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PROPAGATION OF PERTURBATIONS FOR A SIXTH-ORDER THIN FILM EQUATION

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Abstract. We consider an initial-boundary problem for a sixth-order thin film equation, which arises in the industrial application of the isolation oxidation of silicon. Relying on some necessary uniform estimates of the approximate solutions, we prove the existence of radial symmetric solutions to this problem in the two-dimensional space. The nonnegativity and the finite speed of propagation of perturbations of solutions are also discussed.

1. Introduction

This article is devoted to the radial symmetric solutions for a sixth-order thin film equation

$$\frac{\partial u}{\partial t} = \operatorname{div}[|u|^n \nabla \Delta^2 u], \quad x \in B,$$

with the boundary value conditions

$$\frac{\partial u}{\partial \boldsymbol{\nu}}\Big|_{\partial B} = \frac{\partial \Delta u}{\partial \boldsymbol{\nu}}\Big|_{\partial B} = \frac{\partial \Delta^2 u}{\partial \boldsymbol{\nu}}\Big|_{\partial B} = 0,$$

and the initial value condition

$$u\big|_{t=0} = u_0(x),$$

where B is the unit ball in $\mathbb{R}^2$, $n > 0$ is a constant, and $\boldsymbol{\nu}$ is the outward unit normal to ∂B.

The equation is a typical higher order equation, which has a sharp physical background and a rich theoretical connotation. It was first introduced in [13, 14] in the case $n = 3$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(u^3 \frac{\partial^5 u}{\partial x^5}\Big).$$

It describes the spreading of a thin viscous fluid under the driving force of an elastica (or light plate).

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During the past years, only a few works have been devoted to the sixth-order thin film equation [4, 9, 10, 13, 14]. Bernis and Friedman [4] have studied the initial boundary value problems to the thin film equation

$$\frac{\partial u}{\partial t} + (-1)^{m-1}\frac{\partial}{\partial x}\Big(f(u)\frac{\partial^{2m+1}u}{\partial x^{2m+1}}\Big) = 0,$$

where $f(u) = |u|^n f_0(u)$, $f_0(u) > 0$, $n \geq 1$ and proved existence of weak solutions preserving nonnegativity. Barrett, Langdon and Nuernberg [2] considered the above equation with $m = 2$. A finite element method is presented which proves to be well posed and convergent. Numerical experiments illustrate the theory.

Recently, Jüngel and Milišić [12] studied the sixth-order nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Big[u\Big(\frac{1}{u}(u(\ln u)_{xx})_{xx} + \frac{1}{2}((\ln u)_{xx})^2\Big)_x\Big]_x.$$

They proved the global-in-time existence of weak nonnegative solutions in one space dimension with periodic boundary conditions.

Evans, Galaktionov and King [6, 7] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}\big[|u|^n \nabla\Delta^2 u\big] - \Delta(|u|^{p-1}u), n > 0, \quad p > 1.$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + \frac{4}{N}$ for $n \in (0, \frac{5}{4})$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-\frac{N}{nN+6}} f_k(y)$, $k = 1, 2, \cdots$, $y = \frac{x}{(T-t)^{\frac{1}{nN+6}}}$, where $T > 0$ is the blow-up time.

We also refer the following relevant equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\Big(u^n\frac{\partial^3 u}{\partial x^3}\Big),$$

which has been extensively studied. Bernis and Friedman [4] studied the initial boundary value problems to the thin film equation $n > 0$ and proved existence of weak solutions preserving nonnegativity (see also [3, 15, 16, 18]). They proved that if $n \geq 2$ the support of the solutions $u(\cdot, t)$ is nondecreasing with respect to t.

Our purpose in this paper is to study the radial symmetric solutions for the equation. We will study the problem in two-dimensional case, which has particular physical derivation of modeling the oil film spreading over a solid surface, see [17]. After introducing the radial variable $r = |x|$, we see that the radial symmetric solution satisfies

$$\frac{\partial(ru)}{\partial t} = \frac{\partial}{\partial r}\{r|u|^n\frac{\partial W}{\partial r}\}, \quad rW = \frac{\partial}{\partial r}\Big(r\frac{\partial V}{\partial r}\Big), \quad rV = \frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big), \tag{1.1}$$

$$\frac{\partial u}{\partial r}\Big|_{r=0} = \frac{\partial u}{\partial r}\Big|_{r=1} = \frac{\partial V}{\partial r}\Big|_{r=0} = \frac{\partial V}{\partial r}\Big|_{r=1} = \frac{\partial W}{\partial r}\Big|_{r=0} = \frac{\partial W}{\partial r}\Big|_{r=1} = 0, \tag{1.2}$$

$$u\Big|_{t=0} = u_0(r). \tag{1.3}$$

It should be noticed that the equation (1.1) is degenerate at the points where $r = 0$ or $u = 0$, and hence the arguments for one-dimensional problem can not be applied directly. Because of the degeneracy, the problem does not admit classical solutions in general. So, we introduce the weak solutions in the following sense

Definition 1.1. A function u is said to be a weak solution of the problem (1.1)–(1.3), if the following conditions are fulfilled:

(1) $ru(r,t)$ is continuous in $\overline{Q}_T$, where $Q_T = (0,1)\times(0,T)$;
(2) $\sqrt{r}|u|^{n/2}\frac{\partial W}{\partial r} \in L^2(Q_T)$;
(3) For any $\varphi \in C^1(\overline{Q}_T)$, the following integral equality holds

$$\int_0^1 ru(r,T)\varphi(r,T)\,dr - \int_0^1 ru_0(r)\varphi(r,0)\,dr - \iint_{Q_T} ru\frac{\partial\varphi}{\partial t}\,dr\,dt + \iint_{Q_T} r|u|^n\frac{\partial W}{\partial r}\frac{\partial\varphi}{\partial r}\,dr\,dt = 0.$$

Our interest lies in the existence of weak solutions. Because of the degeneracy, we will first consider the regularized problem. Based on the uniform estimates for the approximate solutions, we obtain the existence. Owing to the background, we are much interested in the nonnegativity of the weak solutions and the solutions with the property of finite speed of propagation of perturbations. Using weighted Nirenberg's inequality and Hardy's inequality, we proved these properties. This paper is arranged as follows. We shall prove several preliminary lemmas and obtain some a priori estimates on the solutions of regularized problem in Section 2, and then establish the existence in Section 3. Subsequently, we discuss the nonnegativity of weak solutions in Section 4 and the finite speed of propagation in Section 5.

2. Regularized problem

Bernis and Friedman [4] obtained several uniform estimations for the regularized solutions of fourth order thin film equation with the initial boundary value problems. To discuss the existence of weak solutions of problem (1.1)-(1.3), we adopt the method of parabolic regularization, namely, the desired solution will be obtained as the limit of some subsequence of solutions of the following regularized problem

$$\frac{\partial(r_\varepsilon u)}{\partial t} = \frac{\partial}{\partial r}\{r_\varepsilon m_\varepsilon(u)\frac{\partial W}{\partial r}\}, \quad r_\varepsilon W = \frac{\partial}{\partial r}\Big(r_\varepsilon\frac{\partial V}{\partial r}\Big), \quad r_\varepsilon V = \frac{\partial}{\partial r}\Big(r_\varepsilon\frac{\partial u}{\partial r}\Big), \tag{2.1}$$

$$\frac{\partial u}{\partial r}\Big|_{r=0} = \frac{\partial u}{\partial r}\Big|_{r=1} = \frac{\partial V}{\partial r}\Big|_{r=0} = \frac{\partial V}{\partial r}\Big|_{r=1} = \frac{\partial W}{\partial r}\Big|_{r=0} = \frac{\partial W}{\partial r}\Big|_{r=1} = 0, \tag{2.2}$$

$$u\Big|_{t=0} = u_{0\varepsilon}(r), \tag{2.3}$$

where $r_\varepsilon = r+\varepsilon$, $m_\varepsilon(u) = (|u|^2+\varepsilon)^{n/2}$ and $u_{0\varepsilon}(r)$ is a smooth approximation of the initial data $u_0(r)$.

