $[0,T]$ with exponent $1-\alpha-\beta$, and similarly, this holds for $q(t)$. So by (H1') it is easy to deduce that $x(\cdot)$ is Hölder continuous on $[0,T]$. On the other hand, it has been shown in [15] that the Lipschitz continuity of $A(t)F(\cdot,\cdot,\cdot)$ (condition (H1)) implies $A(\cdot)F(\cdot,\cdot,\cdot)$ is locally Hölder continuous. Hence conditions (H4), (H5) assure that
$$s\mapsto A(s)F(s,x(s))\quad\text{and}\quad s\mapsto G(s,x(r(s)))$$
are both Hölder continuous in X on $[0,T]$. Thus, from the proof of [22, Theorem 5.7.1] it is not difficult to see that $p(t)\in D(A)$, $q(t)\in D(A)$, and
$$\begin{gathered}
p'(t)=A(t)F(t,x(t))-A(t)\int_0^t U(t,s)A(s)F(s,x(s))ds\\
q'(t)=G(t,x(r(t)))-A(t)\int_0^t U(t,s)G(s,x(r(s)))ds,
\end{gathered}$$

Moreover, $p(t), q(t) \in C^1([\epsilon, T] : X)$. On the other hand, $f(t) \in C^1([0, T]$. Consequently, x is differentiable on $(0, T]$ and satisfies

$$\begin{aligned}
&\frac{d}{dt}[x(t) + F(t, x(t))] \\
&= \frac{d}{dt}U(t,0)[x_0 + F(0, x(0)) - g(x)] + p'(t) + q'(t) \\
&= A(t)U(t,0)[x_0 + F(0, x(0)) - g(x)] \\
&\quad + A(t)F(t, x(t)) - A(t)p(t) + G(t, x(r(t))) - A(t)q(t) \\
&= -A(t)x(t) + G(t, x(r(t)))
\end{aligned}$$

This shows that $x(\cdot)$ is a strict solution of the nonlocal Cauchy problem (1.1). Thus the proof is complete. □

4. Example

To illustrate the applications of Theorems 3.2 and 3.4, we consider the following example:

$$\begin{gathered}
\frac{\partial}{\partial t}[z(t,x) + \int_0^\pi \int_0^t b(s,y,x)(z(s,y) + \frac{\partial}{\partial y}z(s,y))dsdy] \\
= \frac{\partial^2}{\partial x^2}z(t,x) + a(t)z(t,x) + h(t, z(t\sin t, x), \frac{\partial}{\partial y}z(t\sin t, x)), \\
0 \le t \le T,\ 0 \le x \le \pi, \\
z(t,0) = z(t,\pi) = 0, \\
z(0,x) + \sum_{i=1}^{p} g_1(z(t_i, x)) = z_0(x), \quad 0 \le x \le \pi,
\end{gathered} \tag{4.1}$$

where $a(t) < 0$ is a continuous function and is Hölder continuous in t with parameter $0 < \delta < 1$. $T \le \pi$, p a positive integer, $0 < t_0 < t_1 < \dots < t_p < T$. $z_0(x) \in X := L^2([0, \pi])$.

Let $A(t)$ be defined by

$$A(t)f = -f'' - a(t)f$$

with domain

$$D(A) = \{f(\cdot) \in X : f, f' \text{absolutely continuous}, f'' \in X, f(0) = f(\pi) = 0\}.$$

Then it is not difficult to verify that $A(t)$ generates an evolution operator $U(t,s)$ satisfying assumptions $(B_1) - (B_4)$ and

$$U(t,s) = T(t-s)\exp\Big(\int_s^t a(\tau)d\tau\Big),$$

where $T(t)$ is the compact analytic semigroup generated by the operator $-A$ with $-Af = -f''$ for $f \in D(A)$. It is easy to compute that, A has a discrete spectrum, the eigenvalues are $n^2, n \in \mathbb{N}$, with the corresponding normalized eigenvectors $z_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Thus for $f \in D(A)$, there holds

$$-A(t)f = \sum_{n=1}^{\infty}(-n^2 + a(t))\langle f, z_n\rangle z_n,$$

and clearly the common domain coincides with that of the operator A. Furthermore, we may define $A^{\alpha}(t_0)$ ($t_0 \in [0,T]$) for self-adjoint operator $A(t_0)$ by the classical spectral theorem and it is easy to deduce that

$$A^{\alpha}(t_0)f = \sum_{n=1}^{\infty}(n^2 - a(t_0))^{\alpha}\langle f, z_n\rangle z_n$$

on the domain $D[A^{\alpha}] = \{f(\cdot) \in X, \sum_{n=1}^{\infty}(n^2 - a(t_0))^{\alpha}\langle f, z_n\rangle z_n \in X\}$. Particularly,

$$A^{1/2}(t_0)f = \sum_{n=1}^{\infty}\sqrt{n^2 - a(t_0)}\langle f, z_n\rangle z_n.$$

