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MULTIPLICITY OF POSITIVE SOLUTIONS FOR QUASILINEAR ELLIPTIC P-LAPLACIAN SYSTEMS

ASADOLLAH AGHAJANI, JAMILEH SHAMSHIRI

Abstract. We study the existence and multiplicity of solutions to the elliptic system

$$\begin{gathered}
-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+m_1(x)|u|^{p-2}u=\lambda g(x,u) \quad x\in\Omega,\\
-\operatorname{div}(|\nabla v|^{p-2}\nabla v)+m_2(x)|v|^{p-2}v=\mu h(x,v) \quad x\in\Omega,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n}=f_u(x,u,v), \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n}=f_v(x,u,v),
\end{gathered}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded and smooth domain. Using fibering maps and extracting Palais-Smale sequences in the Nehari manifold, we prove the existence of at least two distinct nontrivial nonnegative solutions.

1. Introduction

In this article, we study the existence and multiplicity of positive solutions of the quasilinear elliptic system

$$\begin{gathered}
-\operatorname{div}(|\nabla u|^{p-2}\nabla u)+m_1(x)|u|^{p-2}u=\lambda g(x,u) \quad x\in\Omega,\\
-\operatorname{div}(|\nabla v|^{p-2}\nabla v)+m_2(x)|v|^{p-2}v=\mu h(x,v) \quad x\in\Omega,\\
|\nabla u|^{p-2}\frac{\partial u}{\partial n}=f_u(x,u,v), \quad |\nabla v|^{p-2}\frac{\partial v}{\partial n}=f_v(x,u,v) x\in\partial\Omega,
\end{gathered}\tag{1.1}$$

where $\lambda,\mu>0$, $p>2$, $\Omega\subset\mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ with the smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ is the outer normal derivative, $m_1,m_2\in C(\bar\Omega)$ are positive bounded functions together with the following assumptions on the functions f,g and h:

- (A1) $\frac{\partial^2}{\partial t^2}f(x,t|u|,t|v|)|_{t=1}\in C(\partial\Omega\times\mathbb{R}^2)$ and for $u,v\in L^p(\partial\Omega)$, the integral $\int_{\partial\Omega}\frac{\partial^2}{\partial t^2}\big(f(x,t|u|,t|v|)\big)dx$ has the same sign for every $t>0$.
- (A2) There exists $C_1>0$ such that
$$f(x,u,v)\le\frac{1}{r}\frac{\partial}{\partial t}f(x,tu,tv)|_{t=1}\le\frac{1}{r(r-1)}\frac{\partial^2}{\partial t^2}f(x,tu,tv)|_{t=1}\le C_1(u^r+v^r),$$
where $p<r<p^*$ ($p^*=\frac{pN}{N-p}$ if $N>2$, $p^*=\infty$ if $N\le p$) for all $(x,u,v)\in\partial\Omega\times\mathbb{R}^+\times\mathbb{R}^+$.

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(A3) $\frac{\partial}{\partial t}f(x,tu,tv)|_{t=0}\geq 0$ and $\lim_{t\to\infty}\frac{\frac{\partial}{\partial t}f(x,tu,tv)}{t^{p-1}}=\eta(x,u,v)$ uniformly respect to (x,u,v), where $\eta(x,u,v)\in C(\bar{\Omega}\times\mathbb{R}^2)$ and $|\eta(x,u,v)|>\theta>0$, a.e. for all $(x,u,v)\in\Omega\times\mathbb{R}^+\times\mathbb{R}^+$.

(A4) $g(x,u),h(x,v)\in C^1(\Omega\times\mathbb{R})$ such that $g(x,0)\geq 0$, $h(x,v)\geq 0$, $g(x,0)\not\equiv 0$ and there exist $C_2>0$, $C_3>0$ such that, $|g(x,u)|\leq C_2(1+u^{p-1})$ and $|h(x,v)|\leq C_3(1+v^{p-1})$, where $x\in\Omega$, $u,v\in\mathbb{R}^+$ and $p>2$.

(A5) For $u,v\in W^{1,p}(\Omega)$, $\int_\Omega\frac{\partial}{\partial u}g(x,t|u|)u^2dx$ and $\int_\Omega\frac{\partial}{\partial v}h(x,t|v|)v^2dx$ have the same sign for every $t>0$ and there exist $C_4>0$, $C_5>0$ such that $|g_u(x,u)|\leq C_4u^{p-2}$ and $|h_v(x,v)|\leq C_5v^{p-2}$ for all $(x,u,v)\in\Omega\times\mathbb{R}^+\times\mathbb{R}^+$.

Remark 1.1. Equations involving positively homogeneous functions have been considered in many papers, such as [3, 10, 17, 18, 28, 29]. It is clear that, if $f(x,u,v)$ is a positively homogeneous function of degree $r(r>p>2)$, that is, $f(x,tu,tv)=t^rf(x,u,v)$ $(t>0)$, then it satisfies conditions (A1)–(A3). Note that for such an f we have

$$f_uu+f_vv=rf(x,u,v)\leq rK_f(|u|^r+|v|^r),$$

where

$$K_f=\max\{f(x,u,v):(x,u,v)\in\overline{\Omega}\times\mathbb{R}^2,\ |u|^r+|v|^r=1\}.$$

In recent years, there have been many papers concerned with the existence and multiplicity of positive solutions for the elliptic equations (systems) with nonlinear boundary conditions. The results relating to these problems can be found in [1, 4, 6, 9, 11, 12, 13, 16, 19, 20, 21, 22, 24, 25, 26, 27, 30, 31, 32] and the references therein. For instance, Drabek and Schindler [15] showed the existence of positive, bounded and smooth solutions of the following p-Laplacian equation

$$\begin{gathered}-\Delta_pu+b|u|^{p-2}u=f(.,u)\quad\text{in }\Omega,\\ \Re u=0\quad\text{on }\partial\Omega,\end{gathered}$$

where $\Re u=|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}+b_0|u|^{p-2}u$, $\Omega\subset\mathbb{R}^N$ is a bounded domain and $1<p<N$.

Brown and Wu [7] considered the semilinear elliptic system

$$\begin{gathered}-\Delta u+u=\frac{\alpha}{\alpha+\beta}f(x)|u|^{\alpha-2}u|v|^{\beta}\quad\text{in }\Omega,\\ -\Delta v+v=\frac{\beta}{\alpha+\beta}f(x)|u|^{\alpha}|v|^{\beta-2}v\quad\text{in }\Omega,\\ \frac{\partial u}{\partial n}=\lambda g(x)|u|^{q-2}u,\quad\frac{\partial v}{\partial n}=\mu h(x)|v|^{q-2}v\quad\text{on }\partial\Omega,\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\alpha,\beta>1$, $2<\alpha+\beta<2^*$ and the functions f,g,h satisfy the following conditions:

- $f\in C(\overline{\Omega})$ with $\|f\|_\infty=1$ and $f^+=max\{f,0\}\not\equiv 0$,
- $g,\ h\in C(\partial\Omega)$ with $\|g\|_\infty=\|h\|_\infty=1$, $g^\pm=\max\{\pm g,0\}\not\equiv 0$ and $h^\pm=\max\{\pm h,0\}\not\equiv 0$.