From the classical approach [4], it is not difficult to conclude that the problem (2.1)–(2.3) admits a global classical solution. We need some uniform estimates on the classical solutions.

We first introduce some notation. Let $I = (0,1)$ and for any fixed $\varepsilon \ge 0$ denote by $W^{1,2}_{*,\varepsilon}(I)$ the class of all functions satisfying

$$\|u\|_{*,\varepsilon} = \Big(\int_0^1 (r+\varepsilon)|u'(r)|^2dr\Big)^{1/2} + \Big(\int_0^1 (r+\varepsilon)|u(r)|^2dr\Big)^{1/2} < +\infty.$$

It is obvious that $W^{1,2}(I) \subset W^{1,2}_{*,0}(I)$, but the class $W^{1,2}_{*,0}(I)$ is quite different from $W^{1,2}(I)$. In particular, we notice that the functions in $W^{1,2}_{*,0}(I)$ may not be bounded.

For $\varepsilon > 0$ the spaces $W^{1,2}(I)$ and $W^{1,2}_{*,\varepsilon}(I)$ coincide. However, it is not difficult to prove that for $u \in W^{1,2}_{*,\varepsilon}(I)$, the following properties hold:

Lemma 2.1. *If $0 < \alpha \le 1$, and $u \in W^{1,2}_{*,\varepsilon}(I)$. Then*

$$\sup_{0<r\le 1} ((r+\varepsilon)^\alpha |u(r)|) \le C\|u\|_{*,\varepsilon},$$

where C is a constant depending only on α;

Proof. First we consider the case that $0 < r < 1/2$. From the mean value theorem, we obtain immediately

$$\int_r^1 u(x)dx = u(\xi)(1-r)$$

for some $\xi \in [r, 1]$. Thus

$$\begin{aligned} |u(r)| &\le |u(r) - u(\xi)| + |u(\xi)| \le \Big|\int_r^\xi u'(x)dx\Big| + \frac{1}{1-r}\Big|\int_r^1 u(x)dx\Big| \\ &\le \int_r^1 |u'(x)|dx + 2\int_r^1 |u(x)|dx. \end{aligned}$$

It follows that

$$\begin{aligned} (r+\varepsilon)^\alpha |u(r)| &\le \int_r^1 (r+\varepsilon)^\alpha |u'(x)|dx + 2\int_r^1 (r+\varepsilon)^\alpha |u(x)|dx \\ &\le \int_0^1 (r+\varepsilon)^\alpha |u'(r)|dr + 2\int_0^1 (r+\varepsilon)^\alpha |u(r)|dr \\ &\le \Big(\int_0^1 (r+\varepsilon)|u'(r)|^2 dr\Big)^{1/2} \Big(\int_0^1 (r+\varepsilon)^{2\alpha-1} dr\Big)^{1/2} \\ &\quad + 2\Big(\int_0^1 (r+\varepsilon)|u(r)|^2 dr\Big)^{1/2} \Big(\int_0^1 (r+\varepsilon)^{2\alpha-1} dr\Big)^{1/2} \\ &\le C(\alpha)\|u\|_{*,\varepsilon}. \end{aligned}$$

Finally, we discuss the case that $1/2 \le r \le 1$, we have

$$\begin{aligned} &(r+\varepsilon)^\alpha |u(r)| \le (1+\varepsilon)^\alpha |u(r)| \\ &\le (1+\varepsilon)^\alpha \Big[\int_{1/2}^1 |u'(x)|dx + 2\int_{1/2}^1 |u(y)|dy\Big] \\ &\le (1+\varepsilon)^\alpha \Big[\Big(\int_0^1 (r+\varepsilon)|u'(r)|^2 dr\Big)^{1/2} + 2\Big(\int_0^1 (r+\varepsilon)|u(r)|^2 dr\Big)^{1/2}\Big] \\ &\le C(\alpha)\|u\|_{*,\varepsilon}. \end{aligned}$$

The proof is complete. □

Lemma 2.2. *If $0 < \alpha \le 1/2$, and $u \in W^{1,2}_{*,\varepsilon}(I)$. Then for any $\beta < \alpha$,*

$$|(r_1+\varepsilon)^\alpha u(r_1) - (r_2+\varepsilon)^\alpha u(r_2)| \le C|r_1 - r_2|^\beta \|u\|_{*,\varepsilon},$$

where C is a constant depending only on α and β.

Proof. For fixed $0 < r_2 < r_1 < 1$, we have

$$|u(r_2) - u(r_1)| \le \int_{r_2}^{r_1} |u'(t)|dt.$$

It follows that

$$
\begin{aligned}
(r_2+\varepsilon)^\alpha|u(r_1)-u(r_2)| &\le (r_2+\varepsilon)^\alpha\int_{r_2}^{r_1}|u'(t)|dt \le \int_{r_2}^{r_1}(t+\varepsilon)^\alpha|u'(t)|dt\\
&\le \Big(\int_{r_2}^{r_1}(t+\varepsilon)|u'(t)|^2dt\Big)^{1/2}\Big(\int_{r_2}^{r_1}(t+\varepsilon)^{2\alpha-1}dt\Big)^{1/2}\\
&\le C(\alpha)\|u\|_{*,\varepsilon}((r_1+\varepsilon)^{2\alpha}-(r_2+\varepsilon)^{2\alpha})^{1/2}\\
&\le C(\alpha)\|u\|_{*,\varepsilon}|r_1-r_2|^\alpha.
\end{aligned}
$$

On the other hand, from Lemma 2.1, we have

$$
\begin{aligned}
|((r_1+\varepsilon)^\alpha-(r_2+\varepsilon)^\alpha)u(r_1)| &\le 2|r_1-r_2|^\alpha|u(r_1)|\\
&\le 2|r_1-r_2|^\beta(r_1+\varepsilon)^{\alpha-\beta}|u(r_1)|\\
&\le C|r_1-r_2|^\beta\|u\|_{*,\varepsilon}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|(r_1+\varepsilon)^\alpha u(r_1)-(r_2+\varepsilon)^\alpha u(r_2)|\\
&\le |((r_1+\varepsilon)^\alpha-(r_2+\varepsilon)^\alpha)u(r_1)|+(r_2+\varepsilon)^\alpha|u(r_2)-u(r_1)|\\
&\le C|r_1-r_2|^\beta\|u\|_{*,\varepsilon}
\end{aligned}
$$

with C depending only on α and β. The proof is complete. □

Remark 2.3. Let $0<\beta<1/2$ and $u\in W^{1,2}_{*,\varepsilon}(I)$. Then

$$
|(r_1+\varepsilon)u(r_1)-(r_2+\varepsilon)u(r_2)|\le C(\beta)|r_1-r_2|^\beta\|u\|_{*,\varepsilon},
$$

where $C(\beta)$ is a constant depending only on β.

Lemma 2.4. *Let u be a smooth solution of problem (2.1)–(2.3) and for any $\alpha\in(0,1/2]$ and $\beta<\alpha$, there is a constant M independent of ε such that*

$$
\begin{gathered}
|r_\varepsilon^\alpha u(r,t)-s_\varepsilon^\alpha u(s,t)|\le M|r-s|^\beta,\\
|r_\varepsilon u(r,t)-s_\varepsilon u(s,t)|\le M|r-s|^\beta
\end{gathered}
$$

for all $r,s\in(0,1)$, where $s_\varepsilon=s+\varepsilon$.

