Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 113, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PERSISTENCE OF SPREADING SPEED FOR THE DELAYED FISHER EQUATION

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ABSTRACT. This article concerns the long time behavior of the delayed Fisher equation without quasimonotonicity. When the time delay is small and the instantaneous self-limitation effect exists, it is proved that the spreading speed is the same as that of the classical Fisher equation.

1. Introduction

The geographic expansion mode of biological invasion and pathophoresis is one of the most important topics in population dynamics, and much evidence indicates that the mode can be described by the asymptotic spreading. For example, we refer to Lewis et al. [6] and Murray [12, Chapter 1] for the spatial spreading of the grey squirrel in the UK. Mathematically, it has been proved that a feature of the asymptotic spreading of some evolutionary models can be formulated by the asymptotic speed of spreading (for short, spreading speed) which was first introduced by Aronson and Weinberger [1] for the Fisher equation. After that, this concept has been widely studied and some important results have been established for reaction-diffusion equations, lattice differential equations, discrete-time recursions and integral equations, see Diekmann [4], Hsu and Zhao [5], Li et al. [7], Liang and Zhao [8], Lin et al. [10], Thieme and Zhao [14], Weinberger et al. [16], Zhao [18] and the references cited therein. For the sake of convenience, we first show the following definition.

Definition 1.1. Assume that $u(x,t)$ is a nonnegative function for all $x \in \mathbb{R}$, $t > 0$. Then $c_* > 0$ is called the *asymptotic speed of spreading* of $u(x,t)$ if

(a) $\lim_{t\to\infty} \sup_{|x|>(c_*+\epsilon)t} u(x,t) = 0$ for any given $\epsilon > 0$;
(b) $\liminf_{t\to\infty} \inf_{|x|<(c_*-\epsilon)t} u(x,t) > 0$ for any given $\epsilon \in (0, c_*)$.

At the same time, Berestycki et al. in [2] and in [3] presented some examples to illustrate the possible complexity of spreading speed of an unknown function formulated by a scalar equation.

From the viewpoint of monotone dynamical systems, the results mentioned above can be applied to an evolutionary system admitting a proper comparison principle (see Berestycki et al. [2], Berestycki et al. [3], Liang and Zhao [8], Weinberger et

2000 *Mathematics Subject Classification.* 35C07, 35K57, 37C65.
Key words and phrases. Nonmonotone equation; spreading speed; delayed equation.

Submitted April 18, 2012. Published July 5, 2012.

al. [16]) or controlled by two systems generating monotone semiflows (see Hsu and Zhao [5], Li et al. [7]). However, if an equation does not have a comparison principle near the unstable steady state, then these methods fail and the study of asymptotic spreading will be very hard. Recently, Lin [9] considered the spreading speed of the following delayed equation

$$\frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + ru(x,t)\left[1 - u(x,t) - au(x,t-\tau)\right], \tag{1.1}$$

where $u \in \mathbb{R}$, $x \in \mathbb{R}$, $t > 0$, $d > 0$, $r > 0$, $\tau \geq 0$ and $a > 0$ such that the results mentioned above cannot be applied. For $a \in (0,1)$, $u(x,t)$ defined by (1.1) has a spreading speed $2\sqrt{dr}$ if the initial value has nonempty compact support, which is the same as that of the classical Fisher equation (see Lin [9]). Biologically, $a \in (0,1)$ implies that the instantaneous self-limitation effect dominates the corresponding delayed effect.

In this article, we shall further consider the asymptotic spreading of (1.1) if $a \geq 1$ such that the instantaneous self-limitation effect does not dominate the corresponding delayed effect. Using the theory of abstract functional differential equations, some properties of the corresponding initial value problem of (1.1) are proved. In particular, we obtain a positive constant τ_0 such that for each fixed $c < 2\sqrt{dr}$, we prove that

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) > 0$$

when $\tau < \tau_0$ is true and the initial value has nonempty support. At the same time, when $c > 2\sqrt{dr}$ and $\tau \geq 0$, we also confirm that

$$\limsup_{t\to\infty} \sup_{|x|>ct} u(x,t) = 0$$

if (1.1) has an initial value admitting nonempty compact support. Therefore, for small delay, we obtain the persistence of spreading speed (if $\tau = 0$ holds and the initial value has nonempty compact support, then the spreading speed of $u(x,t)$ defined by (1.1) is $2\sqrt{dr}$, see Lemma 2.4).

