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SOLUTIONS TO SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH WEIGHTED SELF-REFERENCE AND HEREDITY

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Abstract. This article studies the existence of solutions to systems of non-linear integro-differential self-referred and heredity equations. We show the existence of a global solution and the uniqueness of a local solution to a system of integro-differential equations with given initial conditions.

1. Introduction

Self-referred and hereditary phenomena play an important role in applied sciences, especially that in studying evolution processes of biology. Mathematically, these phenomena can be described by the following model: let $A : X \to \mathbb{R}$ and $B : X \to \mathbb{R}$ be two functionals defined on a function space X. Consider the equation

$$Au(x,t) = u(Bu(x,t), t), \tag{1.1}$$

where $u = u(x,t)$, $(x,t) \in \mathbb{R}\times[0,+\infty)$ is an unknown function satisfying some initial data at $t = 0$, A and B are differential or/and integral operators. For example, if

$$Bu(x,t) = \int_0^t u(x,\tau)d\tau, \tag{1.2}$$

then B is called a *hereditary* operator. As the unknown function u in the right-hand side of the equation (1.1) depends on itself, equation (1.1) may be called a self-reference equation.

Some special cases of (1.1) were originally studied by Volterra in the 20 century (see [9, 10] and references therein). It is noticeable to say that some authors considered the variable t as the complex one. In the simple case when B is an identity operator, Eder [4] obtained the existence, uniqueness, analyticity of solutions, and the analytic dependence of solutions of the real-variable equation

$$u'(t) = u(u(t)).$$

Si and Cheng [12] investigated a more general functional-differential equation

$$u'(t) = u(at + bu(t)), \tag{1.3}$$

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where $a \neq 1$ and $b \neq 0$ are complex numbers, and $u : \mathbb{C} \to \mathbb{C}$ is the unknown complex-variable function. In particular, by constructing a convergent power series solution $v(t)$ of a companion equation of the form

$$\beta v'(\beta t) = v'(t)[v(\beta^2 t) - av(\beta t) + a],$$

the authors [12] obtained the analytic solution of (1.3) which is of the form

$$\frac{v(\beta v^{-1}(t)) - at}{b}.$$

As a development of (1.3), Cheng, Si, and Wang [23] studied the equation

$$\alpha t + \beta u'(t) = u\big(at + bu'(t)\big),$$

where α and β are complex numbers. The main results of [23] are the existence theorems for the analytic solutions, and an explicit solution via symmetric methods.

Equations of the form (1.1) attract attention of many authors. More investigations can be found in [13, 14, 15, 16, 17, 18, 7, 2, 5, 8, 3, 9, 10, 11, 6, 20, 21, 1], and references therein.

In recent years, Pascali and Miranda obtained many results concerning the self-referred functional-differential equations [8, 9, 10, 11]. For instance, the authors in [9] studied the initial-value problem

$$\begin{gathered} \frac{\partial}{\partial t} u(x,t) = u\Big(\frac{1}{t}\int_0^t u(x,s)ds, t\Big), \quad x \in \mathbb{R},\ t \in [0,T], \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{gathered} \tag{1.4}$$

The authors claimed that under some suitable conditions problem (1.4) has a unique bounded and continuous solution. Observe that the unknown u in the right-hand side of (1.4) contains a weighted hereditary operator

$$(Bu)(t) := \frac{1}{t}\int_0^t u(x,s)ds.$$

Motivated by the long list of works on self-referred functional-differential equations as mentioned above, we study the following system of two partial-differential equations with self-reference and weighted heredity

$$\begin{gathered} \frac{\partial}{\partial t} u(x,t) = u\Big(f\big(u(x,t)\big) + v\Big(\frac{1}{t}\int_0^t u(x,s)ds + \varphi(u(x,t)), t\Big), t\Big) \\ \frac{\partial}{\partial t} v(x,t) = v\Big(g\big(v(x,t)\big) + u\Big(\frac{1}{t}\int_0^t v(x,s)ds + \psi(v(x,t)), t\Big), t\Big), \end{gathered} \tag{1.5}$$

associated with the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{1.6}$$

where f, g, φ, ψ, u_0 and v_0 are given functions satisfying some suitable conditions. This work is devoted to the uniqueness of local solution, and the existence of a global solution of this problem.

2. Preliminaries

To study problem (1.5)-(1.6), we reduce it to the following system of two integral equations

$$
\begin{aligned}
u(x,t) &= u_0(x) + \int_0^t u\Big(f\Big(u(x,s)\Big) + v\Big(\frac{1}{s}\int_0^s u(x,\tau)d\tau + \varphi(u(x,s)), s\Big), s\Big)ds,\\
v(x,t) &= v_0(x) + \int_0^t v\Big(g\Big(v(x,s)\Big) + u\Big(\frac{1}{s}\int_0^s v(x,\tau)d\tau + \psi(v(x,s)), s\Big), s\Big)ds.
\end{aligned} \tag{2.1}
$$

Proposition 2.1. *If problem* (2.1) *has a solution* (u,v)*, then the pair of functions* (u,v) *solves problem* (1.5)-(1.6).