Proof. Multiplying (2.1) by W and integrating with respect to r over $(0,1)$, we obtain

$$
\begin{aligned}
0&=\int_0^1\Big\{\frac{\partial r_\varepsilon u}{\partial t}W-\frac{\partial}{\partial r}\big[r_\varepsilon m_\varepsilon(u)\frac{\partial W}{\partial r}\big]W\Big\}dr\\
&=\int_0^1\Big\{\frac12\frac{\partial r_\varepsilon V^2}{\partial t}+r_\varepsilon m_\varepsilon(u)\big(\frac{\partial W}{\partial r}\big)^2\Big\}dr.
\end{aligned}
$$

Hence, integrating also with respect to t, we hve

$$
\int_0^1 r_\varepsilon V^2dr\le C, \tag{2.4}
$$

$$
\iint_{Q_T} r_\varepsilon m_\varepsilon(u)\big|\frac{\partial W}{\partial r}\big|^2dr\le C. \tag{2.5}
$$

It is easy to see that

$$
\int_0^1 r_\varepsilon V^2dr=\int_0^1\frac{\partial}{\partial r}\big(r_\varepsilon\frac{\partial u}{\partial r}\big)\cdot\frac{1}{r_\varepsilon}\frac{\partial}{\partial r}\big(r_\varepsilon\frac{\partial u}{\partial r}\big)\,dr\le C.
$$

A simple calculation shows that

$$\int_0^1 r_\varepsilon\Big(\frac{\partial^2 u}{\partial r^2}\Big)^2 dr + \int_0^1 \frac{1}{r_\varepsilon}\Big(\frac{\partial u}{\partial r}\Big)^2 dr + 2\int_0^1 \frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial r} dr \le C.$$

Using boundary condition (2.2), we have

$$\int_0^1 \frac{\partial^2 u}{\partial r^2}\frac{\partial u}{\partial r} dr = 0.$$

Hence

$$\int_0^1 r_\varepsilon\Big(\frac{\partial u}{\partial r}\Big)^2 dr \le \int_0^1 \frac{1}{r_\varepsilon}\Big(\frac{\partial u}{\partial r}\Big)^2 dr \le C. \tag{2.6}$$

$$\int_0^1 r_\varepsilon\Big(\frac{\partial^2 u}{\partial r^2}\Big)^2 dr \le C, \tag{2.7}$$

On the other hand, integrating the equation (2.1) on $Q_t = (0,1)\times(0,t)$, we have

$$\int_0^1 r_\varepsilon u(r,t)\, dr = \int_0^1 r_\varepsilon u_{0\varepsilon}(r)\, dr. \tag{2.8}$$

Note that, for any $\rho \in (0,1)$,

$$\begin{aligned}
&\frac{1+2\varepsilon}{2}u(\rho,t) - \int_0^1 s_\varepsilon u(s,t)ds \\
&= \int_0^1 s_\varepsilon[u(\rho,t)-u(s,t)]ds \\
&= \int_0^1\int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)\, dr\, ds = \int_0^\rho\int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)\, dr\, ds + \int_\rho^1\int_s^\rho s_\varepsilon \frac{\partial u}{\partial r}(r,t)\, dr\, ds \\
&= \int_0^\rho\int_0^r s_\varepsilon \frac{\partial u}{\partial r}(r,t) ds dr + \int_\rho^1\int_r^1 s_\varepsilon \frac{\partial u}{\partial r}(r,t) ds dr \\
&= \int_0^\rho \Big(\frac{r^2}{2}+\varepsilon r\Big)\frac{\partial u}{\partial r}(r,t)\, dr + \int_\rho^1 \Big[\frac{1}{2}(1-r^2)+\varepsilon(1-r)\Big]\frac{\partial u}{\partial r}(r,t)\, dr \\
&\le \int_0^\rho r_\varepsilon\Big|\frac{\partial u}{\partial r}(r,t)\Big|dr + 2\int_\rho^1 \Big|\frac{\partial u}{\partial r}(r,t)\Big|dr.
\end{aligned}$$

Setting $\rho_\varepsilon = \rho+\varepsilon$ and multiplying the above inequality by $2\rho_\varepsilon^{1/2}$, we obtain

$$\begin{aligned}
&\big|(1+2\varepsilon)\rho_\varepsilon^{1/2}u(\rho,t) - 2\rho_\varepsilon^{1/2}\int_0^1 s_\varepsilon u(s,t)ds\big| \\
&\le 2\rho_\varepsilon^{1/2}\int_0^\rho r_\varepsilon\Big|\frac{\partial u}{\partial r}(r,t)\Big|dr + 4\rho_\varepsilon^{1/2}\int_\rho^1 \Big|\frac{\partial u}{\partial r}(r,t)\Big|dr \\
&\le 2\rho_\varepsilon^{1/2}\int_0^\rho r_\varepsilon\Big|\frac{\partial u}{\partial r}(r,t)\Big|dr + 4\int_\rho^1 r_\varepsilon^{1/2}\Big|\frac{\partial u}{\partial r}(r,t)\Big|dr \\
&\le C\Big(\int_0^1 r_\varepsilon\Big|\frac{\partial u}{\partial r}(r,t)\Big|^2 dr\Big)^{1/2}.
\end{aligned} \tag{2.9}$$

From (2.6), (2.8) and (2.9), we see that $r_\varepsilon^{1/2}u(r,t)$ is uniformly bounded on Q_T. Furthermore $u(\cdot,t)\in W^{1,2}_{*,\varepsilon}(I)$ for any fixed $t\in(0,T)$, with $\|u(\cdot,t)\|_{*,\varepsilon}$ bounded by a constant C independent of ε. The desired estimates then follow from the properties of $W^{1,2}_{*,\varepsilon}(I)$ mentioned above.

From the above results and using the Remark, we conclude the second inequality. The proof is complete. $\square$

Lemma 2.5. *For any $\alpha > 0$, there is a constant M independent of ε such that*

$$r_\varepsilon^\alpha |u(r,t)| \le M, \quad \|u\|_{*,\varepsilon} \le M, \tag{2.10}$$

$$|r_\varepsilon u(r,t_2) - r_\varepsilon u(r,t_1)| \le M|t_2 - t_1|^{1/16} \tag{2.11}$$

for all $r \in (0,1)$, $t_1, t_2 \in (0,T)$.

Proof. The first two estimates have already been seen from the arguments in Lemma 2.1. Now, we begin to show (2.11). Without loss of generality, we assume that $t_1 < t_2$ and set $\Delta t = t_2 - t_1$. Integrating both sides of the equation (2.1) over $(t_1,t_2) \times (y, y + (\Delta t)^\alpha)$ and then integrating the resulting relation with respect to y over $(x, x + (\Delta t)^\alpha)$, we obtain

$$\begin{aligned}
&(\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (y + \theta(\Delta t)^\alpha + \varepsilon)\Big[u(y+\theta(\Delta t)^\alpha, t_2) - u(y+\theta(\Delta t)^\alpha, t_1)\Big]\,d\theta\,dy \\
&= \int_x^{x+(\Delta t)^\alpha} \int_y^{y+(\Delta t)^\alpha} \int_{t_1}^{t_2} \frac{\partial}{\partial r}\Big\{r_\varepsilon m_\varepsilon(u)\frac{\partial W}{\partial r}\Big\}\,d\tau\,dr\,dy \\
&= \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \Big[(y + (\Delta t)^\alpha + \varepsilon) m_\varepsilon(u(y+(\Delta t)^\alpha)) \\
&\quad \times \frac{\partial}{\partial y} W(y+(\Delta t)^\alpha, \tau) - (y+\varepsilon)m_\varepsilon(u(y))\frac{\partial}{\partial y}W(y,\tau)\Big]\,d\tau\,dy.
\end{aligned}$$