2. Initial Value Problem

In this section, we present some results on the corresponding initial value problem of (1.1). Let

$$X = \{u(x) : u : \mathbb{R} \to \mathbb{R} \text{ is bounded and uniformly continuous}\}.$$

It is well known that X is a Banach space with respect to the standard supremum norm $|\cdot|$. We also denote

$$X^+ = \{u : u \in X \text{ and } u(x) \geq 0 \text{ for all } x \in \mathbb{R}\}.$$

If $a < b$, then

$$X_{[a,b]} = \{u : a \leq u(x) \leq b \text{ for all } x \in \mathbb{R}\}.$$

For each $t > 0$, define $T(t) : X \to X$ as follows

$$T(t)u(x) = \frac{1}{\sqrt{4\pi dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4dt}} u(y)dy, \quad u(x) \in X.$$

Then $T(t) : X \to X$ is an analytic semigroup [13], and $T(t) : X^+ \to X^+$ is a positive semigroup. To use the theory of abstract functional differential equations, we regard $u(x,t) : \mathbb{R}^+ \to X$, and $u(t) \in X$ implies that $u(x,t) =: (u(t))(x) \in X$. Therefore,

$$T(t)u(s) =: T(t)u(x,s) = \frac{1}{\sqrt{4\pi dt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4dt}} u(y,s)dy, u(s) \in X.$$

We now consider the following Cauchy problem

$$\begin{gathered} \frac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + ru(x,t)\left[1 - u(x,t) - au(x,t-\tau)\right], \\ u(x,s) = \phi(x,s), \quad x \in \mathbb{R},\ t > 0,\ s \in [-\tau, 0], \end{gathered} \tag{2.1}$$

where all the parameters are the same as those in (1.1).

In Smith and Zhao [13], the authors discussed the initial value problem of delayed reaction-diffusion equations by the theory in Martin and Smith [11]. Similar to that in [13, Proof of Theorem 2.2], we give the following existence and uniqueness of the mild solution of (2.1).

Lemma 2.1. *Assume that $\phi : [-\tau, 0] \to X$ is continuous. Then there exists $b \in (0, \infty]$ such that* (2.1) *has a unique mild solution $u : [-\tau, b) \to X$, which can also be formulated by the following integral equation*

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)F(u_s)ds \tag{2.2}$$

with

$$F(u_s) = ru(s)\left[1 - u(s) - au(s-\tau)\right], \quad u(s-\theta) \in X, \theta \in [0, \tau].$$

If $t > \tau$, then $u(x,t)$ is a classical solution satisfying (2.1). *Moreover, if b is bounded, then $|u(t)| \to \infty$ if $t \to b-$.*

Recently, by the condition of quasipositivity and the positivity of $T(t)$, Lin [9] also proved the following result for any $a \geq 0$.

Lemma 2.2. *Assume that $\phi : [-\tau, 0] \to X^+$ is continuous. Then* (2.1) *has a unique mild solution $u : [-\tau, \infty) \to X$ satisfying*

$$u(t) \in X^+ \text{ for each } t \in [-\tau, \infty).$$

Moreover, if $\phi : [-\tau, 0] \to X_{[0,1]}$, then

$$u(t) \in X_{[0,1]} \text{ for each } t \in [-\tau, \infty).$$

Note that the proof of Wang et al. [15, Proposition 4.3] only depends on the boundedness of solutions, so we have the following conclusion by Lemma 2.2.

Lemma 2.3. *Assume that $\phi : [-\tau, 0] \to X_{[0,1]}$ is continuous. Then there exists a constant C independent of τ such that $|\frac{\partial u(x,t)}{\partial t}| < C$ and $|\frac{\partial u(x,t)}{\partial x}| < C$ for all $t \geq 3\tau + 2$.*

Before ending this section, we state some results on the classical Fisher equation (we refer to Aronson and Weinberger [1], Smith and Zhao [13], Ye and Li [17]).

Lemma 2.4. *Assume that d_1, r_1 and M are positive constants. Consider the initial value problem*

$$\begin{gathered} \frac{\partial w(x,t)}{\partial t} = d_1\Delta w(x,t) + r_1 w(x,t)\left[1 - Mw(x,t)\right], \quad x \in \mathbb{R},\ t > 0, \\ w(x,0) = \varphi(x) \in X^+, \quad x \in \mathbb{R}. \end{gathered} \tag{2.3}$$

(A1) *For all* $t > 0$, (2.3) *has a classical solution* $w(\cdot, t) \in X^+$. *If* $\varphi(x)$ *admits nonempty support, then* $w(x,t) > 0$ *for all* $x \in \mathbb{R}, t > 0$.