They found that the above problem has at least two nonnegative solutions if the pair (λ,μ) belongs to a certain subset of $\mathbb{R}^2$. Note that the function $\frac{1}{\alpha+\beta}f(x)|u|^\alpha|v|^\beta$ with the above given conditions is positively homogeneous of degree $r=\alpha+\beta$ and clearly satisfies our conditions (A1)–(A3).

Recently, Shen and Zhang [29] considered the semilinear p-Laplacian system

$$\begin{gathered}
-\Delta_p u = \frac{1}{p^*}\frac{\partial F(x,u,v)}{\partial u} + \lambda |u|^{q-2}u \quad \text{in } \Omega,\\
-\Delta_p v = \frac{1}{p^*}\frac{\partial F(x,u,v)}{\partial v} + \mu |v|^{q-2}v \quad \text{in } \Omega,\\
u > 0, \quad v > 0 \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

where Ω is bounded domain in $\mathbb{R}^N$ with smooth boundary, $F \in C^1(\overline{\Omega}, \times(\mathbb{R}^+)^2)$ is positively homogeneous of degree p^*, they proved that this system has at least two positive solutions when the pair of parameters (λ,μ) belongs to certain subset of $\mathbb{R}^2$.

In this article, the main difficulty will be the nonlinearity of $f(x,u,v)$, $g(x,u)$ and $h(x,v)$ in problem (1.1) and the lack of separability. To overcome this difficultly, we need to restrict the problem (1.1) to assumptions (A1) and (A5). Here we present some examples for $f(x,u,v)$ satisfying the conditions (A1)–(A3).

$$\begin{gathered}
f_1(x,u,v) \in C^1(\partial\Omega \times \mathbb{R}^2, \mathbb{R}),\\
f_1(x,tu,tv) = t^r f_1(x,u,v) \quad \text{for } (x,u,v) \in \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \text{ and } t > 0,\\
f_2(x,u,v) = a_1(x)(-a_2(x) + \sqrt[q]{(a_2(x)^q + u^{qr} + v^{qr})},\\
a_i(x) \in C(\partial\Omega), \quad a_i(x) \ge 0, \quad q > 1, \quad q \in \mathbb{N},\\
f_3(x,u,v) = b(x)\frac{u^{q+r} + v^{q+r}}{1 + u^q + v^q}, \quad b(x) \in C(\partial\Omega), \quad b(x) \ge 0, \quad r \ge 0.
\end{gathered}$$

Now we present some examples for $g(x,u)$ and $h(x,v)$ satisfying the conditions (A4) and (A5):

$$Q_1(x,z) = \frac{-a_1(x)z^{p+r}}{1 + a_2(x)z^2} + a_3(x)$$

with $a_i(x) \in C(\overline{\Omega})$, $a_i(x) \ge 0$, $a_3(x) \not\equiv 0$, $\max\{2-p,-1\} \le r \le 1$.

$$Q_2(x,z) = b_1(x)\tan^{-1}(b_2(x)z^{p+k})\ln[1 + z^{2k}] + b_3(x)$$

with $b_i(x) \in C(\overline{\Omega})$, $b_i(x) \ge 0$, $b_3(x) \not\equiv 0$, $\frac{p}{2} \le k \in \mathbb{N}$.

$$Q_3(x,z) = c_1(x)\sqrt[r]{(1 + c_2(x)z^{2k})^{p-1}}$$

with $c_i(x) \ge 0$, $c_i(x) \in C(\overline{\Omega})$, $c_1(x) \not\equiv 0$, $k \in \mathbb{N}$, $0 < 2k \le r$.

$$Q_4(x,z) = \frac{-e_1(x)z^{p-1}}{4 + \cot^{-1}(e_2(x)z^k)} + e_3(x)$$

$e_i(x) \in C(\overline{\Omega})$, $e_i(x) \ge 0$, $e_3(x) \not\equiv 0$, $k \ge 0$.

Here our main tool is the Nehari manifold method which is similar to the fibering method by Drabek and Pohozaev [14]. The main idea in our proofs lies in dividing the Nehari manifold associated with the Euler functional for problem (1.1) into two disjoint parts and then considering the infima of this functional on each part and by extracting Palais-Smale sequences we show that there exists at least one solution on each part.

Define the Sobolev space

$$W := W^{1,p}(\Omega) \times W^{1,p}(\Omega), \tag{1.2}$$

endowed with the norm

$$\|(u,v)\|_W = \Big(\int_\Omega (|\nabla u|^p + m_1(x)|u|^p)dx + \int_\Omega (|\nabla v|^p + m_2(x)|v|^p)dx \Big)^{1/p},$$

which is equivalent to the standard norm. We use the standard $L^r(\Omega)$ spaces whose norms are denoted by $\|u\|_r$. Throughout this paper, we denote S_q and $\bar{S}_q$ the best Sobolev and the best Sobolev trace constants for the embedding of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ and $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$, respectively. So we have

$$\frac{(\|(u,v)\|_W^p)^q}{(\int_{\partial\Omega}(|u|^q + |v|^q)dx)^p} \geq \frac{1}{2^p \bar{S}_q^{pq}} \quad \text{and} \quad \frac{(\|(u,v)\|_W^p)^q}{(\int_\Omega (|u|^q + |v|^q)dx)^p} \geq \frac{1}{2^p S_q^{pq}}. \tag{1.3}$$

Before stating our main results, we mention the following remarks.

Remark 1.2. Notice that using conditions (A4) and (A5), for all $(x,u,v) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+$, we have

- (A6) $(r-1)g(x,u) - ug_u(x,u) \leq C_6(1+u^{p-1})$ and $(r-1)h(x,v) - vh_v(x,v) \leq C_7(1+v^{p-1})$.
- (A7) $G(x,u) - \frac{1}{r}g(x,u)u \leq C_8(1+u^p)$ and $H(x,v) - \frac{1}{r}h(x,v)v \leq C_9(1+v^p)$, where

$$G(x,u) = \int_0^u g(x,s)ds, \quad H(x,v) = \int_0^v h(x,s)ds. \tag{1.4}$$

Remark 1.3. It should be mentioned that using condition (A3) we have

$$|\frac{\partial}{\partial t} f(x, tw_1, tw_2)| \leq (1 + |\eta(x, w_1, w_2)|)t^{r-1}$$

for t sufficiently large and $(x, w_1, w_2) \in \bar{\Omega} \times (\mathbb{R}^+)^2$, hence taking $w_1 = \frac{|u|}{|u|+|v|}$, $w_2 = \frac{|v|}{|u|+|v|}$ and $t = |u| + |v|$ for $|u|$ and $|v|$ sufficiently large we arrive at

$$\begin{aligned} \big|f_u(x,|u|,|v|)|u| + f_v(x,|u|,|v|)|v|\big| &\leq \big(1 + |\eta(x, \frac{|u|}{|u|+|v|}, \frac{|v|}{|u|+|v|})|\big)(|u|+|v|)^r \\ &\leq A_0(|u|^r + |v|^r), \end{aligned}$$

where $A_0 = 2^r \max\{1 + |\eta(x,|u|,|v|)| : |u| + |v| = 1\}$ and $(x,u,v) \in \bar{\Omega} \times \mathbb{R}^2$. Furthermore, if we assume that $f \in C^2(\bar{\Omega} \times {\mathbb{R}^+}^2)$, then there exists $A_1 > 0$ such that

$$|\frac{\partial}{\partial t} f(x, tu, tv)|_{t=1} = |f_u(x,u,v)u + f_v(x,u,v)v| \leq A_1(1 + |u|^r + |v|^r), \tag{1.5}$$

where $(x,u,v) \in \bar{\Omega} \times (\mathbb{R}^+)^2$.