By the mean value theorem, there exists $x^* = y^* + \theta^*(\Delta t)^\alpha$, $y^* \in (x, x + (\Delta t)^\alpha)$, $\theta^* \in (0,1)$ such that the left hand side of the above equality can be expressed by

$$\begin{aligned}
&(\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (y + \theta(\Delta t)^\alpha + \varepsilon)\,[u(y+\theta(\Delta t)^\alpha, t_2) - u(y+\theta(\Delta t)^\alpha, t_1)]\,d\theta\,dy \\
&= (\Delta t)^{2\alpha}(y^* + \theta^*(\Delta t)^\alpha + \varepsilon)\Big[u(y^*+\theta^*(\Delta t)^\alpha, t_2) - u(y^*+\theta^*(\Delta t)^\alpha, t_1)\Big].
\end{aligned}$$

For the right hand side, we have

$$\begin{aligned}
&\int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \Big[(y + (\Delta t)^\alpha + \varepsilon) m_\varepsilon(u(y+(\Delta t)^\alpha)) \\
&\quad \times \frac{\partial}{\partial y} W(y+(\Delta t)^\alpha, \tau) - (y+\varepsilon)m_\varepsilon(u(y))\frac{\partial}{\partial y}W(y,\tau)\Big]\,d\tau\,dy \\
&= \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} (r+\varepsilon)m_\varepsilon(u(r))\frac{\partial}{\partial r}W(r,\tau)\,d\tau\,dr \\
&\quad - \int_x^{x+(\Delta t)^\alpha} (r+\varepsilon)m_\varepsilon(u(r))\frac{\partial}{\partial r}W(r,\tau)\,d\tau\,dr \\
&\le \int_{x+(\Delta t)^\alpha}^{x+2(\Delta t)^\alpha} \int_{t_1}^{t_2} r_\varepsilon m_\varepsilon(u(r))\Big|\frac{\partial}{\partial r}W(r,\tau)\Big|\,d\tau\,dr.
\end{aligned}$$

By (2.5), we see that

$$|x_\varepsilon^* u(x^*, t_2) - x_\varepsilon^* u(x^*, t_1)| \le C(\Delta t)^{\frac{1-3\alpha}{2}},$$

which implies, by setting $\alpha = 1/4$ and using the properties of the functions in $W^{1,2}_{*,\varepsilon}(I)$, that

$$|r_\varepsilon u(r,t_2) - r_\varepsilon u(r,t_1)| \le C(\Delta t)^{1/16}.$$

The proof is complete. □

3. Existence

After the discussion of the regularized problem, we can now turn to the investigation of the existence of weak solutions of the problem (1.1)–(1.3). The main existence result is the following

Theorem 3.1. *If* $u_0(r) \in H^2(I)$*, then problem* (1.1)–(1.3) *admits at least one weak solution.*

Proof. Let u_ε be the approximate solution of (2.1)–(2.3) constructed in the previous section. Using the estimates in Lemma 2.4 and 2.5, for any $\beta < \frac{1}{2}$, and $(r_1,t_2),(r_2,t_1) \in Q_T$, we have

$$|r_{1\varepsilon} u_\varepsilon(r_1,t_2) - r_{2\varepsilon} u_\varepsilon(r_2,t_1)| \le C(|r_1-r_2|^\beta + |t_1-t_2|^{\beta/4})$$

with constant C independent of ε. So, we may extract a subsequence from $\{r_\varepsilon u_\varepsilon\}$, denoted also by $\{r_\varepsilon u_\varepsilon\}$, such that

$$r_\varepsilon u_\varepsilon(r,t) \to ru(r,t) \text{ uniformly in } \overline{Q}_T,$$

and the limiting function $ru \in C^{1/4,1/16}(\overline{Q}_T)$.

Now we prove that $u(r,t)$ is a weak solution of problem (1.1)-(1.3), let $\delta > 0$ be fixed and set $P_\delta = \{(r,t) : r|u|^n > \delta\}$. We choose $\varepsilon_0(\delta) > 0$, such that

$$r_\varepsilon(|u_\varepsilon|^2+\varepsilon)^{n/2} \ge \frac{\delta}{2}, \quad (r,t) \in P_\delta,\ 0<\varepsilon<\varepsilon_0(\delta). \tag{3.1}$$

Then from (2.5)

$$\iint_{P_\delta} \Big(\frac{\partial W_\varepsilon}{\partial r}\Big)^2 \,dr\,dt \le \frac{C}{\delta}. \tag{3.2}$$

To prove the integral equality in the definition of solutions, it suffices to pass the limit as $\varepsilon \to 0$ in

$$\begin{aligned} &\int_0^1 r_\varepsilon u_\varepsilon(r,T)\varphi(r,T)\,dr - \int_0^1 r_\varepsilon u_{0\varepsilon}\varphi(r,0)\,dr - \iint_{Q_T} r_\varepsilon u_\varepsilon \frac{\partial \varphi}{\partial t}\,dr\,dt \\ &+ \iint_{Q_T} r_\varepsilon(u_\varepsilon^2+\varepsilon)^{n/2}\frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt = 0. \end{aligned}$$

The limits

$$\begin{gathered} \lim_{\varepsilon\to 0}\int_0^1 r_\varepsilon u_\varepsilon(r,T)\varphi(r,T) = \int_0^1 ru(r,T)\varphi(r,T)\,dr, \\ \lim_{\varepsilon\to 0}\int_0^1 r_\varepsilon u_{0\varepsilon}(r)\varphi(r,0)\,dr = \int_0^1 u_0(r)\varphi(0,r)\,dr, \\ \lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon u_\varepsilon \frac{\partial\varphi}{\partial t}\,dr\,dt = \iint_{Q_T} ru\frac{\partial\varphi}{\partial t}\,dr\,dt, \end{gathered}$$

are obvious. It remains to show that

$$\lim_{\varepsilon\to 0}\iint_{Q_T} r_\varepsilon(u_\varepsilon^2+\varepsilon)^{n/2}\frac{\partial W_\varepsilon}{\partial r}\frac{\partial\varphi}{\partial r}\,dr\,dt = \iint_{Q_T} r|u|^n\frac{\partial W}{\partial r}\frac{\partial\varphi}{\partial r}\,dr\,dt. \tag{3.3}$$

In fact, for any fixed $\delta > 0$,

$$
\begin{aligned}
&\Big|\iint_{Q_T} r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt - \iint_{Q_T} r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \\
&\le \Big|\iint_{P_\delta} r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt - \iint_{P_\delta} r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \\
&\quad + \Big|\iint_{Q_T\setminus P_\delta} r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| + \Big|\iint_{Q_T\setminus P_\delta} r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big|.
\end{aligned}
$$

From the estimates (2.5), we have

$$
\Big|\iint_{Q_T\setminus P_\delta} r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \le C\delta \sup\Big|\frac{\partial \varphi}{\partial r}\Big|, \quad 0<\varepsilon<\varepsilon_0(\delta),
$$

$$
\Big|\iint_{Q_T\setminus P_\delta} r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \le C\delta \sup\Big|\frac{\partial \varphi}{\partial r}\Big|,
$$

$$
\begin{aligned}
&\Big|\iint_{P_\delta} r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt - \iint_{P_\delta} r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \\
&\le \iint_{P_\delta} \big|r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} - r|u|^n\big|\Big|\frac{\partial W_\varepsilon}{\partial r}\Big|\Big|\frac{\partial \varphi}{\partial r}\Big|\,dr\,dt \\
&\quad + \Big|\iint_{P_\delta} r|u|^n \Big(\frac{\partial W_\varepsilon}{\partial r} - \frac{\partial W}{\partial r}\Big)\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \\
&\le \sup|r_\varepsilon (u_\varepsilon^2+\varepsilon)^{n/2} - r|u|^n|\Big|\frac{\partial \varphi}{\partial r}\Big|\frac{C}{\sqrt{\delta}} + \Big|\iint_{P_\delta} r|u|^n \Big(\frac{\partial W_\varepsilon}{\partial r} - \frac{\partial W}{\partial r}\Big)\frac{\partial \varphi}{\partial r}\,dr\,dt\Big|
\end{aligned}
$$

and hence

$$
\limsup_{\varepsilon\to 0}\Big|\iint_{Q_T} r_\varepsilon \frac{\partial W_\varepsilon}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt - \iint_{Q_T} r\frac{\partial W}{\partial r}\frac{\partial \varphi}{\partial r}\,dr\,dt\Big| \le C\delta \sup\Big|\frac{\partial \varphi}{\partial r}\Big|.
$$

By the arbitrariness of δ, we see that the limit (3.3) holds. The proof is complete. □

4. Nonnegativity

Just as mentioned by several authors, it is much interesting to discuss the physical solutions. For the two-dimensional problem (1.1)–(1.3), a very typical example is the modeling of oil films spreading over a solid surface, where the unknown function u denotes the height from the surface of the oil film to the solid surface. Motivated by this idea, we devote this section to the discussion of the nonnegativity of solutions.