We omit the proof of this proposition, as it is quite simple. Therefore, we shall investigate problem (2.1) hereafter. We now define the sequences of real functions $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ as follows:

$$
\begin{aligned}
u_1(x,t) &= u_0(x) + \int_0^t u_0\Big(f\Big(u_0(x)\Big) + v_0\Big(u_0(x) + \varphi\big(u_0(x)\big)\Big)\Big)ds,\\
v_1(x,t) &= v_0(x) + \int_0^t v_0\Big(g\Big(v_0(x)\Big) + u_0\Big(v_0(x) + \psi\big(v_0(x)\big)\Big)\Big)ds,\\
u_{n+1}(x,t) &= u_0(x) + \int_0^t u_n\Big(f\Big(u_n(x,s)\Big)\\
&\quad + v_n\Big(\frac{1}{s}\int_0^s u_n(x,\tau)d\tau + \varphi\big(u_n(x,s)\big), s\Big), s\Big)ds,\\
v_{n+1}(x,t) &= v_0(x) + \int_0^t v_n\Big(g\Big(v_n(x,s)\Big)\\
&\quad + u_n\Big(\frac{1}{s}\int_0^s v_n(x,\tau)d\tau + \psi\big(v_n(x,s)\big), s\Big), s\Big)ds,
\end{aligned} \tag{2.2}
$$

for $x \in \mathbb{R}$ and $t > 0$.

We should give the following additional conditions on the functions u_0, v_0 and f, g, φ, ψ:

- (A1) u_0 and v_0 are bounded and Lipschitz continuous on $\mathbb{R}$.
- (A2) f, g, φ and ψ are Lipschitz continuous on $\mathbb{R}$.

The functional inequalities in the next lemma are useful for proving the main results.

Lemma 2.2. *Assume that the functions* u_0, v_0 *and* f, g, φ*, and* ψ *satisfy conditions as in* (A1)–(A2)*. For any* $n \geq 1$ *there exist two continuous, non-negative functions defined on* $\mathbb{R}^+$*, say* $M_n(t)$ *and* $N_n(t)$*, such that the following two inequalities hold:*

$$
\begin{aligned}
|u_{n+1}(x,t) - u_{n+1}(y,t)| &\leq M_{n+1}(t)|x-y|, \quad n \in \mathbb{N},\ x,y \in \mathbb{R}\\
|v_{n+1}(x,t) - v_{n+1}(y,t)| &\leq N_{n+1}(t)|x-y|, \quad n \in \mathbb{N},\ x,y \in \mathbb{R}.
\end{aligned}
$$

Moreover, there is a positive constant T_1 *such that the non-negative function sequences* $\{M_n(t)\}_{n\geq 1}$, $\{N_n(t)\}_{n\geq 1}$ *are uniformly bounded on the interval* $(0, T_1]$*; i.e., there exists a constant* $G_0 > 0$ *such that* $0 < M_n(t), N_n(t) \leq G_0$ *for every* $t \in (0, T_1]$*, and for any* $n \geq 1$.

Proof. For $n = 0$, we have

$$|u_0(x) - u_0(y)| \le M_0|x - y|, \quad |v_0(x) - v_0(y)| \le N_0|x - y|$$

for any $x, y \in \mathbb{R}$ and for some $M_0 > 0, N_0 > 0$. Let $P, Q, \varpi, \sigma > 0$ such that

$$\begin{aligned}
|f(\alpha_1) - f(\alpha_2)| &\le P|\alpha_1 - \alpha_2|, \quad \alpha_1, \alpha_2 \in \mathbb{R} \\
|g(\beta_1) - g(\beta_2)| &\le Q|\beta_1 - \beta_2|, \quad \beta_1, \beta_2 \in \mathbb{R} \\
|\varphi(\gamma_1) - \varphi(\gamma_2)| &\le \varpi|\gamma_1 - \gamma_2|, \quad \gamma_1, \gamma_2 \in \mathbb{R} \\
|\psi(\eta_1) - \psi(\eta_2)| &\le \sigma|\eta_1 - \eta_2|, \quad \eta_1, \eta_2 \in \mathbb{R}.
\end{aligned} \tag{2.3}$$

For $n = 1$ we have

$$|u_1(x, t) - u_1(y, t)| \le M_1(t)|x - y|,$$

where

$$M_1(t) = M_0 + t(M_0^2 P + M_0^2 N_0 + M_0^2 N_0 \varpi),$$

and

$$|v_1(x, t) - v_1(y, t)| \le N_1(t)|x - y|,$$

where

$$N_1(t) = N_0 + t(N_0^2 Q + M_0 N_0^2 + M_0 N_0^2 \sigma).$$

For $n = 2$, we derive

$$|u_2(x, t) - u_2(y, t)| \le M_2(t)|x - y|,$$

where

$$M_2(t) = M_0 + \int_0^t \Big(M_1^2(s)P + N_1(s)\frac{1}{2s}\frac{d}{ds}\Big(\int_0^s M_1(\tau)d\tau\Big)^2 + M_1^2(s)N_1(s)\varpi\Big)ds,$$

and

$$|v_2(x, t) - v_2(y, t)| \le N_2(t)|x - y|,$$

where

$$N_2(t) = N_0 + \int_0^t \Big(N_1^2(s)Q + M_1(s)\frac{1}{2s}\frac{d}{ds}\Big(\int_0^s N_1(\tau)d\tau\Big)^2 + M_1(s)N_1^2(s)\sigma\Big)ds.$$