The purpose of this paper is to prove the following results.

Theorem 1.4. *There exists $K^* \subset (\mathbb{R}^+)^2$ such that for each $(\lambda,\mu) \in K^*$ problem* (1.1) *has at least one positive solution.*

Theorem 1.5. *There exists $K^{**} \subset K^*$ such that for each $(\lambda,\mu) \in K^{**}$ problem* (1.1) *has at least two distinct positive solutions.*

This paper is organized as follows. In section 2 we point out some notation and preliminary results and give some properties of Nehari manifold and fibering maps. In section 3 a fairly complete description of the Nehari manifold and fibering maps associated with the problem is given, and finally Theorems 1.4 and 1.5 are proved in Section 4.

2. Preliminaries and auxiliary results

First, we define the weak solution of problem (1.1) as follows.

Definition 2.1. A pair of functions $(u, v) \in W$ (W is given by (1.2)) is said to be a weak solution of (1.1), whenever

$$
\begin{aligned}
&\int_\Omega \big(|\nabla u|^{p-2}\nabla u.\nabla\varphi_1 + m_1(x)|u|^{p-2}u\varphi_1\big)dx \\
&- \lambda\int_\Omega g(x,u)\varphi_1 dx - \int_{\partial\Omega} f_u(x,u,v)\varphi_1 dx = 0, \\
&\int_\Omega \big(|\nabla v|^{p-2}\nabla v.\nabla\varphi_2 + m_2(x)|v|^{p-2}v\varphi_2\big)dx \\
&- \mu\int_\Omega h(x,v)\varphi_2 dx - \int_{\partial\Omega} f_v(x,u,v)\varphi_2 dx = 0,
\end{aligned}
$$

for all $(\varphi_1, \varphi_2) \in W$.

Associated with problem (1.1), we consider the energy functional $J_{\lambda,\mu} : W \to \mathbb{R}$

$$
J_{\lambda,\mu}(u,v) = \frac{1}{p}M(u,v) - F(u,v) - \lambda\int_\Omega G(x,|u|)dx - \mu\int_\Omega H(x,|v|)dx, \tag{2.1}
$$

where $G(x, u)$ and $H(x, v)$ are introduced in (1.4) and

$$
\begin{gathered}
M(u,v) = \int_\Omega (|\nabla u|^p + m_1(x)|u|^p)dx + \int_\Omega (|\nabla v|^p + m_2(x)|v|^p)dx, \\
F(u,v) = \int_{\partial\Omega} f(x,|u|,|v|)dx.
\end{gathered} \tag{2.2}
$$

If $J_{\lambda,\mu}$ is bounded from below and $J_{\lambda,\mu}$ has a minimizer on W, then this minimizer is a critical point of $J_{\lambda,\mu}$, so it is a solution of (1.1). Since $J_{\lambda,\mu}$ is unbounded from below on whole space W, it is useful to consider the functional on the Nehari manifold

$$
\mathcal{N}_{\lambda,\mu}(\Omega) = \{(u,v) \in W \setminus \{(0,0)\} : \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}, \tag{2.3}
$$

where $\langle , \rangle$ denotes the usual duality between W and W^{-1} (W^{-1} is the dual space of the Sobolev space W). We recall that any nonzero solution of problem (1.1) belongs to $\mathcal{N}_{\lambda,\mu}(\Omega)$. Moreover, by definition, we have that $(u, v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$ if and only if

$$
\begin{aligned}
&M(u,v) - \int_{\partial\Omega} \big(f_u(x,|u|,|v|)|u| + f_v(x,|u|,|v|)|v|\big)dx \\
&- \lambda\int_\Omega g(x,|u|)|u|dx - \mu\int_\Omega h(x,|v|)|v|dx = 0.
\end{aligned} \tag{2.4}
$$

Furthermore, we have the following result.

Theorem 2.2. *$J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$ for λ and μ sufficiently small.*

Proof. Let $(u, v) \in \mathcal{N}_{\lambda,\mu}(\Omega)$, then by (A2), (A7), (1.3) and (2.1)–(2.4), we obtain

$$
J_{\lambda,\mu}(u,v) \ge (\frac{1}{p} - \frac{1}{r})M(u,v) - \lambda\int_\Omega \Big(G(x,|u|) - \frac{1}{r}g(x,|u|)|u|\Big)dx
$$

$$
\begin{aligned}
&- \mu \int_\Omega \Big(H(x,|v|) - \frac{1}{r} h(x,|v|)|v| \Big) dx \\
&\geq (\frac{1}{p} - \frac{1}{r}) \|(u,v)\|_W^p - \lambda \int_\Omega C_8(1+|u|^p)dx - \mu \int_\Omega C_9(1+|v|^p)dx \\
&\geq \frac{r-p}{rp} \|(u,v)\|_W^p - (C_8\lambda + C_9\mu)|\Omega| - (C_8\lambda + C_9\mu) 2S_p^p \|(u,v)\|_W^p,
\end{aligned}
$$

thus $J_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$ provided that $0 \leq (C_8\lambda + C_9\mu)2S_p^p < (r-p)/(rp)$. $\square$

It can be proved that the points in $\mathcal{N}_{\lambda,\mu}(\Omega)$ correspond to the stationary points of the fibering map $\phi_{u,v}(t) : [0,\infty) \to \mathbb{R}$ defined by $\phi_{u,v}(t) = J_{\lambda,\mu}(tu,tv)$, which were introduced by Drabek and Pohozaev in [14] and also discussed in Brown and Zhang [8]. Using (2.1) for $(u,v) \in W$, we have

$$
\begin{aligned}
\phi_{u,v}(t) &= J_{\lambda,\mu}(tu,tv) \\
&= \frac{t^p}{p} M(u,v) - F(tu,tv) - \lambda \int_\Omega G(x,t|u|)dx - \mu \int_\Omega H(x,t|v|)dx,
\end{aligned}
$$

$$
\begin{aligned}
\phi'_{u,v}(t) &= t^{p-1} M(u,v) - \int_{\partial\Omega} \nabla f(x,t|u|,t|v|).(|u|,|v|)dx \\
&\quad - \lambda \int_\Omega g(x,t|u|)|u|dx - \mu \int_\Omega h(x,t|v|)|v|dx, \\
\phi''_{u,v}(t) &= (p-1)t^{p-2} M(u,v) - \int_{\partial\Omega} \mathcal{F}(x,tu,tv)dx \\
&\quad - \lambda \int_\Omega g_u(x,t|u|)u^2dx - \mu \int_\Omega h_v(x,t|v|)v^2dx,
\end{aligned} \tag{2.5}
$$

where

$$
\begin{gathered}
\nabla f(x,u,v) := (\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}), \\
\mathcal{F}(x,tu,tv) := \frac{\partial^2}{\partial t^2} \big(f(x,t|u|,t|v|) = f_{uu}u^2 + f_{vv}v^2 + 2f_{uv}|uv|.
\end{gathered} \tag{2.6}
$$

Hence it is natural to divide $\mathcal{N}_{\lambda,\mu}$ into three subsets $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^-_{\lambda,\mu}$ *and* $\mathcal{N}^0_{\lambda,\mu}$ which correspond to local minima, local maxima and points of inflection of the fibering maps and so we define

$$
\begin{aligned}
\mathcal{N}^+_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \phi''_{u,v}(1) > 0\}, \\
\mathcal{N}^-_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \phi''_{u,v}(1) < 0\}, \\
\mathcal{N}^0_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu}(\Omega) : \phi''_{u,v}(1) = 0\}.
\end{aligned} \tag{2.7}
$$

The following lemma shows that minimizers for $J_{\lambda,\mu}(u,v)$ on $N_{\lambda,\mu}(\Omega)$ are usually critical points for $J_{\lambda,\mu}$, as proved by Brown and Zhang in [8] or in Aghajani et al. [2].

