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EXISTENCE AND UNIQUENESS OF WEAK AND ENTROPY SOLUTIONS FOR HOMOGENEOUS NEUMANN BOUNDARY-VALUE PROBLEMS INVOLVING VARIABLE EXPONENTS

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Abstract. In this article we study the nonlinear homogeneous Neumann boundary-value problem

$$\begin{gathered} b(u) - \operatorname{div} a(x, \nabla u) = f \quad \text{in } \Omega \\ a(x, \nabla u).\eta = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N \geq 3$ and η the outer unit normal vector on $\partial\Omega$. We prove the existence and uniqueness of a weak solution for $f \in L^\infty(\Omega)$ and the existence and uniqueness of an entropy solution for L^1-data f. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

1. Introduction

The paper is motivated by phenomena which are described by a homogeneous Neumann boundary value problem of the type

$$\begin{gathered} b(u) - \operatorname{div} a(x, \nabla u) = f \quad \text{in } \Omega, \\ a(x, \nabla u).\eta = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N \geq 3$ and η the outer unit normal vector on $\partial\Omega$.

The study of problems involving variable exponents has received considerable attention in recent years (see [5], [7]-[17], [19]-[23], [26]-[30], [33]-[36]) due to the fact that they can model various phenomena which arise in the study of elastic mechanics (see [4]), electrorheological fluids (see [11, 22, 28, 29]) or image restauration (see [9]).

When the boundary value condition is a Neumann boundary condition in the context of variable exponent, we must work in general with the space $W^{1,p(\cdot)}(\Omega)$ instead of the common space $W_0^{1,p(\cdot)}(\Omega)$. The main difficulty which appears in this case for the existence and also the uniqueness of solutions is that the famous

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Poincaré inequality does not apply (see [8]). The same can be said for the Poincaré-Wirtinger inequality which does not apply for general data f considered in this work (see [27]). Recently, Ouaro and Soma [27] studied the problem

$$\begin{gathered} -\operatorname{div} a(x,\nabla u)+|u|^{p(x)-2}u=f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

under the assumption

$$p(\cdot):\Omega\to\mathbb{R} \text{ is a measurable function and } 1<p_-\le p_+<+\infty, \tag{1.3}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

For the vector fields $a(\cdot,\cdot)$ in [27], the authors assumed that $a(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is Carathéodory and is the continuous derivative with respect to ξ of the mapping $A:\Omega\times\mathbb{R}^N\to\mathbb{R}$, $A=A(x,\xi)$; i.e., $a(x,\xi)=\nabla_\xi A(x,\xi)$ such that:

- for almost every $x\in\Omega$,
$$A(x,0)=0; \tag{1.4}$$
- there exists a positive constant C_1 such that
$$|a(x,\xi)|\le C_1(j(x)+|\xi|^{p(x)-1}) \tag{1.5}$$
for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$ where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $1/p(x)+1/p'(x)=1$;
- the following inequality hold for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^N$ with $\xi\ne\eta$,
$$(a(x,\xi)-a(x,\eta)).(\xi-\eta)>0; \tag{1.6}$$
- for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$,
$$|\xi|^{p(x)}\le a(x,\xi).\xi\le p(x)A(x,\xi) \tag{1.7}$$

Under assumptions (1.3)-(1.7), Ouaro and Soma [27] proved the existence and uniqueness of entropy solutions to (1.2) for L^1-data f. The assumption on the function A and the use of the quantity $|u|^{p(x)-2}u$ allowed them in particular to use a minimization method for the proof of the existence of a weak solution for (1.2) when the right-hand side is in $L^\infty(\Omega)$ (see [27, Theorem 3.1]). Note also that the uniqueness of weak and entropy solutions u of [27, (1.2)] is due to the fact that $s\mapsto |s|^{p(x)-2}s$ is increasing.

In this article we improve the result in [27]. We make restrictive assumptions on the data a and b. For this reason, we can not use the minimization methods used in [27] to get our existence result of weak solutions. We use an auxiliary result due to Le (see [21, Theorem 3.1]). Indeed, Le [21] proved in particular some existence results of weak solutions for the Neumann and Robin boundary value problem

$$\begin{gathered} -\operatorname{div} a(x,\nabla u)+f(x,u)=0 \quad \text{in } \Omega, \\ a(x,\nabla u).\eta=-g(x,u) \quad \text{on } \partial\Omega, \end{gathered}$$

where $a:\Omega\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|a(x,\xi)|\le a_1(x)+b_1|\xi|^{p(x)-1}, \quad \text{for a. e. } x\in\Omega \text{ and all } \xi\in\mathbb{R}^N,$$

with $p\in C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}) \text{ such that } p(x)>1 \text{ for } x\in\overline{\Omega}\}$, $a_1\in L^{p'(\cdot)}(\Omega)$, $p'(\cdot)$ is the Hölder conjugate of $p(\cdot)$ and $b_1>1$. Moreover, a is monotone; i.e.,

$$(a(x,\xi)-a(x,\xi')).(\xi-\xi')\ge 0, \quad \text{for a. e. } x\in\Omega \text{ and all } \xi,\xi'\in\mathbb{R}^N,$$

and coercive in the following sense: there exist $a_2 \in L^1(\Omega)$ and $b_2 > 0$ such that

$$a(x,\xi).\xi \geq b_2|\xi|^{p(x)} - a_2(x), \quad \text{for a. e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^N.$$