We can inductively prove that

$$|u_{n+1}(x, t) - u_{n+1}(y, t)| \le M_{n+1}(t)|x - y|, \tag{2.4}$$

where

$$M_{n+1}(t) = M_0 + \int_0^t \Big(M_n^2(s)P + N_n(s)\frac{1}{2s}\frac{d}{ds}\Big(\int_0^s M_n(\tau)d\tau\Big)^2 + M_n^2(s)N_n(s)\varpi\Big)ds,$$

and

$$|v_{n+1}(x, t) - v_{n+1}(y, t)| \le N_{n+1}(t)|x - y|, \tag{2.5}$$

where

$$N_{n+1}(t) = N_0 + \int_0^t \Big(N_n^2(s)Q + M_n(s)\frac{1}{2s}\frac{d}{ds}\Big(\int_0^s N_n(\tau)d\tau\Big)^2 + M_n(s)N_n^2(s)\sigma\Big)ds.$$

Clearly, the functions $M_{n+1}(t)$ and $N_{n+1}(t)$ are non-negative and continuous on $\mathbb{R}$. We shall prove that each one of the function sequences $\{M_{n+1}\}_{n\ge1}(t)$ and $\{N_{n+1}\}_{n\ge1}(t)$ is uniformly bounded on some $(0, T_1]$. Indeed, by choosing constants K_0, H_0, and $I_0 > 0$ fulfilling the following conditions

$$N_0 + K_0 \le H_0 \quad M_0 + K_0 \le I_0 \quad G_0 = \max\{H_0, I_0\},$$

there exists a number $T_1 > 0$ such that

$$\begin{gathered} (M_0^2 P + M_0^2 N_0 + M_0^2 N_0 \varpi)t \le K_0, \quad \forall t \in (0, T_1] \\ (N_0^2 Q + M_0 N_0^2 + M_0 N_0^2 \sigma)t \le K_0, \quad \forall t \in (0, T_1] \\ (G_0^2 P + G_0^3 + G_0^3 \varpi)t \le K_0, \\ (G_0^2 Q + G_0^3 + G_0^3 \sigma)t \le K_0. \end{gathered} \tag{2.6}$$

Then

$$\begin{gathered} 0 \le M_1(t) - M_0 = (M_0^2 P + M_0^2 N_0 + M_0^2 N_0 \varpi)t \le K_0, \\ 0 \le N_1(t) - N_0 = (N_0^2 Q + M_0 N_0^2 + M_0 N_0^2 \sigma)t \le K_0. \end{gathered} \tag{2.7}$$

It follows that

$$\begin{gathered} 0 \le M_1(t) \le K_0 + M_0 \le I_0 \le G_0, \\ 0 \le N_1(t) \le K_0 + N_0 \le H_0 \le G_0. \end{gathered} \tag{2.8}$$

Similarly,

$$\begin{gathered} 0 \le M_2(t) - M_0 \le \int_0^t (G_0^2 P + G_0^3 + G_0^3 \varpi) ds = (G_0^2 P + G_0^3 + G_0^3 \varpi)t \le K_0, \\ 0 \le N_2(t) - N_0 \le \int_0^t (G_0^2 Q + G_0^3 + G_0^3 \sigma) ds = (G_0^2 Q + G_0^3 + G_0^3 \sigma)t \le K_0. \end{gathered} \tag{2.9}$$

From these inequalities, we have

$$\begin{gathered} 0 \le M_2(t) \le M_0 + K_0 \le I_0 \le G_0, \\ 0 \le N_2(t) \le N_0 + K_0 \le H_0 \le G_0. \end{gathered} \tag{2.10}$$

By induction on n we obtain

$$\begin{gathered} 0 \le M_{n+1}(t) \le M_0 + K_0 \le G_0, \\ 0 \le N_{n+1}(t) \le N_0 + K_0 \le G_0, \end{gathered} \tag{2.11}$$

for every $t \in (0, T_1]$, $T_1 > 0$. The lemma is proved. □

We can see that Lemma 2.2 concerns the properties of the functions $\{u_n(x,t)\}$ and $\{v_n(x,t)\}$, while Lemma 2.3 concerns the recursive sequences $\{u_{n+1}(x,t) - u_n(x,t)\}$ and $\{v_{n+1}(x,t) - v_n(x,t)\}$.

Lemma 2.3. *Assume that the functions u_0, v_0 and f, g, φ, and ψ satisfy conditions as in* (A1)–(A2). *For any $n \ge 1$ there exist two nonnegative, continuous functions, say $A_n(t)$ and $B_n(t)$, satisfying the following two inequalities:*

$$\begin{gathered} |u_{n+1}(x,t) - u_n(x,t)| \le A_{n+1}(t), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ |v_{n+1}(x,t) - v_n(x,t)| \le B_{n+1}(t), \quad x \in \mathbb{R},\ t \in \mathbb{R}^+. \end{gathered}$$

Moreover, there is a positive constant T_2 such that the both series with general terms $A_n(t)$, and $B_n(t)$ are uniformly convergent on $(0, T_2]$.