Theorem 4.1. *The weak solution u obtained in Section 3 satisfies $u(r,t) \ge 0$, if $u_0(r) \ge 0$.*

Proof. Suppose the contrary, that is, the set

$$
E = \{(r,t) \in \overline{Q}_T; u(r,t) < 0\} \tag{4.1}
$$

is nonempty. For any fixed $\delta > 0$, choose a C^∞ function $H_\delta(s)$ such that $H_\delta(s) = -\delta$ for $s \ge -\delta$, $H_\delta(s) = -1$, for $s \le -2\delta$ and that $H_\delta(s)$ is nondecreasing for $-2\delta < s < -\delta$. Also, we extend the function $u(r,t)$ to be defined in the whole plane $\mathbb{R}^2$ such that the extension $\bar{u}(r,t) = 0$ for $t \ge T+1$ and $t \le -1$. Let $\alpha(s)$ be the kernel of mollifier in one-dimension, that is, $\alpha(s) \in C^\infty(\mathbb{R})$, $\operatorname{supp}\alpha = [-1,1]$, $\alpha(s) > 0$ in $(-1,1)$, and $\int_{-1}^1 \alpha(s)ds = 1$. For any fixed $k > 0, \delta > 0$, define

$$u^h(r,t) = \int_{\mathbb{R}} \bar{u}(s,r)\alpha_h(t-s)ds,$$
$$\beta_\delta(t) = \int_t^{+\infty} \alpha\Big(\frac{s-\frac{T}{2}}{\frac{T}{2}-\delta}\Big)\frac{1}{\frac{T}{2}-\delta}ds,$$

where $\alpha_h(s) = \frac{1}{h}\alpha(s/h)$.

The function

$$\varphi_\delta^h(r,t) \equiv [\beta_\delta(t)H_\delta(u^h)]^h$$

is clearly an admissible test function, that is, the following integral equality holds

$$\begin{aligned} &\int_0^1 ru(r,T)\varphi_\delta^h(T,r)\,dr - \int_0^1 ru_0(r)\varphi_\delta^h(r,0)\,dr \\ &- \iint_{Q_T} ru\frac{\partial\varphi_\delta^h}{\partial t}\,dr\,dt + \iint_{Q_T} r|u|^n\frac{\partial W}{\partial r}\frac{\partial\varphi_\delta^h}{\partial r}\,dr\,dt = 0. \end{aligned} \tag{4.2}$$

To proceed further, we give an analysis on the properties of the test function $\varphi_\delta^h(r,t)$. The definition of $\beta_\delta(t)$ implies that

$$\varphi_\delta^h(r,t) = 0, \quad t \ge T - \frac{\delta}{2}, \quad h < \frac{\delta}{2}. \tag{4.3}$$

Since $\bar{u}(r,t)$ is continuous, for fixed δ, there exists $\eta_1(\delta) > 0$, such that

$$u^h(r,t) \ge -\frac{\delta}{2}, \quad t \le \eta_1(\delta), \quad 0 \le r \le 1, \quad h < \eta_1(\delta), \tag{4.4}$$

which together with the definition of $\beta_\delta(t)$ and $H_\delta(s)$ imply

$$H_\delta(u^h(r,t)) = -\delta, \quad t \le \eta_1(\delta), \quad 0 \le r \le 1, \quad h < \eta_1(\delta) \tag{4.5}$$

and hence

$$\varphi_\delta^h = -\delta, \quad t \le \frac{1}{2}\eta_1(\delta), \quad 0 \le r \le 1, \quad h < \frac{1}{2}\eta_1(\delta). \tag{4.6}$$

We note also that for any functions $f(t), g(t) \in L^2(R)$,

$$\begin{aligned} \int_{\mathbb{R}} f(t)g^h(t)dt &= \int_{\mathbb{R}} f(t)dt\int_{\mathbb{R}} g(s)\alpha_n(t-s)ds = \int_{\mathbb{R}} f(t)\int_{\mathbb{R}} g(s)\alpha_n(s-t)ds \\ &= \int_{\mathbb{R}} g(s)ds\int_{\mathbb{R}} f(t)\alpha_n(s-t)dt = \int_{\mathbb{R}} f^h(t)g(t)dt. \end{aligned}$$

Taking this into account and using (4.3), (4.5), (4.6), we have

$$\begin{aligned} \iint_{Q_T} ru\frac{\partial}{\partial t}\varphi_\delta^h\,dr\,dt &= \int_{-\infty}^{+\infty} dt\int_0^1 ru\big[\frac{\partial}{\partial t}\big(\beta_\delta(t)H_\delta(u^h)\big)\big]^h dr \\ &= \iint_{Q_T} (ru)^h\frac{\partial}{\partial t}\big(\beta_\delta(t)H_\delta(u^h)\big)\,dr\,dt \end{aligned}$$

and hence by integrating by parts

$$\begin{aligned} &\iint_{Q_T} (ru)^h\frac{\partial}{\partial t}\left(\beta_\delta(t)H_\delta(u^h)\right)\,dr\,dt \\ &= \int_0^1 (ru)^h(r,T)\beta_\delta(T)H_\delta(u^h(r,T))\,dr - \int_0^1 (ru)^h(r,0)\beta_\delta(0)H_\delta(u^h(r,0))\,dr \end{aligned}$$

$$
\begin{aligned}
&- \iint_{Q_T} \beta_\delta(t) H_\delta(u^h) \frac{\partial (ru)^h}{\partial t}\,dr\,dt \\
&= \delta \int_0^1 (ru)^h(r,0)\,dr - \iint_{Q_T} r\beta_\delta(t) \frac{\partial}{\partial t} F_\delta(u^h)\,dr\,dt,
\end{aligned}
$$

where $F_\delta(s) = \int_0^s H_\delta(\sigma)d\sigma$.

Again by (4.5)

$$
\begin{aligned}
F_\delta(u^h(r,0)) &= \int_0^{u^h(r,0)} H_\delta(\sigma)d\sigma = \int_0^1 H_\delta(\lambda u^h(r,0))d\lambda \cdot u^h(r,0) \\
&= -\delta u^h(r,0)
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\iint_{Q_T} (ru)^h \frac{\partial}{\partial t}(\beta_\delta(t) H_\delta(u^h))\,dr\,dt \\
&= \delta \int_0^1 (ru)^h(r,0)\,dr + \int_0^1 r\beta_\delta(0) F_\delta(u^h(r,0))\,dr + \iint_{Q_T} rF_\delta(u^h)\beta_\delta'(t)\,dr\,dt \quad (4.7)\\
&= -\frac{1}{\frac{T}{2}-\delta} \iint_{Q_T} rF_\delta(u^h)\alpha\Big(\frac{t-\frac{T}{2}}{\frac{T}{2}-\delta}\Big)\,dr\,dt.
\end{aligned}
$$

From (4.3), (4.6) it is clear that

$$
\int_0^1 ru(r,T)\varphi_\delta^h(T,r)\,dr = 0, \quad 0 < h < \frac{1}{2}\eta_1(\delta), \tag{4.8}
$$

$$
-\int_0^1 ru_0(r)\varphi_\delta^h(r,0)\,dr = \delta \int_0^1 ru_0(r)\,dr. \tag{4.9}
$$