Lemma 2.3. *Let (u_0,v_0) be a local minimizer for $J_{\lambda,\mu}(u,v)$ on $\mathcal{N}_{\lambda,\mu}(\Omega)$. If (u_0,v_0) is not in $\mathcal{N}^0_{\lambda,\mu}(\Omega)$, then (u_0,v_0) is a critical point of $J_{\lambda,\mu}$.*

Motivated by the above lemma, we give conditions for $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.

Lemma 2.4. *There exists $K_0 \subset (\mathbb{R}^+)^2$ such that for all $(\lambda,\mu) \in K_0$, we have $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

Proof. Suppose the contrary, that is there exists (λ,μ) such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$. Then for $(u,v) \in \mathcal{N}^0_{\lambda,\mu}$ by (2.5)–(2.7) we have

$$\begin{aligned} \phi'_{u,v}(1) = M(u,v) &- \int_{\partial\Omega} \big(\nabla f(x,|u|,|v|).(|u|,|v|)\big)dx \\ &- \lambda \int_\Omega g(x,|u|)|u|dx - \mu \int_\Omega h(x,|v|)|v|dx = 0, \end{aligned} \tag{2.8}$$

and by (2.7) $\phi''_{u,v}(1) = 0$, so

$$\begin{aligned} &(p-1)M(u,v) - \int_{\partial\Omega} \big(f_{uu}u^2 + f_{vv}v^2 + 2f_{uv}|uv|\big)dx \\ &- \lambda \int_\Omega g_u(x,|u|)u^2dx - \mu \int_\Omega h_v(x,|v|)v^2dx = 0, \end{aligned} \tag{2.9}$$

using (A2) in (2.9) we obtain

$$\begin{aligned} &(p-1)M(u,v) - (r-1)\int_{\partial\Omega} \nabla f(x,|u|,|v|).(|u|,|v|)dx \\ &- \lambda \int_\Omega g_u(x,|u|)u^2dx - \mu \int_\Omega h_v(x,|v|)v^2dx \geq 0. \end{aligned} \tag{2.10}$$

Using (1.3), (2.8), (2.10) and condition (A6) we obtain

$$\begin{aligned} (r-p)M(u,v) &\leq \lambda \int_\Omega \big((r-1)g(x,|u|) - g_u(x,|u|)|u|\big)|u|dx \\ &\quad + \mu \int_\Omega \big((r-1)h(x,|v|) - h_v(x,|v|)|v|\big)|v|dx \\ &\leq 2\lambda C_6 \int_\Omega (1+|u|^p)dx + 2\mu C_7 \int_\Omega (1+|v|^p)dx \\ &\leq (2\lambda C_6 + 2\mu C_7)|\Omega| + (2\lambda C_6 + 2\mu C_7)2S_p^p\|(u,v)\|_W^p, \end{aligned}$$

which concludes

$$M(u,v) \leq \Big(\frac{(2\lambda C_6 + 2\mu C_7)|\Omega|}{(r-p) - (4\lambda C_6 + 4\mu C_7)S_p^p}\Big). \tag{2.11}$$

Moreover, (1.3), (2.2) together (A2) imply

$$\begin{aligned} &\int_{\partial\Omega} \big(f_{uu}u^2 + f_{vv}v^2 + 2f_{uv}|uv|\big)dx \\ &\leq r(r-1)\int_{\partial\Omega} C_1(|u|^r + |v|^r)dx \leq 2r(r-1)C_1\bar{S}_r^r\|(u,v)\|_W^r, \end{aligned} \tag{2.12}$$

hence using (2.12) in (2.9) and taking into account (A5) and (1.3) we obtain

$$M(u,v) \leq L\|(u,v)\|_W^r + (\lambda L' + \mu L'')M(u,v), \tag{2.13}$$

where

$$L = \frac{2r(r-1)C_1\bar{S}_r^r}{p-1}, \quad L' = \frac{C_4S_p^p}{p-1}, \quad L'' = \frac{C_5S_p^p}{p-1}. \tag{2.14}$$

From (2.13) we obtain

$$M(u,v) \geq \Big(\frac{1-\lambda L' - \mu L''}{L}\Big)^{\frac{p}{r-p}}, \tag{2.15}$$

so using (2.11) we must have

$$\Big(\frac{1-\lambda L'-\mu L''}{L}\Big)^{\frac{p}{r-p}} \le \Big(\frac{(2\lambda C_6+2\mu C_7)|\Omega|}{(r-p)-(4\lambda C_6+4\mu C_7)S_p^p}\Big),$$

which is a contradiction for λ,μ sufficiently small. So there exists $K_0 \subset (\mathbb{R}^+)^2$ such that for $(\lambda,\mu)\in K_0$, $\mathcal{N}^0_{\lambda,\mu}=\emptyset$. $\square$

Definition 2.5. A sequence $y_n=(u_n,v_n)\subset W$ is called a Palais-Smale sequence if $I_{\lambda,\mu}(y_n)$ is bounded and $I'_{\lambda,\mu}(y_n)\to 0$ as $n\to\infty$. If $I_{\lambda,\mu}(y_n)\to c$ and $I'_{\lambda,\mu}(y_n)\to 0$, then y_n is a $(PS)_c$-sequence. It is said that the functional $I_{\lambda,\mu}$ satisfies the Palais-Smale condition (or $(PS)_c$-condition), if each Palais-Smale sequence ($(PS)_c$-sequence) has a convergent subsequence.

Now we prove the boundedness of Palais-Smale sequences.