Proof. We have

$$\begin{gathered} |u_1(x,t) - u_0(x)| \le t\|u_0\|_{L^\infty} := A_1(t), \\ |v_1(x,t) - v_0(x)| \le t\|v_0\|_{L^\infty} := B_1(t). \end{gathered}$$

Similarly,

$$|u_2(x,t) - u_1(x,t)| \le \int_0^t \Big(A_1(s)\Big(1 + M_0 P + M_0 N_0 \varpi\Big) + M_0 B_1(s)$$

$$+ M_0 N_0 \frac{1}{s}\int_0^s A_1(\tau)d\tau\Big)ds := A_2(t),$$

and

$$\begin{aligned}|v_2(x,t)-v_1(x,t)| \le \int_0^t \Big(B_1(s)\Big(1+N_0Q+M_0N_0\sigma\Big)+N_0A_1(s)\\ + M_0N_0\frac{1}{s}\int_0^s B_1(\tau)d\tau\Big)ds := B_2(t).\end{aligned}$$

By induction on n, we conclude that

$$|u_{n+1}(x,t)-u_n(x,t)| \le A_{n+1}(t), \tag{2.12}$$

where

$$\begin{aligned}A_{n+1}(t) = \int_0^t \Big(A_n(s)\Big(1+M_{n-1}(s)P+M_{n-1}(s)N_{n-1}(s)\varpi\Big)\\ + B_n(s)M_{n-1}(s)+M_{n-1}(s)N_{n-1}(s)\frac{1}{s}\int_0^s A_n(\tau)d\tau\Big)ds;\end{aligned}$$

and

$$|v_{n+1}(x,t)-v_n(x,t)| \le B_{n+1}(t), \tag{2.13}$$

where

$$\begin{aligned}B_{n+1}(t) = \int_0^t \Big(B_n(s)\Big(1+N_{n-1}(s)Q+M_{n-1}(s)N_{n-1}(s)\sigma\Big)\\ + A_n(s)N_{n-1}(s)+M_{n-1}(s)N_{n-1}(s)\frac{1}{s}\int_0^s B_n(\tau)d\tau\Big)ds.\end{aligned}$$

For a number $h \in (0,1/2)$, we can choose $T_2 > 0$ such that the following two inequalities hold for any $t \in (0,T_2]$,

$$\begin{aligned}(1+G_0P+G_0+G_0^2\varpi+G_0^2)t \le h < \frac{1}{2},\\ (1+G_0Q+G_0+G_0^2\sigma+G_0^2)t \le h < \frac{1}{2},\end{aligned} \tag{2.14}$$

By (2.14) and Lemma 2.2,

$$\begin{aligned}0 \le A_{n+1}(t) &\le (1+G_0P+G_0^2\varpi+G_0^2)t\|A_n\|_{L^\infty}+G_0t\|B_n\|_{L^\infty}\\ &\le h(\|A_n\|_{L^\infty}+\|B_n\|_{L^\infty}),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}0 \le B_{n+1}(t) &\le (1+G_0Q+G_0^2\sigma+G_0^2)t\|B_n\|_{L^\infty}+G_0t\|A_n\|_{L^\infty}\\ &\le h(\|A_n\|_{L^\infty}+\|B_n\|_{L^\infty}).\end{aligned} \tag{2.16}$$

By induction on n, we obtain

$$0 \le A_{n+1}(t), B_{n+1}(t) \le h^n\Big(\|A_1\|_\infty+\|B_1\|_\infty\Big),$$

for $t \in (0,T_2]$. Therefore, the series with general terms $A_n(.)$ and $B_n(.)$ uniformly converge on the interval $(0,T_2]$. Lemma 2.3 is proved. □

Remark 2.4. It is easy to prove inductively that

$$|u_{n+1}(x,t)| \le e^t \|u_0\|_\infty, \quad |v_{n+1}(x,t)| \le e^t \|v_0\|_\infty.$$

If we consider T such that $0 < T \le \min\{T_1, T_2\}$, the functions $u_n(x,t)$, $v_n(x,t)$ are bounded uniformly with respect to variable $x \in \mathbb{R}$, for $t \in (0,T]$. On the other hand, due to (2.8) and Lemma 2.2, the functions $u_n(x,t)$, $v_n(x,t)$ are uniformly Lipschitz continuous with respect to each of the variables $x \in \mathbb{R}$ and $t \in (0,T]$.

For serving the existence of a global solution to problem (2.1), we propose some assumptions on the functions u_0, v_0 and f, g, φ, ψ, that are different from (A1)–(A2) in Lemma 2.3. Namely, assume that

- (B1) u_0 and v_0 are non-negative, non-decreasing, bounded and lower semi-continuous on $\mathbb{R}$.
- (B2) f, g, φ and ψ are non-decreasing and lower semi-continuous.

Lemma 2.5. *Suppose that the functions u_0, v_0 and f, g, φ and ψ fulfill the conditions as in* (B1)–(B2). *Then the functions $\{u_n(x,t)\}_{n\ge1}$ and $\{v_n(x,t)\}_{n\ge1}$ possess the following properties:*

- (C1) *u_n and v_n are non-negative.*
- (C2) *u_n and v_n are non-decreasing with respect to each one of variables $x \in \mathbb{R}$, $t \in (0,T]$; more precisely $u_{n+1} \ge u_n, v_{n+1} \ge v_n$.*
- (C3) *u_n and v_n are lower semi-continuous with respect to x, for every $t \in (0,+\infty)$.*
- (C4) *u_n and v_n are Lipschitz continuous with respect to t, uniformly bounded with respect to $x \in \mathbb{R}$.*