Therefore, for each $f \in X$,

$$U(t,s)f = \sum_{n=1}^{\infty} e^{-n^2(t-s)+\int_s^t a(\tau)d\tau}\langle f, z_n\rangle z_n,$$

$$A^{\alpha}(t_0)A^{-\beta}(t_0)f = \sum_{n=1}^{\infty}(n^2 - a(t_0))^{\alpha-\beta}\langle f, z_n\rangle z_n,$$

$$A^{\alpha}(t_0)U(t,s)f = \sum_{n=1}^{\infty}(n^2 - a(t_0))^{\alpha}e^{-n^2(t-s)+\int_s^t a(\tau)d\tau}\langle f, z_n\rangle z_n.$$

Then

$$\|A^{\alpha}(t)A^{-\beta}(s)\| \le (1 + \|a(\cdot)\|)^{\alpha}, \quad \|A^{\beta}(t)U(t,s)A^{-\beta}(s)\| \le (1 + \|a(\cdot)\|)^{\beta}, \tag{4.2}$$

for $t, s \in [0,T]$, $0 < \alpha < \beta$. Also

$$\begin{aligned}
&\|A^{\beta}(t)U(t,s)f\|^2 \\
&= \sum_{n=1}^{\infty}(n^2 - a(t))^{2\beta}e^{-2n^2(t-s)+2\int_s^t a(\tau)d\tau}|\langle f, z_n\rangle|^2 \\
&= (t-s)^{-2\beta}\sum_{n=1}^{\infty}[(n^2 - a(t))(t-s)]^{2\beta}e^{-2(n^2-a(t))(t-s)-2a(t)(t-s)+2\int_s^t a(\tau)d\tau}|\langle f, z_n\rangle|^2 \\
&= (t-s)^{-2\beta}\sum_{n=1}^{\infty}e^{2\beta\log[(n^2-a(t))(t-s)]-2(n^2-a(t))(t-s)-2a(t)(t-s)+2\int_s^t a(\tau)d\tau}|\langle f, z_n\rangle|^2 \\
&\le (t-s)^{-2\beta}\sum_{n=1}^{\infty}\beta^{2\beta}e^{-2a(t)(t-s)+2\int_s^t a(\tau)d\tau}|\langle f, z_n\rangle|^2;
\end{aligned}$$

(note that $c\log x - x \le c\log c - c$), which shows that

$$\|A^{\beta}(t)U(t,s)\| \le \frac{C_{\beta}}{(t-s)^{\beta}} \tag{4.3}$$

for $C_{\beta} = \beta^{\beta}\max\left\{e^{-2a(t)(t-s)+2\int_s^t a(\tau)d\tau} : t, s \in [0,T]\right\} > 0$.

Now define the abstract functions $F, G : X_{1/2}(t_0) \to X$ by

$$F(t, Z(t,\cdot))(x) = \int_0^{\pi}\int_0^t b(s,y,x)[Z(s,y) + \frac{\partial}{\partial y}Z(s,y)]dsdy,$$

$$G(t, Z(t,x))(x) = h(t, Z(t,x), \frac{\partial}{\partial x}Z(t,x)),$$

and $g : C([0,T]; X_{1/2}(t_0) \to X$ by

$$g(Z(t,x))(x) = \sum_{i=1}^{p} g_1(z(t_i,x)).$$

Then system (4.1) is rewritten in the form (1.1).

For System (4.1) we assume that the following conditions hold:

(C1) The function $b(\cdot,\cdot,\cdot)$ is a C^2 function, and $b(y,0) = b(y,\pi) = 0$;

(C2) For the function $h : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the following three conditions are satisfied:

(1) For each $t \in [0,T]$, $h(t,\cdot,\cdot)$ is continuous, and $h(\cdot,\cdot,\cdot)$ is measurable in t,

(2) There are positive functions $h_1, h_2 \in C([0,T])$ such that for all $(t,z) \in [0,T] \times X$,

$$|h(t,z)| \le h_1(t)|z| + h_2(t)$$

(C3) g_1 takes values in $D(A)$ and $A(t_0)g_1$ is a continuous map and there is a positive constants L such that $\|g_1(x)\|_{1/2} \le L$.

Condition (C1) implies that $R(F) \subset D(A)$. Clearly, $A(t)F(\cdot)$ the Lipschitz continuous on X. Observe that, for any $z_1,\ z_2 \in X_{1/2}(t_0)$,

$$\begin{aligned} \|z_2(x) - z_1(x)\|^2 &= \sum_{n=1}^{\infty} \langle z_2 - z_1, z_n\rangle^2 \\ &\le \sum_{n=1}^{\infty} (n^2 + a(t_0))\langle z_2 - z_1, z_n\rangle^2 \\ &\le {\|z_2(x) - z_1(x)\|_{1/2}}^2, \end{aligned}$$

it follows that the above conditions ensure that F, G and g verify Assumptions (H1)–(H3) respectively. Consequently, for any $z_0 \in X_\beta(t_0)$ ($\frac{1}{2} < \beta \le 1$), by Theorem 3.2, system (4.1) has a mild solution on $[0,T]$ under these assumptions, provided that (3.4) and (3.5) hold (note that the constants $C_\beta, C'_\beta, C_{(\alpha,\beta)}$ are given by (4.2) and (4.3) explicitly).

Furthermore, if we suppose that

(C4) The function $h(t,z)$ is Lipschitz continuous.

Then it is not difficult to verify that the conditions (including condition (H1')) of Theorem 3.4 are satisfied and so the mild solution is also a strict solution of (4.1) for given $z_0 \in D(A)$.

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