Lemma 2.6. *If $\{(u_n,v_n)\}$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$, then $\{(u_n,v_n)\}$ is bounded in W provided that $(\lambda,\mu)\in K_1=\{(\lambda,\mu): r-p-4r(C_8\lambda+C_9\mu)S_p^p>0\}$.*

Proof. Using (1.3), (2.5), (A2) and (A7) we have

$$\begin{aligned}
&J_{\lambda,\mu}(u_n,v_n)-\frac{1}{r}\langle J'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle\\
&\ge \frac{r-p}{rp}M(u_n,v_n)-\lambda\int_\Omega (G(x,|u_n|)-\frac{1}{r}g(x,|u_n|)|u_n|)dx\\
&\quad -\mu\int_\Omega (H(x,|v_n|)-\frac{1}{r}h(x,|v_n|)|v_n|)dx\\
&\ge \frac{r-p}{rp}M(u_n,v_n)-\lambda\int_\Omega C_8(1+|u_n|^p)dx-\mu\int_\Omega C_9(1+|v_n|^p)dx\\
&\ge \frac{r-p-4r(C_8\lambda+C_9\mu)S_p^p}{rp}\|(u_n,v_n)\|_W^p-(C_8\lambda+C_9\mu)|\Omega|,
\end{aligned}$$

so for $(\lambda,\mu)\in K_1$, $\{(u_n,v_n)\}$ is bounded in W. $\square$

Lemma 2.7. *There exists $K_2\subset\mathbb{R}^2$ such that if $(\lambda,\mu)\in K_2$ and $(u,v)\in N^-_{\lambda,\mu}$, then $\int_{\partial\Omega}\mathcal{F}(x,u,v)dx>0$, where $\mathcal{F}(x,u,v)$ is defined by (2.6).*

Proof. Suppose otherwise, then $-\int_{\partial\Omega}\mathcal{F}(x,u,v)dx\ge 0$ and from (2.5) and (2.7) we obtain

$$\begin{aligned}
\phi''_{u,v}(1)&=(p-1)M(u,v)-\int_{\partial\Omega}\mathcal{F}(x,u,v)dx\\
&\quad -\lambda\int_\Omega g_u(x,|u|)u^2dx-\mu\int_\Omega h_v(x,|v|)v^2dx<0,
\end{aligned}$$

so by (1.3), (2.2), (2.14) and condition (A5) we have

$$\begin{aligned}
\|(u,v)\|_W^p&\le \frac{\lambda}{p-1}\int_\Omega g_u(x,|u|)u^2dx+\frac{\mu}{p-1}\int_\Omega h_v(x,|v|)v^2dx\\
&\le(\lambda L'+\mu L'')\|(u,v)\|_W^p,
\end{aligned}$$

which is a contradiction for $(\lambda,\mu)\in K_2=\{(\lambda,\mu):\ \lambda L'+\mu L\text{"}<1\}$. $\square$

3. Properties of Nehari manifold and fibering maps

To obtain a better understanding of the behavior of fibering maps, we will describe the nature of the derivative of the fibering maps for all possible signs of $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx$ (by (A1) and (2.6), $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx$ has the same sign for every $t > 0$). Define the functions $R(t)$ and $S(t)$ as follows

$$R(t) := \frac{1}{p} t^p M(u, v) - F(tu, tv) \quad (t > 0), \tag{3.1}$$

$$S(t) := \lambda \int_\Omega G(x, t|u|)dx + \mu \int_\Omega H(x, t|v|)dx \quad (t > 0), \tag{3.2}$$

then from (2.5) it follows that $\phi_{u,v}(t) = R(t) - S(t)$. Moreover, $\phi'_{u,v}(t) = 0$ if and only if $R'(t) = S'(t)$, where

$$R'(t) = t^{p-1} M(u, v) - \int_{\partial\Omega} \Big(f_u(x, t|u|, t|v|)|u| + f_v(x, t|u|, t|v|)|v| \Big) dx, \tag{3.3}$$

and

$$S'(t) = \lambda \int_\Omega g(x, t|u|)|u|dx + \mu \int_\Omega h(x, t|v|)|v|dx. \tag{3.4}$$

In the next result we see that, $\phi_{u,v}$ and $\phi'_{u,v}$ take on positive values for all nonzero $(u, v) \in W$ whenever, λ and μ belong to a certain subset of $\mathbb{R}^2$.

Lemma 3.1. *There exists $K_3 \subset (\mathbb{R}^+)^2$ such that for all nonzero $(u, v) \in W$, $\phi_{u,v}(t)$ and $\phi'_{u,v}(t)$ take on positive values whenever $(\lambda, \mu) \in K_3$.*

Proof. First we show that $\phi_{u,v}(t)$ takes on positive values, for all possible signs of $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx$. If $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx \le 0$, then by (3.1) $R''(t) \ge 0$ and using (3.2), $R(t) > S(t)$ for t sufficiently large, so $\phi_{u,v}(t) > 0$ for t sufficiently large . Now, suppose there exists $(u, v) \in W$ such that $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx \ge 0$. Condition (A2) together (3.1) imply that

$$R(t) \ge \frac{1}{p} t^p M(u, v) - C_1 t^r \int_{\partial\Omega} (|u|^r + |v|^r)dx.$$

Define

$$K(t) := \frac{1}{p} t^p M(u, v) - C_1 t^r \int_{\partial\Omega} (|u|^r + |v|^r)dx \quad (t > 0), \tag{3.5}$$

we obtain $R(t) \ge K(t)$, and by elementary calculus, we see that $K(t)$ takes a maximum value at

$$t_{\max} = \Big(\frac{M(u, v)}{rC_1 \int_{\partial\Omega} (|u|^r + |v|^r)dx)} \Big)^{\frac{1}{r-p}}, \tag{3.6}$$

then follows by (3.1), (3.6), (1.3) and (2.2) that

$$\begin{aligned} R(t_{\max}) \ge K(t_{\max}) &= \frac{r-p}{rp} \Big(\frac{(\|(u, v)\|_W^p)^r}{\big(rC_1 \int_{\partial\Omega} (|u|^r + |v|^r)dx\big)^p} \Big)^{\frac{1}{r-p}} \\ &\ge \frac{r-p}{rp} \Big(\frac{1}{(2rC_1)^p \bar{S}_r^{rp}} \Big)^{\frac{1}{r-p}} = \delta_1, \end{aligned} \tag{3.7}$$

where δ_1 is independent of (u,v). Now from (3.6), (3.7) and (1.3) for $1 \le \alpha < p^*$, we deduce

$$
\begin{aligned}
(t_{\max})^\alpha \int_\Omega (|u|^\alpha + |v|^\alpha)dx &\le 2S_\alpha^\alpha \Big(\frac{\|(u,v)\|_W^p}{rC_1\int_{\partial\Omega}(|u|^r+|v|^r)dx}\Big)^{\frac{\alpha}{r-p}} (\|(u,v)\|_W^p)^{\frac{\alpha}{p}} \\
&= 2S_\alpha^\alpha \Big(\frac{(\|(u,v)\|_W^p)^r}{\big(rC_1\int_{\partial\Omega}(|u|^r+|v|^r)dx\big)^p}\Big)^{\frac{\alpha}{p(r-p)}} \\
&\le 2S_\alpha^\alpha \big(\frac{rp}{r-p}\big)^{\frac{\alpha}{p}} \big(R(t_{\max})\big)^{\frac{\alpha}{p}} = c_1 \big(R(t_{\max})\big)^{\frac{\alpha}{p}}.
\end{aligned} \tag{3.8}
$$