Proof. We have

$$\begin{aligned} u_1(x,t) &\ge u_0(x) \ge 0, \quad \forall x \in \mathbb{R},\ t \in (0,+\infty), \\ v_1(x,t) &\ge v_0(x) \ge 0, \quad \forall x \in \mathbb{R},\ t \in (0,+\infty). \end{aligned} \tag{2.17}$$

For $t_1, t_2 \in (0,+\infty)$, $t_1 < t_2$, and for $x \in \mathbb{R}$, we have

$$\begin{aligned} u_1(x,t_2) &= u_0(x) + \int_0^{t_2} u_0\Big(f\Big(u_0(x)\Big) + v_0\Big(u_0(x) + \varphi\Big(u_0(x)\Big)\Big)\Big) ds \\ &\ge u_0(x) + \int_0^{t_1} u_0\Big(f\Big(u_0(x)\Big) + v_0\Big(u_0(x) + \varphi\Big(u_0(x)\Big)\Big)\Big) ds \\ &= u_1(x,t_1), \end{aligned} \tag{2.18}$$

$$\begin{aligned} v_1(x,t_2) &= v_0(x) + \int_0^{t_2} v_0\Big(g\Big(v_0(x)\Big) + u_0\Big(v_0(x) + \psi\Big(v_0(x)\Big)\Big)\Big) ds \\ &\ge v_0(x) + \int_0^{t_1} v_0\Big(g\Big(v_0(x)\Big) + u_0\Big(v_0(x) + \psi\Big(v_0(x)\Big)\Big)\Big) ds \\ &= v_1(x,t_1). \end{aligned} \tag{2.19}$$

Similarly, for all $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, for all $t \in (0,+\infty)$, we derive

$$\begin{aligned} u_1(x_1,t) &= u_0(x_1) + \int_0^{t} u_0\Big(f\big(u_0(x_1)\big) + v_0\big(u_0(x_1) + \varphi\big(u_0(x_1)\big)\big)\Big) ds \\ &\le u_0(x_2) + \int_0^{t} u_0\Big(f\big(u_0(x_2)\big) + v_0\big(u_0(x_2) + \varphi\big(u_0(x_2)\big)\big)\Big) ds \\ &= u_1(x_2,t), \end{aligned} \tag{2.20}$$

$$\begin{aligned} v_1(x_1,t) &= v_0(x_1) + \int_0^t v_0\Big(g\big(v_0(x_1)\big) + u_0\big(v_0(x_1) + \psi\big(v_0(x_1)\big)\big)\Big)ds \\ &\le v_0(x_2) + \int_0^t v_0\Big(g\big(v_0(x_2)\big) + u_0\big(v_0(x_2) + \psi\big(v_0(x_2)\big)\big)\Big)ds \\ &= v_1(x_2,t). \end{aligned} \tag{2.21}$$

Using (2.18)–(2.21), we can prove inductively that

$$\begin{gathered} u_n(x,t_2) \ge u_n(x,t_1), \quad \forall x \in \mathbb{R},\ t_2 > t_1, \\ u_n(x_2,t) \ge u_n(x_1,t), \quad \forall t \in (0,+\infty),\ x_2 > x_1, \\ v_n(x,t_2) \ge v_n(x,t_1), \quad \forall x \in \mathbb{R},\ t_2 > t_1, \\ v_n(x_2,t) \ge v_n(x_1,t), \quad \forall t \in (0,+\infty),\ x_2 > x_1. \end{gathered} \tag{2.22}$$

Also, we can prove that (see also remark 2.4)

$$\begin{gathered} 0 \le u_n(x,t) \le u_{n+1}(x,t) \le e^T \|u_0\|_{L^\infty}, \\ 0 \le v_n(x,t) \le v_{n+1}(x,t) \le e^T \|v_0\|_{L^\infty}, \end{gathered} \tag{2.23}$$

for all $x \in \mathbb{R}$, $t \in (0,T]$ and $n \in \mathbb{N}$. On the other hand,

$$|u_{n+1}(x,t_1) - u_{n+1}(x,t_2) \le \Big|\int_{t_1}^{t_2} \|u_0\|_{L^\infty} e^T ds\Big| \le \|u_0\|_{L^\infty} e^T |t_2 - t_1|, \tag{2.24}$$

$$|v_{n+1}(x,t_1) - v_{n+1}(x,t_2)| \le \Big|\int_{t_1}^{t_2} \|v_0\|_{L^\infty} e^T ds\Big| \le \|v_0\|_{L^\infty} e^T |t_2 - t_1|. \tag{2.25}$$

Relations (2.24) and (2.25) ensure that u_n and v_n satisfy (C_4). Since the sequences (u_n) and (v_n) are non decreasing, above and upper bounded, there exist the limits

$$u_\infty(x,t) = \lim_n u_n(x,t), \quad v_\infty(x,t) = \lim_n v_n(x,t). \tag{2.26}$$

Since u_0, v_0, f, g, φ and ψ are lower semi-continuous and non-decreasing, the functions $f(u_0), g(v_0), v_0\big(u_0 + \varphi(u_0)\big)$, and $u_0\big(v_0 + \psi(v_0)\big)$ are lower semi-continuous and non-decreasing (see [21, Lemma 3]). Hence, $u_0\big(f(u_0) + v_0\big(u_0 + \varphi(u_0)\big)\big)$, and $v_0\big(g(v_0)+u_0\big(v_0+\psi(v_0)\big)\big)$ are lower semi-continuous and non-decreasing, too. Thus, the lower semi-continuity and the decrease of $u_1(x,t)$ and $v_1(x,t)$ are established. By induction on n, we can conclude that $u_n(x,t)$ and $v_n(x,t)$ are lower semi-continuous and non-decreasing. Lemma 2.5 is proved. □