Substituting (4.7), (4.8) and (4.9) into (4.2), we have

$$
\begin{aligned}
&\frac{2}{T-2\delta} \iint_{Q_T} rF_\delta(u^h)\alpha\Big(\frac{t-\frac{T}{2}}{\frac{T}{2}-\delta}\Big)\,dr\,dt + \delta \int_0^1 ru_0(r)\,dr \\
&+ \iint_P r|u|^n \frac{\partial W}{\partial r}\frac{\partial \varphi_\delta^h}{\partial r}\,dr\,dt = 0.
\end{aligned} \tag{4.10}
$$

By the uniform continuity of $u(r,t)$ in $\overline{Q}_T$, there exists $\eta_2(\delta) > 0$, such that

$$
u(r,t) \ge -\frac{\delta}{2} \quad \forall (r,t) \in P^\delta, \tag{4.11}
$$

where $P^\delta = \{(r,t); \mathrm{dist}((r,t),P) < \eta_2(\delta)\}$. Here we have used the fact that $u(r,t) > 0$ in P. Thus

$$
H_\delta(u^h(r,t)) = -\delta, \quad \forall (r,t) \in P^{\delta/2}, \quad 0 < h < \frac{1}{2}\eta_2(\delta)
$$

where $P^{\delta/2} = \{(r,t); \mathrm{dist}((r,t),P) < \frac{1}{2}\eta_2(\delta)\}$.

By this and the definition of $u^h, H_\delta(s)$ shows that the function $\varphi_\delta^h(r,t)$ is only a function of t in P, whenever $h < \frac{1}{2}\eta_2(\delta)$. Therefore

$$
D\varphi_\delta^h(r,t) = 0, \quad \forall (r,t) \in P, \quad 0 < h < \frac{1}{2}\eta_2(\delta) \tag{4.12}
$$

and so (4.10) becomes

$$\delta \int_0^1 r u_0(r)\, dr + \frac{2}{T-2\delta} \iint_{Q_T} r F_\delta(u^h) \alpha\Big(\frac{2t-T}{T-2\delta}\Big)\, dr\, dt = 0, \tag{4.13}$$

where $\eta(\delta) = \min(\eta_1(\delta), \eta_2(\delta))$. Letting h tend to zero, we have

$$\delta \int_0^1 r u_0(r)\, dr + \frac{2}{T-2\delta} \iint_{Q_T} r F_\delta(u) \alpha\Big(\frac{2t-T}{T-2\delta}\Big)\, dr\, dt = 0. \tag{4.14}$$

From the definition of $F_\delta(s)$ and $H_\delta(s)$, it is easily seen that

$$F_\delta(u(r,t)) \to -\chi_E(r,t) u(r,t) \quad (\delta \to 0)$$

and so by letting δ tend to zero in (4.14), we have

$$\iint_E |u(r,t)| \alpha\Big(\frac{2t-T}{T}\Big)\, dr\, dt = 0,$$

which contradicts the fact that $\alpha\big(\frac{2t-T}{T}\big) > 0$ for $0 < t < T$. We have thus proved the theorem. $\square$

Lemma 4.2. *Let u be the limit function of the approximate solutions obtained above. Then the following integral inequality holds*

$$\int_0^1 r u^{2-n} dr + \iint_{Q_t} r \Big(\frac{\partial V}{\partial r}\Big)^2\, dr\, ds \le \int_0^1 r u_0^{2-n} dr.$$

Proof. Let u_ε be the solution of the problem (2.1)-(2.3). Denote

$$g_\varepsilon(u) = \int_0^u \frac{dr}{(|r|^2 + \varepsilon)^{n/2}}, \quad G_\varepsilon(u) = \int_0^u g_\varepsilon(r)\, dr.$$

Multiplying both sides of the equation (2.1) by $g_\varepsilon(u_\varepsilon)$, and then integrating over Q_t, we obtain

$$\int_0^1 (r+\varepsilon) G_\varepsilon(u_\varepsilon(r,t))\, dr + \iint_{Q_t} (r+\varepsilon) \frac{\partial W}{\partial r} \frac{\partial u_\varepsilon}{\partial r}\, dr\, ds = \int_0^1 (r+\varepsilon) G_\varepsilon(u_{0\varepsilon}(r))\, dr. \tag{4.15}$$

Integrating by parts, we obtain

$$\int_0^1 (r+\varepsilon) G_\varepsilon(u_\varepsilon(r,t))\, dr + \iint_{Q_t} (r+\varepsilon) \big(\frac{\partial V}{\partial r}\big)^2\, dr\, ds = \int_0^1 G_\varepsilon(u_{0\varepsilon}(r))\, dr.$$

Letting $\varepsilon \to 0$ and using the fact that $G_\varepsilon(u_\varepsilon) \to u^{2-n}/(1-n)(2-n)$ and $u_\varepsilon \to u$ pointwise and the lower semi-continuity of the integrals, we immediately get the conclusion of the lemma. The proof is complete. $\square$

Theorem 4.3. *Suppose that $u_0(r) > 0$ and $n \ge 6$, then the weak solution u satisfies $u(r,t) > 0$.*

Proof. Since we have proved that $u(r,t) \ge 0$, if the conclusion were false, then there would exist a point $(r_0, t_0) \in Q_T$, such that $u(r_0, t_0) = 0$. From the Hölder continuity of ru, we see that

$$r u(r, t_0) \le C |r - r_0|^{1/4}.$$

Since $n \ge 6$, we have

$$\int_0^1 r u(r, t_0)^{2-n} dr \ge C \int_0^1 |r - r_0|^{(2-n)/4} dr = \infty.$$

On the other hand, by Lemma 4.2,

$$\int_0^1 ru(r,t_0)^{2-n}dr \le C,$$

which is a contradiction. The proof is complete. □

5. Finite speed of propagation of perturbations

As is well known, one of the important properties of solutions of the porous medium equation is the finite speed of propagation of perturbations. So from the point of view of physical background, it seems to be natural to investigate this property for thin film equation. Bernis and Friedman [4], Bernis [5] considered this property for thin film equation. On the other hand, the mathematical description of this property is that if $\operatorname{supp} u_0$ is bounded, then for any $t > 0$, $\operatorname{supp} u(\cdot, t)$ is also bounded. So from the point of view of mathematics, this problem seems to be quite interesting. We adopt the weighted energy method and the main technical tools are weighted Nirenberg's inequality and Hardy's inequality.

Theorem 5.1. *Assume* $0 < n < 1$, $u_0 \in H_0^1(I) \cap H^2(I)$, $u_0 \ge 0$, $\operatorname{supp} u_0 \subset [r_1, r_2]$, $0 < r_1 < r_2 < 1$, *and* u *is the weak solution of the problem* (1.1)-(1.3), *then for any fixed* $t > 0$, *we have*

$$\operatorname{supp} u(x,\cdot) \subset [r_1(t), r_2(t)] \cap [0,1],$$

where $r_1(t) = r_1 - C_1 t^\gamma$, $r_2(t) = r_2 + C_2 t^\gamma$, $C_1, C_2, \gamma > 0$.

We need some uniform estimates on such approximate solutions u_ε.