Combining (A4), (A7), (1.3) and (3.8) imply that

$$
\begin{aligned}
S(t_{\max}) &= \lambda \int_\Omega G(x, t_{\max}|u|)dx + \mu \int_\Omega H(x, t_{\max}|v|)dx \\
&\le \frac{\lambda}{r} \int_\Omega rC_8(1+|t_{\max}u|^p) + C_2(|t_{\max}u| + |t_{\max}u|^p)dx \\
&\quad + \frac{\mu}{r} \int_\Omega rC_9(1+|t_{\max}v|^p) + C_3(|t_{\max}v| + |t_{\max}v|^p)dx \\
&\le \lambda b_0 \int_\Omega (1+|t_{\max}u|^p)dx + \mu b_1 \int_\Omega (1+|t_{\max}v|^p)dx \\
&\le \lambda B_0(1+R(t_{\max})) + \mu B_1(1+R(t_{\max})),
\end{aligned} \tag{3.9}
$$

where B_0 and B_1 are independent of (u,v). Using (3.9) together with (3.7) and (2.5), we obtain

$$
\begin{aligned}
\phi_{u,v}(t_{\max}) &= R(t_{\max}) - S(t_{\max}) \\
&\ge R(t_{\max})\Big(1 - (\lambda B_0 + \mu B_1)(R(t_{\max})^{-1} + 1)\Big) \\
&\ge \delta_1\big(1 - (\lambda B_0 + \mu B_1)(\delta_1^{-1} + 1)\big).
\end{aligned} \tag{3.10}
$$

So we conclude that if $2(\lambda B_0 + \mu B_1)(1+\delta_1) < \delta_1$, then $\phi_{u,v}(t_{\max}) > 0$ for all nonzero $(u,v) \in W$.

Now we prove that $\phi'_{u,v}(t)$ takes on positive values. If $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx \le 0$, then using (3.1), (3.2) $\phi'_{u,v}(t) \ge 0$ for t sufficiently large. Suppose that, there exists $(u,v) \in W$ such that $\int_{\partial\Omega} \mathcal{F}(x, tu, tv)dx \ge 0$. To verify that $\phi'_{u,v}(t) > 0$, it is sufficient to show that $tR'(t) > tS'(t)$. Using (A2) and (3.3) we have

$$
tR'(t) \ge t^p M(u,v) - rC_1 t^r \int_{\partial\Omega} (|u|^r + |v|^r)dx.
$$

In view of (3.5), we write

$$
\bar{K}(t) := t^p M(u,v) - rC_1 t^r \int_{\partial\Omega} (|u|^r + |v|^r)dx \quad (t > 0), \tag{3.11}
$$

so $tR'(t) > \bar{K}(t)$ and by elementary calculus we can show that $\bar{K}(t)$ achieves its maximum at

$$
\tau_{\max} = \Big(\frac{pM(u,v)}{r^2 C_1 \int_{\partial\Omega}(|u|^r + |v|^r)dx}\Big)^{\frac{1}{r-p}}. \tag{3.12}
$$

Using (1.3), (3.3), (3.11) and (3.12), we arrive at

$$\begin{aligned}\tau_{\max}R'(\tau_{\max}) &= (\frac{p}{r^2C_1})^{\frac{p}{r-p}}(\frac{r-p}{r})\Big(\frac{(\|(u,v)\|^p)^r}{(\int_{\partial\Omega}(|u|^r+|v|^r)dx)^p}\Big)^{\frac{1}{r-p}} \\ &\geq (\frac{p}{r^2C_1})^{\frac{p}{r-p}}(\frac{r-p}{r})\Big(\frac{1}{2^p\bar{S}_r^{rp}}\Big)^{\frac{1}{r-p}} = \delta_2 > 0,\end{aligned} \tag{3.13}$$

where δ_2 is independent of (u,v). Using (1.3), (3.12) and (3.13), and by some calculations very Similar to (3.8), we obtain

$$(\tau_{\max})^\beta\int_\Omega(|u|^\beta+|v|^\beta)dx \leq c_2(\tau_{\max}R'(\tau_{\max}))^{\frac{\beta}{p}}, \tag{3.14}$$

for $1\leq\beta<2^*$. Then using (1.3), (3.4), (3.14) and condition (A4) we find

$$\begin{aligned}\tau_{\max}S'(\tau_{\max}) &= \lambda\tau_{\max}\int_\Omega g(x,\tau_{\max}|u|)|u|dx + \mu\tau_{\max}\int_\Omega h(x,\tau_{\max}|v|)|v|dx \\ &\leq \lambda\int_\Omega C_1(|t_{\max}u|+|t_{\max}u|^p)dx + \mu\int_\Omega C_3(|t_{\max}v|+|t_{\max}v|^p)dx \\ &\leq (\lambda e_0+\mu e_1)\Big((t_{\max}R'(t_{\max}))^{\frac{1}{p}} + t_{\max}R'(t_{\max})\Big),\end{aligned}$$

where e_0 and e_1 are independent of (u,v), so from the above inequality and (3.13), we obtain

$$\begin{aligned}&\tau_{\max}\phi'_{u,v}(\tau_{\max}) = \tau_{\max}R'(\tau_{\max}) - \tau_{\max}S'(\tau_{\max}) \\ &\geq \tau_{\max}R'(\tau_{\max})\Big(1-(\lambda e_0+\mu e_1)\big((\tau_{\max}R'_\lambda(\tau_{\max}))^{\frac{1-p}{p}}+1\big)\Big) \\ &\geq \delta_2\Big(1-(\lambda e_0+\mu e_1)\big(\delta_2^{\frac{1-p}{p}}+1\big)\Big),\end{aligned}$$

Clearly for all nonzero $(u,v)\in W$, $\tau_{\max}\phi'_{u,v}(\tau_{\max}) > 0$ provided that $2(\lambda e_0 + \mu e_1)\big(\delta_2^{\frac{1}{p}}+\delta_2\big) < \delta_2$.

Using the above inequality and (3.10), we obtain that if $(\lambda,\mu)\in K_3$, where

$$K_3 = \{(\lambda,\mu) : 2(\lambda B_0+\mu B_1)(1+\delta_1) < \delta_1 \text{ and } 2(\lambda e_0+\mu e_1)\big(\delta_2^{\frac{1}{p}}+\delta_2\big) < \delta_2\}, \tag{3.15}$$

then $\phi_{u,v}(t)$ and $\phi'_{u,v}(t)$ take on positive values for all nonzero $(u,v)\in W$ and this completes the proof. □

Corollary 3.2. *If $(\lambda,\mu)\in K_2\cap K_3$, then there exists $\varepsilon>0$ such that $J_{\lambda,\mu}(u,v)>\epsilon$ for all $(u,v)\in\mathcal{N}^-_{\lambda,\mu}$.*

Proof. If $(u,v)\in N^-_{\lambda,\mu}$, then by lemma 2.7, $\int_{\partial\Omega}\mathcal{F}(x,u,v)dx>0$. Also due to (A1) and (A5), $\phi_{u,v}$ has a positive global maximum at $t=1$ and so by (2.5), (3.10) and (3.15)

$$J_{\lambda,\mu}(u,v) = \phi_{u,v}(1) \geq \phi_{u,v}(t_{\max}) \geq \delta_1\big(1-(\lambda B_0+\mu B_1)(\delta_1^{-1}+1)\big) \geq \delta_1/2 = \varepsilon > 0.$$

□

From (A1) and (2.6), $\int_{\partial\Omega}\mathcal{F}(x,tu,tv)dx$ has the same sign for every $t>0$, so we have the following corollary.