3. Main results

Theorem 3.1 (Uniqueness of local solutions). *Assume that the functions f, g, φ, ψ, u_0, and v_0 satisfy* (A1)–(A2). *Then there exists a positive constant $T_\star$ such that* (2.1) *has a unique solution on $R \times (0, T_*]$ denoted by $\{u_*, v_*\}$. Moreover, the functions u_∞, v_∞ are Lipschitz continuous and bounded with respect to each of the variables $x \in \mathbb{R}$, and $t \in (0, T_\star]$.*

Theorem 3.2 (Existence of global solutions). *Assume that f, g, φ, ψ, u_0 and v_0 satisfy* (B1)–(B2). *There exist two functions $u_\infty, v_\infty : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ satisfying* (2.1) *for $t \in (0,+\infty)$. Moreover, these solutions have the properties similar to those of $\{u_n(x,t)\}_{n\ge1}$ and $\{v_n(x,t)\}_{n\ge1}$ as in Lemma 2.5; namely, the functions u_∞, v_∞ possess the properties* (C1)–(C4).

Proof of Theorem 3.1. Write $T_* := \min\{T_1, T_2\}$. By Lemmas 2.2, and 2.3, the limits $u_\infty(x,t)$, $v_\infty(x,t)$ of the sequences $\{u_n(x,t)\}_{n\geq 1}$, $\{v_n(x,t)\}_{n\geq 1}$ are bounded on $\mathbb{R}\times(0,T_*]$, Lipschitz continuous with respect to each of variables, and satisfy problem (2.1).

Now, suppose that $(u_\star, v_\star)$ is another solution of (2.1) on $\mathbb{R}\times(0,T_*]$ with the same given data. We have

$$\begin{aligned}
&\Big|u_\star\Big(f(u_\star(x,t)) + v_\star\Big(\frac{1}{t}\int_0^t u_\star(x,s)ds + \varphi(u_\star(x,t)), t\Big), t\Big)\\
&- u_\infty\Big(f(u_\infty(x,t)) + v_\infty\Big(\frac{1}{t}\int_0^t u_\infty(x,s)ds + \varphi(u_\infty(x,t)), t\Big), t\Big)\Big|\\
&\leq \|u_\star - u_\infty\|_{L^\infty} + \Big|u_\infty\Big(f(u_\star(x,t)) + v_\star\Big(\frac{1}{t}\int_0^t u_\star(x,s)ds + \varphi(u_\star(x,t)), t\Big), t\Big)\\
&\quad - u_\infty\Big(f(u_\infty(x,t)) + v_\infty\Big(\frac{1}{t}\int_0^t u_\infty(x,s)ds + \varphi(u_\infty(x,t)), t\Big), t\Big)\Big|\\
&\leq (1 + M_\infty(t)P + M_\infty(t)N_\infty(t) + M_\infty(t)N_\infty(t)\varpi)\|u_\star - u_\infty\|_{L^\infty}\\
&\quad + M_\infty(t)\|v_\star - v_\infty\|_{L^\infty}.
\end{aligned}$$

From the above inequality and Lemma (2.2) we obtain

$$\begin{aligned}
&|u_\star(x,t) - u_\infty(x,t)|\\
&\leq \Big(1 + G_0P + G_0^2 + G_0^2\varpi\Big)t\|u_\star - u_\infty\|_{L^\infty} + G_0t\|v_\star - v_\infty\|_{L^\infty}.
\end{aligned} \tag{3.1}$$

In addition, we have

$$\begin{aligned}
&\Big|v_\star\Big(g(u_\star, v_\star) + u_\star\Big(\frac{1}{t}\int_0^t v_\star(x,s)ds + \psi(v_\star(x,t)), t\Big), t\Big)\\
&- v_\infty\Big(g(u_\infty, v_\infty) + u_\infty\Big(\frac{1}{t}\int_0^t v_\infty(x,s)ds + \psi(v_\infty(x,t)), t\Big), t\Big)\Big|\\
&\leq \Big(1 + N_\infty(t)Q + M_\infty(t)N_\infty(t)\sigma\Big)\|v_\star - v_\infty\|_{L^\infty} + N_\infty(t)\|u_\star - u_\infty\|_{L^\infty}.
\end{aligned} \tag{3.2}$$

By (3.2) and Lemma (2.2), we find

$$\begin{aligned}
&|v_\star(x,t) - v_\infty(x,t)|\\
&\leq \Big(1 + G_0Q + G_0^2 + G_0^2\sigma\Big)t\|v_\star - v_\infty\|_{L^\infty} + G_0t\|u_\star - u_\infty\|_{L^\infty}.
\end{aligned} \tag{3.3}$$

Combining (3.1) and (3.3), we obtain

$$\begin{aligned}
&|u_\star(x,t) - u_\infty(x,t)|\\
&\leq \Big(1 + G_0P + G_0 + G_0^2 + G_0^2\varpi\Big)t\max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\},
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&|v_\star(x,t) - v_\infty(x,t)|\\
&\leq \Big(1 + G_0Q + G_0 + G_0^2 + G_0^2\sigma\Big)t\max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\}.
\end{aligned} \tag{3.5}$$