Lemma 5.2. *Let* u *be the limit function of the approximate solutions obtained above. Then for any* $y \in \mathbb{R}^+$, *the following integral inequality holds*

$$\begin{aligned}
&\int_0^1 r(r-y)_+^\alpha u^{2-n}dr + \frac{1}{2}\iint_{Q_t} r(r-y)_+^\alpha \Big(\frac{\partial^3 u}{\partial r^3}\Big)^2 dr\,ds \\
&\le C\iint_{Q_t} r(r-y)_+^{\alpha-4}\Big(\frac{\partial u}{\partial r}\Big)^2 dr\,ds + C\iint_{Q_t} r(x-y)_+^{\alpha-2}\Big(\frac{\partial^2 u}{\partial r^2}\Big)^2 dr\,ds \\
&\quad + C\int_0^1 r(r-y)_+^\alpha |u_0|^{2-n}dr + C\Big(\iint_{Q_t} r|u_0|^{2-n}\,dr\,ds\Big)^{1/2},
\end{aligned}$$

where C *depends only on* n, u_0 *and* $\alpha \ge 2p-1$, *where* $(r-y)_+$ *denotes the positive part of* $r-y$.

Proof. Let $g_\varepsilon(u)$ and $G_\varepsilon(u)$ be defined as in the proof of Lemma 4.2. Let u_ε be the approximate solutions derived from the problem (2.1)–(2.3). Then, using the equation (2.1) and integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 r(r-y)_+^\alpha G_\varepsilon(u_\varepsilon)\,dr - \int_0^1 r(r-y)_+^\alpha G_\varepsilon(u_0)\,dr \\
&= -\iint_{Q_t} r_\varepsilon(|u_\varepsilon|^2+\varepsilon)^{n/2}\frac{\partial W}{\partial r}\frac{\partial}{\partial r}\left[(r-y)_+^\alpha g_\varepsilon(u_\varepsilon)\right] dr\,ds \\
&= -\iint_{Q_t} r_\varepsilon \frac{\partial W}{\partial r}(r-y)_+^\alpha \frac{\partial u_\varepsilon}{\partial r}\,dr\,ds \\
&\quad - \iint_{Q_t} r_\varepsilon(|u_\varepsilon|^2+\varepsilon)^{n/2}\frac{\partial W}{\partial r}\left[\alpha(r-y)_+^{\alpha-1} g_\varepsilon(u_\varepsilon)\right] dr\,ds
\end{aligned}$$

$$\equiv I_1 + I_2.$$

As for I_1, integrating by parts, we have

$$\begin{aligned}
I_1 &= -\iint_{Q_t} r_\varepsilon \frac{\partial W}{\partial r}(r-y)_+^\alpha \frac{\partial u_\varepsilon}{\partial r}\,dr\,ds \\
&= \iint_{Q_t} W r_\varepsilon \frac{\partial u_\varepsilon}{\partial r}\alpha(r-y)_+^{\alpha-1}\,dr\,ds + \iint_{Q_t} W\frac{\partial}{\partial r}\Big(r_\varepsilon \frac{\partial u_\varepsilon}{\partial r}\Big)(r-y)_+^\alpha\,dr\,ds \\
&= -\iint_{Q_t} r_\varepsilon \frac{\partial V}{\partial r}\frac{\partial^2 u_\varepsilon}{\partial r^2}\alpha(r-y)_+^{\alpha-1}\,dr\,ds - \iint_{Q_t} r_\varepsilon \frac{\partial V}{\partial r}\frac{\partial u_\varepsilon}{\partial r}\alpha(\alpha-1)(r-y)_+^{\alpha-2}\,dr\,ds \\
&\quad - \iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds - \iint_{Q_t} r_\varepsilon \alpha(r-y)_+^{\alpha-1}\frac{\partial V}{\partial r}V\,dr\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^1 r_\varepsilon (r-y)_+^\alpha G_\varepsilon(u_\varepsilon)\,dr - \int_0^1 r_\varepsilon (r-y)_+^\alpha G_\varepsilon(u_0)\,dr + \iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds \\
&= -\iint_{Q_t} r_\varepsilon \frac{\partial V}{\partial r}\frac{\partial^2 u_\varepsilon}{\partial r^2}\alpha(r-y)_+^{\alpha-1}\,dr\,ds - \iint_{Q_t} r_\varepsilon \frac{\partial V}{\partial r}\frac{\partial u_\varepsilon}{\partial r}\alpha(\alpha-1)(r-y)_+^{\alpha-2}\,dr\,ds \\
&\quad - \iint_{Q_t} r_\varepsilon \alpha(r-y)_+^{\alpha-1}\frac{\partial V}{\partial r}V\,dr\,ds \\
&\quad - \iint_{Q_t} r_\varepsilon (|u_\varepsilon|^2+\varepsilon)^{n/2}\frac{\partial W}{\partial r}\left[\alpha(r-y)_+^{\alpha-1}g_\varepsilon(u_\varepsilon)\right]\,dr\,ds \\
&\equiv I_a + I_b + I_c + I_d.
\end{aligned}$$

Hölder's inequality yields

$$\begin{gathered}
|I_a| \le \frac{1}{8}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds + C\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2}\Big(\frac{\partial^2 u_\varepsilon}{\partial r^2}\Big)^2\,dr\,ds, \\
|I_b| \le \frac{1}{8}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds + C_1\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-4}\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2\,dr\,ds, \\
|I_c| \le \frac{1}{4}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds + C_2\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2}V^2\,dr\,ds.
\end{gathered}$$

Noticing that

$$(|u_\varepsilon|^2+\varepsilon)^{n/2}|g_\varepsilon(u_\varepsilon)| \le \frac{2}{1-n}|u_\varepsilon|,$$

using (2.5), we have

$$\begin{aligned}
|I_d| &\le C\Big(\iint_{Q_t} r_\varepsilon (|u_\varepsilon|^2+\varepsilon)^{n/2}\Big(\frac{\partial W}{\partial r}\Big)^2\,dr\,ds\Big)^{1/2} \\
&\quad\times\Big(\iint_{Q_t} r_\varepsilon (|u_\varepsilon|^2+\varepsilon)^{n/2}g_\varepsilon(u_\varepsilon)^2\,dr\,ds\Big)^{1/2} \\
&\le C\Big(\iint_{Q_t} r_\varepsilon |u_\varepsilon|^{2-n}\,dr\,ds\Big)^{1/2}.
\end{aligned}$$

Summing up, we have

$$\begin{aligned}
&\int_0^1 r_\varepsilon (r-y)_+^\alpha G_\varepsilon(u_\varepsilon)\,dr - \int_0^1 r_\varepsilon (r-y)_+^\alpha G_\varepsilon(u_0)\,dr \\
&+ \frac{1}{2}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds \\
&\le C\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2}\Big(\frac{\partial^2 u_\varepsilon}{\partial r^2}\Big)^2\,dr\,ds + C_1\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-4}\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2\,dr\,ds \\
&\quad + C_2\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2} V^2\,dr\,ds + C_3\Big(\iint_{Q_t} r_\varepsilon |u_\varepsilon|^{2-n}\,dr\,ds\Big)^{1/2}.
\end{aligned}$$

A simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial V}{\partial r}\Big)^2\,dr\,ds \\
&= \frac{1}{2}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\Big(\frac{\partial^3 u_\varepsilon}{\partial r^3}\Big)^2 + 2\frac{\partial^3 u_\varepsilon}{\partial r^3}\frac{\partial}{\partial r}\Big(\frac{1}{r_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}\Big) + \Big(\frac{\partial}{\partial r}\Big(\frac{1}{r_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}\Big)\Big)^2\Big)\,dr\,ds \\
&= \frac{1}{2}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial^3 u_\varepsilon}{\partial r^3}\Big)^2\,dr\,ds + \frac{1}{2}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial}{\partial r}\Big(\frac{1}{r_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}\Big)\Big)^2\,dr\,ds \\
&\quad + \iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \frac{\partial^3 u_\varepsilon}{\partial r^3}\frac{\partial}{\partial r}\Big(\frac{1}{r_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}\Big)\,dr\,ds.
\end{aligned}$$