(A2) *If* $z(\cdot, t) \in X^+$ *with* $t \in (0, b)$ *satisfies*

$$\frac{\partial z(x,t)}{\partial t} \geq (\leq) d_1 \Delta z(x,t) + r_1 z(x,t) \left[1 - Mz(x,t)\right],$$
$$z(x,0) \geq (\leq) \varphi(x),$$

then $z(x,t) \geq (\leq) w(x,t)$ *for all* $x \in \mathbb{R}, t \in (0,b)$.

(A3) *Assume that* $z(t) \in X^+$ *with* $t \in [0, b)$ *satisfies*

$$z(t) \geq (\leq) T(t-s)z(s) + \int_s^t T(t-\theta)[r_1 z(\theta)(1 - Mz(\theta))]d\theta$$

for all $0 \leq s \leq t < b$. *Then* $z(x,t) \geq (\leq) w(x,t)$ *for all* $x \in \mathbb{R}, t \in [0,b)$.

(A4) *If* $\varphi(x)$ *admits nonempty support, then*

$$\liminf_{t\to\infty} \inf_{|c|<ct} w(x,t) = \limsup_{t\to\infty} \sup_{|c|<ct} w(x,t) = 1/M$$

for each $c < 2\sqrt{d_1 r_1}$. *Moreover, if* $\varphi(x)$ *admits nonempty compact support, then*

$$\limsup_{t\to\infty} \sup_{|x|>ct} w(x,t) = 0$$

for each $c > 2\sqrt{d_1 r_1}$.

Remark 2.5. (A2)–(A3) remain true if $M = 0$ or $r_1 < 0$.

3. Main Results

Theorem 3.1. *Assume that* $u(x,t)$ *is the mild solution of* (2.1) *and* $\phi(0) \in X^+$ *admits nonempty support. If* $\tau > 0$ *such that* $aC\tau < 1$, *then*

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) \geq \frac{1 - aC\tau}{1 + a}$$

for each $c < 2\sqrt{dr(1 - aC\tau)}$, *hereafter* C *is given by Lemma 2.3.*

Proof. If $t \geq 4\tau + 3$, then Lemma 2.3 implies that

$$|u(x,t) - u(x,t-\tau)| < C\tau.$$

Thus, we obtain

$$\frac{\partial u(x,t)}{\partial t} \geq d\Delta u(x,t) + ru(x,t)\left[1 - (1+a)u(x,t) - aC\tau\right]. \tag{3.1}$$

On the other hand, Lemma 2.2 and the positivity of semigroup indicate that

$$\begin{aligned} u(t) &= T(t-s)u(s) + \int_s^t T(t-\theta)\left\{ru(\theta)[1 - u(\theta) - au(\theta-\tau)]\right\} d\theta \\ &\geq T(t-s)u(s) + \int_s^t T(t-\theta)[-rau(\theta)]d\theta \end{aligned} \tag{3.2}$$

for any $0 \leq s \leq t < \infty$. Let $v(x,t)$ be defined by

$$\begin{aligned} \frac{\partial v(x,t)}{\partial t} &= d\Delta v(x,t) - rav(x,t), \\ v(x,0) &= \varphi(x) \in X_{[0,1]}. \end{aligned} \tag{3.3}$$

By Remark 2.5, we see that

$$v(x,t) > 0, \quad x \in \mathbb{R}, \ t > 0$$

if $\varphi(x)$ admits nonempty compact support. From Lemma 2.4, we also have

$$u(x,t) > v(x,t) > 0$$

for any $t > 0$. Therefore, $u(x, 4\tau + 3) > 0$ admits nonempty support. Again by Lemma 2.4, the result is evident by (3.1). The proof is complete. ☐

From the above proof and Lemma 2.4, we also have the following result.

Corollary 3.2. *Let $u(x,t)$ be the mild solution of (2.1) and $\delta_1 > 0, \delta_2 > 0$ be constants such that*

$$u(x,0) > \delta_1, |x| \le \delta_2.$$

Then for any $\epsilon < \frac{1-aC\tau}{1+a}$, there exists $T = T(\epsilon, \delta_1, \delta_2)$ such that

$$u(0,t) > \epsilon, t > T.$$

Theorem 3.3. *Assume that $\phi(0) \in X^+$ admits nonempty support and $\tau < \tau_0$. If $u(x,t)$ is the mild solution of (2.1), then*

$$\liminf_{t\to\infty} \inf_{|x|<ct} u(x,t) \ge \frac{1-aC\tau}{1+a}$$

for each fixed $c < 2\sqrt{dr}$.