Corollary 3.3. *for $(u,v)\in W\setminus\{(0,0)\}$ we have*

(i) *If $\int_{\partial\Omega}\mathcal{F}(x,tu,tv)dx\leq 0$, then there exists t_1 such that $(t_1u,t_1v)\in N^+_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<0$.*

(ii) *If* $\int_{\partial\Omega}\mathcal{F}(x,tu,tv)dx \geq 0$ *and* $(\lambda,\mu)\in K_3$*, then there exist* $0<t_1<t_2$ *such that* $(t_1u,t_1v)\in N^+_{\lambda,\mu}$*,* $(t_2u,t_2v)\in N^-_{\lambda,\mu}$ *and* $\phi_{u,v}(t_1)<0$*.*

Proof. (i) From (2.5), (A3), (A4) and the assumptions we obtain $\phi'_{u,v}(0)<0$ and $\lim_{t\to\infty}\phi'_{u,v}(t)=+\infty$, so by the intermediate value theorem, there exists $t_1>0$ such that $\phi'_{u,v}(t_1)=0$. Now using (A1) and (A5), for $0<t<t_1$, $\phi'_{u,v}(t)<0$ and for $t>t_1$, $\phi'_{u,v}(t)>0$, therefore $(t_1u,t_1v)\in N^+_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<\phi_{u,v}(0)=0$.

(ii) Using (2.5), (A3), (A5) and the assumption that $\int_{\partial\Omega}\mathcal{F}(x,tu,tv)dx\geq 0$ we obtain $\lim_{t\to\infty}\phi'_{u,v}(t)=-\infty$, $\phi'_{u,v}(0)<0$ and by Lemma 3.1 we have $\phi'_{u,v}(\tau)>0$ for suitable $\tau>0$, so using again the intermediate value theorem concludes that there exist t_1 and t_2 such that $0<t_1<\tau<t_2$, and $\phi'_{u,v}(t_1)=\phi'_{u,v}(t_2)=0$. Also using the same argument as in the proof of (i) and using (A1) and (A5) we have $(t_1u,t_1v)\in N^+_{\lambda,\mu}$, $(t_2u,t_2v)\in N^-_{\lambda,\mu}$ and $\phi_{u,v}(t_1)<\phi_{u,v}(0)=0$. □

4. Proof of Theorems 1.4 and 1.5

To prove these to theorems, we need to show the existence of local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$. To do this, we need the Remark 4.1, below. Here for simplicity, for a functional ψ defined on a normed space E, and $w\in E$ by $\psi'(w)$ and $\psi''(w)$, we mean $\frac{\partial}{\partial t}\psi(wt)|_{t=1}$, and $\frac{\partial^2}{\partial t^2}\psi(wt)|_{t=1}$, respectively.

Remark 4.1. From Remark 1.3, (2.6) and (A2) we obtain that

$$|\nabla f(x,|u|,|v|).(|u|,|v|)|\leq A_1(1+|u|^r+|v|^r)$$

and $|\mathcal{F}(x,u,v)|\leq A_2(1+|u|^r+|v|^r)$, also from (A4) and (A5) we obtain

$$|g(x,|u|)|\leq C_2(1+|u|^{p-1}),\quad |g_u(x,|u|)|\leq C_4(1+|u|^{p-1}),$$
$$|h(x,|v|)|\leq C_3(1+|v|^{p-1}),\quad |h_v(x,|v|)|\leq C_5(1+|v|^{p-1}),$$

for $r>p\geq 2$. Hence from the compactness of the embeddings $W^{1,p}\hookrightarrow L^\alpha(\Omega)$ and $W^{1,p}\hookrightarrow L^\alpha(\partial\Omega)$ for $1\leq\alpha<p^*$ (the Rellich-Kondrachov Theorem [5]) and the fact that the $g(x,u)$, $h(x,u)$ are continuous and $f(x,u,v)\in C^2(\partial\Omega\times\mathbb{R}^2)$, we conclude that the functionals $I_1(u,v)=\int_{\partial\Omega}f(x,|u|,|v|)dx$, $I_2(u)=\int_\Omega G(x,|u|)dx$ and $I_3(A5)=\int_\Omega H(x,|v|)dx$ are weakly continuous, i.e. if $(u_n,v_n)\rightharpoonup(u,v)$, then $I_1(u_n,v_n)\to I_1(u,v)$, $I_2(u_n)\to I_2(u)$ and $I_3(v_n)\to I_3(A5)$. Moreover the operators $I_1'(u,v)=\int_{\partial\Omega}\nabla f(x,|u|,|v|).|(|u|,|v|)dx$, $I_2'(u)=\int_\Omega g(x,|u|)|u|dx$, $I_3'(A5)=\int_\Omega h(x,|v|)|v|dx$, $I_1''(u,v)=\int_{\partial\Omega}\mathcal{F}(x,u,v)dx$, $I_2'(u)=\int_\Omega g_u(x,|u|)u^2dx$ and $I_3'(A5)=\int_\Omega h_v(x,|v|)v^2dx$ are weak to strong continuous, i.e. if $(u_n,v_n)\rightharpoonup(u,v)$ then $I_1'(u_n,v_n)\to I_1'(u,v)$, $I_1''(u_n,v_n)\to I_1''(u,v)$, $I_2'(u_n)\to I_2'(u)$, $I_2''(u_n)\to I_2''(u)$, $I_3'(v_n)\to I_3'(A5)$ and $I_3''(v_n)\to I_3''(A5)$.

Now, we establish the existence of local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ and $N^-_{\lambda,\mu}$. For simplicity let $K^*=K_0\cap K_1\cap K_3$ and $K^{**}=K_0\cap K_1\cap K_2\cap K_3$, where K_i's $(i=0,1,2,3)$ are given in the previous section.

Lemma 4.2. (i) *For* $(\lambda,\mu)\in K^*$*, there exists a minimizer of* $J_{\lambda,\mu}$ *on* $\mathcal{N}^+_{\lambda,\mu}(\Omega)$*.*

(ii) *For* $(\lambda,\mu)\in K^{**}$*, there exists a minimizer of* $J_{\lambda,\mu}$ *on* $\mathcal{N}^-_{\lambda,\mu}(\Omega)$*.*

Proof. (i) As in Theorem 2.2, $J_{\lambda,\mu}$ is bounded from below on $\mathcal{N}_{\lambda,\mu}(\Omega)$ and so on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$. Let $\{(u_n,v_n)\}$ be a minimizing sequence for $J_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$; i.e.,

$$\lim_{n\to\infty}J_{\lambda,\mu}(u_n,v_n)=\inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}}J_{\lambda,\mu}(u,v).$$

By Ekeland's variational principle [16] we may assume that

$$\langle J'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \to 0,$$

combining the compact embedding Theorem [5] and Lemma 2.6, we obtain that there exists a subsequence $\{(u_n, v_n)\}$ and (u_1, v_1) in W such that

$$\begin{gathered} u_n \rightharpoonup u_1 \quad \text{weakly in } W^{1,p}(\Omega), \\ v_n \rightharpoonup v_1 \quad \text{weakly in } W^{1,p}(\Omega), \\ u_n \to u_1 \quad \text{strongly in } L^m(\Omega),\ 1 \le m < p^*, \\ v_n \to v_1 \quad \text{strongly in } L^m(\partial\Omega),\ 1 \le m < p^*, \end{gathered} \tag{4.1}$$

and $(u_n(x), v_n(x)) \to (u_1(x), v_1(x))$ almost everywhere.