Note that

$$\begin{aligned}
&\Big|\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \frac{\partial^3 u_\varepsilon}{\partial r^3}\frac{\partial}{\partial r}\Big(\frac{1}{r_\varepsilon}\frac{\partial u_\varepsilon}{\partial r}\Big)\,dr\,ds\Big| \\
&= \Big|\iint_{Q_t} (r-y)_+^\alpha \frac{\partial^3 u_\varepsilon}{\partial r^3}\frac{\partial^2 u_\varepsilon}{\partial r^2}\,dr\,ds - \iint_{Q_t} \frac{1}{r_\varepsilon}(r-y)_+^\alpha \frac{\partial^3 u_\varepsilon}{\partial r^3}\frac{\partial u_\varepsilon}{\partial r}\,dr\,ds\Big| \\
&\le \frac{1}{8}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial^3 u_\varepsilon}{\partial r^3}\Big)^2\,dr\,ds + C\iint_{Q_t} \frac{r_\varepsilon}{r_\varepsilon^2}(r-y)_+^\alpha \Big(\frac{\partial^2 u_\varepsilon}{\partial r^2}\Big)^2\,dr\,ds \\
&\quad + \frac{1}{8}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial^3 u_\varepsilon}{\partial r^3}\Big)^2\,dr\,ds + C\iint_{Q_t} \frac{r_\varepsilon^2}{r_\varepsilon^4}(r-y)_+^\alpha \Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2\,dr\,ds \\
&\le \frac{1}{4}\iint_{Q_t} r_\varepsilon (r-y)_+^\alpha \Big(\frac{\partial^3 u_\varepsilon}{\partial r^3}\Big)^2\,dr\,ds + C\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2}\Big(\frac{\partial^2 u_\varepsilon}{\partial r^2}\Big)^2\,dr\,ds \\
&\quad + C\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-4}\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2\,dr\,ds.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2} V^2\,dr\,ds \\
&\le 2\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-2}\Big(\frac{\partial^2 u_\varepsilon}{\partial r^2}\Big)^2\,dr\,ds + 2\iint_{Q_t} r_\varepsilon (r-y)_+^{\alpha-4}\Big(\frac{\partial u_\varepsilon}{\partial r}\Big)^2\,dr\,ds.
\end{aligned}$$

Letting $\varepsilon \to 0$, we immediately get the desired conclusion and complete the proof of the lemma. □

Proof of Theorem 5.1. For any $y \geq r_2$, Lemma 5.2 and Hardy's inequality [11] imply that for any $t \in [0,T]$,

$$
\begin{aligned}
&\int_0^1 r(r-y)_+^{\alpha} u^{2-n} dr + \frac{1}{2}\iint_{Q_t} r(r-y)_+^{\alpha} \big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds \\
&\leq C\iint_{Q_t} r(r-y)_+^{\alpha-4}\big|\frac{\partial u}{\partial r}\big|^2\,dr\,ds + C\iint_{Q_t} r(r-y)_+^{\alpha-2}\big|\frac{\partial^2 u}{\partial r^2}\big|^2\,dr\,ds \qquad (5.1)\\
&\leq C\iint_{Q_t} (r-y)_+^{\alpha-2}\big|\frac{\partial^2 u}{\partial r^2}\big|^2\,dr\,ds.
\end{aligned}
$$

For any positive number m, define

$$
f_m(y) = \int_0^t \int_0^1 r(r-y)_+^m \big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds, \quad f_0(y) = \int_0^t \int_y^1 r\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds.
$$

By $y > r_2 > 0$, then, weighted Nirenberg's inequality [1] and estimate (5.1) imply that

$$
\begin{aligned}
&f_{2p+1}(y)\\
&\leq C\iint_{Q_t} (r-y)_+^{2p-1}\big|\frac{\partial^2 u}{\partial r^2}\big|^2\,dr\,ds\\
&\leq C_1\int_0^t \Big(\int_0^1 (r-y)_+^{2p-1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2 dr\Big)^a \Big(\int_0^1 (r-y)_+^{2p-1}|u|^q dr\Big)^{2(1-a)/q} ds\\
&\leq C\sup_{0<t<T}\Big(\int_0^1 r(r-y)_+^{2p-1}|u|^q dr\Big)^{2(1-a)/q} t^{1-a}\Big(\iint_{Q_t}(r-y)_+^{2p-1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big)^a.
\end{aligned}
$$

Using (5.1) and Hardy's inequality, we have

$$
\sup_{0<t<T}\int_0^1 r(r-y)_+^{2p-1}|u|^q dr \leq C\iint_{Q_t}(r-y)_+^{2p-1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds
$$

and hence

$$
f_{2p+1}(y) \leq Ct^{1-a}\Big(\iint_{Q_t}(r-y)_+^{2p-1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big)^{a+2(1-a)/q},
$$

where $q = 2-n$ and

$$
a = \frac{\frac{1}{2}-\frac{1}{p}-\frac{1}{q}}{\frac{1}{2}-\frac{3}{2p}-\frac{1}{q}}.
$$

Denote $\lambda = 1-a, \mu = a + 2(1-a)/q$, then $\lambda > 0,\ 1 < \mu$. Applying Hölder's inequality, we have

$$
\begin{aligned}
f_{2p+1}(y) &\leq Ct^{\lambda}\Big[\iint_{Q_t}(r-y)_+^{2p-1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big]^{\mu}\\
&\leq Ct^{\lambda}\Big[\iint_{Q_t} r(r-y)_+^{2p-2}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big]^{\mu}\\
&\leq Ct^{\lambda}\Big[\iint_{Q_t} r(r-y)_+^{2p+1}\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big]^{(2p-2)\mu/(2p+1)}\\
&\quad\times\Big[\int_0^t\int_y^1 r\big|\frac{\partial^3 u}{\partial r^3}\big|^2\,dr\,ds\Big]^{3\mu/(2p+1)}
\end{aligned}
$$

$$\leq Ct^{\lambda}[f_{2p+1}(y)]^{(2p-2)\mu/(2p+1)}[f_0(y)]^{3\mu/(2p+1)}.$$

Therefore,

$$f_{2p+1}(y) \leq Ct^{\lambda/\sigma}[f_0(y)]^{3\mu/(2p+1)\sigma}, \quad \sigma = 1 - \frac{2p-2}{2p+1}\mu > 0.$$

Using Hölder's inequality again, we obtain

$$f_1(y) \leq [f_0(y)]^{2p/2p+1}[f_{2p+1}(y)]^{1/2p+1} \leq Ct^{\gamma}[f_0(y)]^{1+\theta},$$

where

$$\gamma = \frac{\lambda}{(2p+1)\sigma}, \quad \theta = \frac{2\mu}{(2p+1)^2\sigma} - \frac{1}{2p+1} > 0.$$

Noticing that $f_1'(y) = -f_0(y)$, we obtain

$$f_1'(y) \leq -Ct^{-\gamma/(\theta+1)}[f_1(y)]^{1/(\theta+1)}.$$

If $f_1(r_2) = 0$, then supp$u \subset [0, r_2]$. If $f_1(r_2) > 0$, then there exists a maximal interval (r_2, r_2^*) in which $f_1(y) > 0$ and

$$\Big[f_1(y)^{\theta/(\theta+1)}\Big]' = \frac{\theta}{\theta+1}\frac{f_1'(y)}{[f_1(y)]^{1/(\theta+1)}} \leq -Ct^{-\gamma/(\theta+1)}.$$

Integrating the above inequality over (r_2, r_2^*), we have

$$f_1(r_2^*)^{\theta/(\theta+1)} - f_1(r_2)^{\theta/(\theta+1)} \leq -Ct^{-\gamma/(\theta+1)}(r_2^* - r_2),$$

which implies

$$r_2^* \leq r_2 + Ct^{\gamma}(f_0(r_2))^{\theta}.$$

Therefore.

$$\sup \operatorname{supp} u(\cdot, t) \leq r_2 + Ct^{\gamma} \equiv r_2(t).$$

We have thus completed the proof of Theorem 5.1. □

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