Taking account of (2.14), (3.4) and (3.5), we have

$$\begin{aligned}
|u_\star(x,t) - u_\infty(x,t)| &\leq h\max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\},\\
|v_\star(x,t) - v_\infty(x,t)| &\leq h\max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\},
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_0]$, $x \in \mathbb{R}$. Finally, we conclude that

$$\max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\} \le h \max\{\|u_\star - u_\infty\|_{L^\infty}, \|v_\star - v_\infty\|_{L^\infty}\}.$$

The last inequality ensures the uniqueness of the solution. Theorem 3.1 is proved. □

Proof of Theorem 3.2. Thanks to (2.22) and (2.23), the following two limits exit:

$$u_\infty(x,t) = \sup_n u_n(x,t), \quad v_\infty(x,t) = \sup_n v_n(x,t). \tag{3.7}$$

We shall prove that $u_\infty(x,t), v_\infty(x,t)$ satisfy (2.1). From (3.7) we have

$$\begin{aligned} &u_{n+1}(x,t) - u_0(x) \\ &= \int_0^t u_n\Big(f\Big(u_n(x,s)\Big) + v_n\Big(\frac{1}{s}\int_0^s u_n(x,\tau)d\tau + \varphi\big(u_n(x,s)\big), s\Big), s\Big)ds \\ &\le \int_0^t u_\infty\Big(f\Big(u_\infty(x,s)\Big) + v_\infty\Big(\frac{1}{s}\int_0^s u_\infty(x,\tau)d\tau + \varphi\big(u_\infty(x,s)\big), s\Big), s\Big)ds, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &v_{n+1}(x,t) - v_0(x) \\ &= \int_0^t v_n\Big(g\Big(v_n(x,s)\Big) + v_n\Big(\frac{1}{s}\int_0^s v_n(x,\tau)d\tau + \psi\big(v_n(x,s)\big), s\Big), s\Big)ds \\ &\le \int_0^t v_\infty\Big(g\Big(v_\infty(x,s)\Big) + u_\infty\Big(\frac{1}{s}\int_0^s v_\infty(x,\tau)d\tau + \psi\big(v_\infty(x,s)\big), s\Big), s\Big)ds. \end{aligned} \tag{3.9}$$

As $u_n(x,t)$, and $v_n(x,t)$ are non-decreasing, we have

$$u_{n+p}\Big(f\Big(u_{n+p}(x,t)\Big) + v_{n+p}\Big(\frac{1}{t}\int_0^t u_{n+p}(x,s)ds + \varphi\big(u_{n+p}(x,t)\big), t\Big), t\Big) \tag{3.10}$$

$$\ge u_n\Big(f\Big(u_{n+p}(x,t)\Big) + v_{n+p}\Big(\frac{1}{t}\int_0^t u_{n+p}(x,s)ds + \varphi\big(u_{n+p}(x,t)\big), t\Big), t\Big), \tag{3.11}$$

and

$$\begin{aligned} &v_{n+p}\Big(g\Big(v_{n+p}(x,t)\Big) + u_{n+p}\Big(\frac{1}{t}\int_0^t v_{n+p}(x,s)ds + \psi\big(v_{n+p}(x,t)\big), t\Big), t\Big) \\ &\ge v_n\Big(g\Big(v_{n+p}(x,t)\Big) + u_{n+p}\Big(\frac{1}{t}\int_0^t v_{n+p}(x,s)ds + \psi\big(v_{n+p}(x,t)\big), t\Big), t\Big). \end{aligned} \tag{3.12}$$

From (3.10) and (3.12) we deduce

$$\begin{aligned} &\lim_{p\to\infty} \int_0^t u_{n+p}\Big(f\Big(u_{n+p}(x,s)\Big) \\ &\quad + v_{n+p}\Big(\frac{1}{s}\int_0^s u_{n+p}(x,\tau)d\tau + \varphi\big(u_{n+p}(x,s)\big), s\Big), s\Big)ds \\ &\ge \int_0^t u_n\Big(f\Big(u_\infty(x,s)\Big) + v_\infty\Big(\frac{1}{s}\int_0^s u_\infty(x,\tau)d\tau + \varphi\big(u_\infty(x,s)\big), s\Big), s\Big)ds, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\lim_{p\to\infty}\int_0^t v_{n+p}\Big(g\Big(v_{n+p}(x,s)\Big)\\
&\quad + u_{n+p}\Big(\frac{1}{s}\int_0^s v_{n+p}(x,\tau)d\tau+\psi\big(v_{n+p}(x,s)\big),s\Big),s\Big)ds \\
&\ge \int_0^t v_n\Big(g\Big(v_\infty(x,s)\Big)+u_\infty\Big(\frac{1}{s}\int_0^s v_\infty(x,\tau)d\tau+\psi\big(v_\infty(x,s)\big),s\Big),s\Big)ds.
\end{aligned}\tag{3.14}$$