By Corollary 3.3 for $(u_1, v_1) \in W \setminus \{(0,0)\}$, there exists t_1 such that $(t_1u_1, t_1v_1) \in N^+_{\lambda,\mu}$ and so $\phi'_{u_1,v_1}(t_1) = 0$. Now we show that $(u_n, v_n) \to (u_1, v_1)$ in W. Suppose this is false, then

$$M(u_1, v_1) < \liminf_{n\to\infty} M(u_n, v_n), \tag{4.2}$$

so from (2.5), (4.1), (4.2) and Remark 4.1, $\phi'_{u_n,v_n}(t_1) > \phi'_{u_1,v_1}(t_1) = 0$ for n sufficiently large. Since $\{(u_n, v_n)\} \subseteq N^+_{\lambda,\mu}(\Omega)$, by considering the possible fibering maps it is easy to see that, $\phi'_{u_n,v_n}(t) < 0$ for $0 < t < 1$ and $\phi'_{u_n,v_n}(1) = 0$ for all n. Hence we must have $t_1 > 1$, but $(t_1u_1, t_1v_1) \in N^+_{\lambda,\mu}$ and so

$$\begin{aligned} &J_{\lambda,\mu}(t_1u_1, t_1v_1) = \phi_{u_1,v_1}(t_1) < \phi_{u_1,v_1}(1) \\ &< \lim_{n\to\infty} \phi_{u_n,v_n}(1) = \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v), \end{aligned}$$

which is a contradiction. Therefore, $(u_nv_n) \to (u_1, v_1)$ in W and this concludes that

$$J_{\lambda,\mu}(u_1, v_1) = \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v).$$

Thus (u_1, v_1) is a minimizer for $J_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}(\Omega)$.

(ii) By Corollary 3.2 we have $J_{\lambda,\mu}(u, v) \ge \varepsilon > 0$ for all $(u, v) \in \mathcal{N}^-_{\lambda,\mu}$, so

$$\inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v) > 0,$$

hence, there exists a minimizing sequence $\{(u_n, v_n)\} \subseteq \mathcal{N}^-_{\lambda,\mu}(\Omega)$ such that

$$\lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v) > 0. \tag{4.3}$$

Similar to the argument in the proof of (i) we find that $\{(u_n, v_n)\}$ is bounded in W and also the results obtained in (4.1) are satisfied for $\{(u_n, v_n)\}$ and $\{(u_2, v_2)\}$.

Since $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$, so by (2.7), $\phi''_{u_n,v_n}(1) < 0$, letting $n \to \infty$, by (2.5), Remark 4.1 and the above argument we see that

$$\begin{aligned} \phi''_{u_2,v_2}(1) &= M(u_2, v_2) - \int_{\partial\Omega} \mathcal{F}(x, u_2, v_2)dx - \lambda \int_\Omega g_u(x, |u_2|)u_2^2dx \\ &\quad - \mu \int_\Omega h_v(x, |v_2|)v_2^2dx \le 0. \end{aligned} \tag{4.4}$$

On the other hand for $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$, by Lemma 2.7, $\int_{\partial\Omega} \mathcal{F}(x, u_n, v_n)dx > 0$. Letting $n \to \infty$, we see that $\int_{\partial\Omega} \mathcal{F}(x, u_2, v_2)dx \ge 0$.

We claim that $\int_{\partial\Omega} \mathcal{F}(x, u_2, v_2)dx \neq 0$. If $\int_{\partial\Omega} \mathcal{F}(x, u_2, v_2)dx = 0$, then by (A4), (1.3), (2.14) and (4.4) we have

$$M(u_2, v_2) \leq \lambda \int_{\partial\Omega} g_u(x, |u_2|)u_2^2 dx + \mu \int_{\partial\Omega} h_v(x, |v_2|)v_2^2 dx \leq (\lambda L + \mu L')M(u_2, v_2),$$

which is a contradiction for $(\lambda, \mu) \in K_2$. So $\int_{\partial\Omega} \mathcal{F}(x, u_2, v_2)dx > 0$ and by Corollary 3.3 (ii) there exists $t_2 > 0$ such that $(t_2 u_2, t_2 v_2) \in \mathcal{N}^-_{\lambda,\mu}(\Omega)$. We claim that $(u_n, v_n) \to (u_2, v_2)$ in W. Suppose that this is false, so we have

$$M(u_2, v_2) < \liminf_{n\to\infty} M(u_n, v_n). \tag{4.5}$$

However, $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ and so $J_{\lambda,\mu}(u_n, v_n) \geq J_{\lambda,\mu}(tu_n, tv_n)$ for all $t \geq 0$. Therefore, considering (2.5), (4.3)–(4.5) and Remark (4.1), we can write

$$\begin{aligned}
&J_{\lambda,\mu}(t_2 u_2, t_2 v_2)\\
&= \frac{t_2^p}{p} M(u_2, v_2) - F(t_2 u_2, t_2 v_2) - \lambda \int_\Omega H(x, t_2|u_2|)dx - \mu \int_\Omega G(x, t_2|v_2|)dx\\
&< \lim_{n\to\infty} \Big(\frac{t_2^p}{p} M(u_n, v_n) - F(t_2 u_n, t_2 v_n) - \lambda \int_\Omega H(x, t_2|u_n|)dx\\
&\quad - \mu \int_\Omega G(x, t_2|v_n|)dx \Big)\\
&= \lim_{n\to\infty} J_{\lambda,\mu}(t_2 u_n, t_2 v_n) \leq \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \inf_{(u,v)\in\mathcal{N}^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v),
\end{aligned}$$

which is a contradiction. So, $(u_n, v_n) \to (u_2, v_2)$ in W and the proof is complete. □

Proof of Theorem 1.4. By Lemma 4.2 (i) there exists $(u_1, v_1) \in N^+_{\lambda,\mu}(\Omega)$ such that $J_{\lambda,\mu}(u_1, v_1) = \inf_{(u,v)\in N^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v)$ and by Lemmas 2.3 and 2.4, (u_1, v_1) is a critical point of $J_{\lambda,\mu}$ on W and hence is a weak solution of problem (1.1). On the other hand $J_{\lambda,\mu}(u, v) = J_{\lambda,\mu}(|u|, |v|)$, so we may assume that (u_1, v_1) is a positive solution and the proof is complete. □

Proof of Theorem 1.5. By Lemma 4.2 there exist $(u_1, v_1) \in N^+_{\lambda,\mu}(\Omega)$ and $(u_2, v_2) \in N^-_{\lambda,\mu}(\Omega)$ such that

$$J_{\lambda,\mu}(u_1, v_1) = \inf_{(u,v)\in N^+_{\lambda,\mu}} J_{\lambda,\mu}(u, v), \quad J_{\lambda,\mu}(u_2, v_2) = \inf_{(u,v)\in N^-_{\lambda,\mu}} J_{\lambda,\mu}(u, v).$$

By Lemmas 2.3 and 2.4, (u_1, v_1) and (u_2, v_2) are critical points of $J_{\lambda,\mu}$ on W and hence are weak solutions of problem (1.1). Similar to the proof of Theorem 1.4, we may assume that (u_1, v_1) and (u_2, v_2) are positive solutions. Also since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that (u_1, v_1) and (u_2, v_2) are distinct and the proof is complete. □

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