Proof. In the subsequent proof, we assume that $c < 2\sqrt{dr}$ is a fixed constant. It suffices to consider $t \ge 4\tau + 3$ such that $u(x,t)$ satisfies (2.1) and

$$\Big|\frac{\partial u(x,t-\tau)}{\partial t}\Big| < C, \quad \Big|\frac{\partial u(x,t-\tau)}{\partial x}\Big| < C.$$

Since $c < 2\sqrt{dr}$, there exists $\epsilon' > 0$ such that

$$8\sqrt{dr(1-a\epsilon')} = c + 6\sqrt{dr}.$$

If $u(x,t-\tau) \le \epsilon'$, then

$$\frac{\partial u(x,t)}{\partial t} \ge d\Delta u(x,t) + ru(x,t)\left[1 - u(x,t) - a\epsilon'\right]. \tag{3.4}$$

If $u(x,t-\tau) > \epsilon'$, then Lemma 2.3 and Corollary 3.2 indicate that there exists $T_1 > 0$ such that

$$u(x, t-\tau+T') > \frac{1-aC\tau}{2(1+a)} \text{ for any } T' > T_1.$$

Note that $u(x,s), s \in [t-\tau, t]$ also satisfies (3.2), then Lemma 2.4 leads to

$$u(x,t) > \delta\epsilon' \tag{3.5}$$

with some fixed $\delta > 0$. In particular, the comparison principle (Lemma 2.4) also confirms that both δ and T_1 are uniform for all x, t satisfying

$$u(x,t-\tau) > \epsilon', t \ge 4\tau + 3.$$

From (3.5) and $\epsilon' < u(x,t-\tau) \le 1$, it is evident that

$$\frac{u(x,t-\tau)}{u(x,t)} < \frac{1}{\delta\epsilon'}, t > 4\tau + 3. \tag{3.6}$$

Combining (3.4) with (3.6), we obtain

$$\frac{\partial u(x,t)}{\partial t} \geq d\Delta u(x,t) + ru(x,t)\left[1 - a\epsilon' - Mu(x,t)\right]$$

with $M = (1 + a/(\delta\epsilon')), x \in \mathbb{R}, t > 4\tau + 3$.

Using Lemma 2.4, there exists $T_2 > 0$ such that

$$\inf_{|x|\leq c_1 t} u(x,t) > \frac{1 - a\epsilon'}{2M}, \quad t > T_2, \quad 2c_1 = c + 2\sqrt{dr(1 - a\epsilon')}. \tag{3.7}$$

From Theorem 3.1 and $c < c_1$, we further have

$$\liminf_{t\to\infty} \inf_{|x|\leq ct} u(x,t) \geq \frac{1 - aC\tau}{1 + a}. \tag{3.8}$$

In fact, $u(x,t)$ also satisfies

$$\frac{\partial u(x,t)}{\partial t} \geq d\Delta u(x,t) + ru(x,t)\left[1 - (1+a)u(x,t) - aC\tau\right],$$

then for any $\varepsilon \in (0,1)$, (3.7) implies that there exists $T_3 > 0$ such that

$$\inf_{|x|\leq c_1 t} u(x, t + T_4) \geq \frac{\varepsilon(1 - aC\tau)}{1 + a}$$

for any $t > T_2$ and $T_4 > T_3$. Moreover, if $|x| < c_1 t$ with large t, then there exists $s = c_1 t/c$ such that $|x| < cs$ and $t + T_3 < s$, which further implies that

$$\inf_{|x|\leq cs} u(x,s) \geq \frac{\varepsilon(1 - aC\tau)}{1 + a}.$$

Furthermore, $t \to \infty$ if and only if $s \to \infty$, and we have

$$\liminf_{s\to\infty} \inf_{|x|\leq cs} u(x,s) \geq \frac{\varepsilon(1 - aC\tau)}{1 + a}.$$

By the arbitrary nature of ε, (3.8) holds and the proof is complete. □

Theorem 3.4. *Assume that $\phi(0) \in X^+$ admits nonempty compact support. Then*

$$\limsup_{t\to\infty} \sup_{|x|>ct} u(x,t) = 0$$

for each $c > 2\sqrt{dr}$.

Proof. Lemma 2.2 and the positivity of semigroup indicate that

$$u(t) \leq T(t-s)u(s) + \int_s^t T(t-\theta)[ru(\theta)[1 - u(\theta)]]d\theta$$

for any $0 \leq s \leq t < \infty$. By Lemma 2.4, the result is clear. □

Before ending this article, we make the following remark.

Remark 3.5. In the proof of this paper and that of Lin [9], the existence of the instantaneous self-limitation effect plays an important role. If the effect does not exist, then the discussion will be very difficult even if the time delay is small enough.

Acknowledgements. The author would like to thank the referee for his/her careful reading and useful comments. The work was supported by the Development Program for Outstanding Young Teachers in Lanzhou University of Technology (1010ZCX019).

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