Hence,

$$\begin{aligned}
&\lim_p[u_{n+p+1}(x,t)-u_0(x)]\\
&=\lim_p\int_0^t u_{n+p}\Big(f\Big(u_{n+p}(x,s)\Big)\\
&\quad + v_{n+p}\Big(\frac{1}{s}\int_0^s u_{n+p}(x,\tau)d\tau+\varphi\big(u_{n+p}(x,s)\big),s\Big),s\Big)ds \\
&\ge \int_0^t u_n\Big(f\Big(u_\infty(x,s)\Big)+v_\infty\Big(\frac{1}{s}\int_0^s u_\infty(x,\tau)d\tau+\varphi\big(u_\infty(x,s)\big),s\Big),s\Big)ds,
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
&\lim_p[v_{n+p+1}(x,t)-v_0(x)]\\
&=\lim_p\int_0^t v_{n+p}\Big(g\Big(v_{n+p}(x,s)\Big)\\
&\quad + u_{n+p}\Big(\frac{1}{s}\int_0^s v_{n+p}(x,\tau)d\tau+\psi\big(v_{n+p}(x,s)\big),s\Big),s\Big)ds \\
&\ge \int_0^t v_n\Big(g\Big(v_\infty(x,s)\Big)+u_\infty\Big(\frac{1}{s}\int_0^s v_\infty(x,\tau)d\tau+\psi\big(v_\infty(x,s)\big),s\Big),s\Big)ds.
\end{aligned}\tag{3.16}$$

By (3.15)-(3.16) we find that

$$\begin{aligned}
u_\infty(x,t)-u_0(x)\ge&\int_0^t u_\infty\Big(f\Big(u_\infty(x,s)\Big)\\
&+v_\infty\Big(\frac{1}{s}\int_0^s u_\infty(x,\tau)d\tau+\varphi\big(u_\infty(x,s)\big),s\Big),s\Big)ds,
\end{aligned}\tag{3.17}$$

and

$$\begin{aligned}
v_\infty(x,t)-v_0(x)\ge&\int_0^t v_\infty\Big(g\Big(v_\infty(x,s)\Big)\\
&+u_\infty\Big(\frac{1}{s}\int_0^s v_\infty(x,\tau)d\tau+\psi\big(v_\infty(x,s)\big),s\Big),s\Big)ds.
\end{aligned}\tag{3.18}$$

Combining (3.8)-(3.9) and (3.17)-(3.18) we obtain

$$\begin{aligned}
u_\infty(x,t)-u_0(x)=&\int_0^t u_\infty\Big(f\Big(u_\infty(x,s)\Big)\\
&+v_\infty\Big(\frac{1}{s}\int_0^s u_\infty(x,\tau)d\tau+\varphi\big(u_\infty(x,s)\big),s\Big),s\Big)ds,
\end{aligned}\tag{3.19}$$

and

$$\begin{aligned} v_\infty(x,t) - v_0(x) = \int_0^t v_\infty\Big(v_\infty(x,s)\Big) \\ + u_\infty\Big(\frac{1}{s}\int_0^s v_\infty(x,\tau)d\tau + \psi\big(v_\infty(x,s)\big), s\Big), s\Big)ds. \end{aligned} \tag{3.20}$$

The above equalities imply that (u_∞, v_∞) is a solution of (2.1).

On the other hand, it is easily seen that u_∞, v_∞ are Lipschitz continuous in t on $(0, +\infty)$. The proof is complete. □

4. Illustrative Example

Consider the initial-value problem for a system of integro-differential equations (1.5)-(1.6) with the following data:

$$u_0(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$v_0(x) = 1 \quad \text{for all } x \in \mathbb{R},$$

$$f(u) = u, \quad g(v) = v, \quad \varphi(u) = \psi(v) = 0.$$

We compute the successive approximations as follows:

$$\begin{aligned} u_1(x,t) &= u_0(x) + \int_0^t u_0\Big(f(u_0(x)) + v_0(u_0(x) + \varphi(u_0(x)))\Big)ds \\ &= u_0(x) + \int_0^t u_0(u_0(x)+1)ds = u_0(x) + \int_0^t 0ds = u_0(x), \\ v_1(x,t) &= v_0(x) + \int_0^t v_0\Big(g(v_0(x)) + u_0(v_0(x) + \psi(v_0(x)))\Big)ds \\ &= 1 + \int_0^t 1ds = 1 + t. \end{aligned} \tag{4.1}$$

Similarly, $u_2(x,t) = u_0(x)$, $v_2(x,t) = 1 + t + (t^2/2)$. Suppose that

$$u_n(x,t) = u_0(x), \quad v_n(x,t) = \sum_{i=0}^n \frac{t^i}{i!}. \tag{4.2}$$

We can prove inductively that

$$u_{n+1}(x,t) = u_0(x), \quad v_{n+1}(x,t) = \sum_{i=0}^{n+1} \frac{t^i}{i!}.$$

Tending n to infinity we obtain

$$u_\star(x,t) = u_0(x), \quad v_\star(x,t) = e^t. \tag{4.3}$$

In fact, we can choose $u_0(x)$ as a nonnegative, Lipschitz continuous function having a compact support, and $v_0(x) = c$ as a constant function. Due to the symmetry of the system, the functions u_0 and v_0 are interchangeable.

Concluding remarks. Mathematically, one can provide acceptable assumptions on equations, and add suitable restrictions on initial data of problems so that the solution exists uniquely. Therefore, both the existence and uniqueness of solutions of self-referred and heredity problems, in general, remain considerable challenges to attempts at generalization, namely (see also [4, 8, 9, 10, 11]),

(1) The uniqueness/non-uniqueness of global solutions, with also relaxed condition on data.
(2) Structure of the solution set.
(3) Numerical solution for the above mentioned system.

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