$f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are Carathéodory functions such that

$$|f(x,u)| \leq a_3(x), \quad |g(\xi,v)| \leq \widetilde{a}_3(\xi)$$

for a. e. $x \in \Omega$, $\xi \in \partial\Omega$, where $a_3 \in L^{q(\cdot)}(\Omega)$, $\widetilde{a}_3 \in L^{\widetilde{q}}(\partial\Omega)$ with $q(x) < p^*(x)$, for all $x \in \overline{\Omega}$, $\widetilde{q}(x) < \widetilde{p}^*(x)$, for all $x \in \partial\Omega$, $q \in C_+(\overline{\Omega})$, $\widetilde{q} \in C_+(\partial\Omega)$. Here, p^* is the Sobolev conjugate exponent of $p(x)$,

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \leq p(x); \end{cases}$$

$$\widetilde{p}^*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \leq p(x). \end{cases}$$

The proof of the existence results in [21] uses the sub and super solution methods.

In this article, our assumptions are the following:

$$p(\cdot) : \overline{\Omega} \to \mathbb{R} \text{ is a continuous function such that } 1 < p_- \leq p_+ < +\infty \tag{1.8}$$

and

$$b : \mathbb{R} \to \mathbb{R} \text{ is a continuous, nondecreasing function, surjective such that } b(0) = 0. \tag{1.9}$$

For the vector field $a(\cdot,\cdot)$ we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory such that:

- there exists a positive constant C_2 with
$$|a(x,\xi)| \leq C_2(j(x) + |\xi|^{p(x)-1}) \tag{1.10}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$;
- there exists a positive constant C_3 such that for every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$, the following two inequalities hold
$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0, \tag{1.11}$$
$$a(x,\xi).\xi \geq C_3|\xi|^{p(x)} \tag{1.12}$$
for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

We remark that [27, Assumption 1.3] is more restrictive than (1.8). This is due to the use of the results in [21] to get the existence of a weak solution to the problem (1.1).

The remaining part of the paper is the following: in section 2, we introduce some notations/functional spaces. In section 3, we prove the existence and uniqueness of a weak solution of (1.1) when the right-hand side $f \in L^\infty(\Omega)$. Using the results of section 3, we study in section 4, the question of the existence and uniqueness of entropy solutions of (1.1) for $f \in L^1(\Omega)$.

2. Assumptions and preliminaries

As the exponent $p(\cdot)$ appearing in (1.10) and (1.12) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u:\Omega\to\mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u):=\int_\Omega |u|^{p(x)}dx$$

is finite. If the exponent is bounded; i.e., if $p_+<+\infty$, then the expression

$$|u|_{p(\cdot)}=\inf\{\lambda>0:\rho_{p(\cdot)}(u/\lambda)\le 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega),|.|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1<p_-\le p_+<+\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$. Finally, we have the Hölder type inequality:

$$|\int_\Omega uv\,dx|\le (\frac{1}{p_-}+\frac{1}{(p')_-})|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{2.1}$$

for all $u\in L^{p(\cdot)}(\Omega)$ and $v\in L^{p'(\cdot)}(\Omega)$.

Let

$$W^{1,p(\cdot)}(\Omega)=\{u\in L^{p(\cdot)}(\Omega):|\nabla u|\in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)}=|u|_{p(\cdot)}+|(|\nabla u|)|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega),\|.\|_{1,p(\cdot)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (see [16]).

Lemma 2.1. *If $u_n,u\in L^{p(\cdot)}(\Omega)$ and $p_+<+\infty$, then the following properties hold:*

(i) $|u|_{p(\cdot)}>1\Rightarrow |u|_{p(\cdot)}^{p_-}\le\rho_{p(\cdot)}(u)\le|u|_{p(\cdot)}^{p_+}$;
(ii) $|u|_{p(\cdot)}<1\Rightarrow |u|_{p(\cdot)}^{p_+}\le\rho_{p(\cdot)}(u)\le|u|_{p(\cdot)}^{p_-}$;
(iii) $|u|_{p(\cdot)}<1$ *(respectively* $=1;>1$*)* $\Leftrightarrow \rho_{p(\cdot)}(u)<1$ *(respectively* $=1;>1$*);*
(iv) $|u_n|_{p(\cdot)}\to 0$ *(respectively* $\to+\infty$*)* $\Leftrightarrow \rho_{p(\cdot)}(u_n)\to 0$ *(respectively* $\to+\infty$*);*
(v) $\rho_{p(\cdot)}(u/|u|_{p(\cdot)})=1$

For a measurable function $u:\Omega\to\mathbb{R}$, we introduce the function

$$\rho_{1,p(\cdot)}(u)=\int_\Omega |u|^{p(x)}\,dx+\int_\Omega|\nabla u|^{p(x)}\,dx.$$

Then we have the following lemma (see [33, 35]).

Lemma 2.2. *If $u\in W^{1,p(\cdot)}(\Omega)$, then the following properties hold:*

(i) $\|u\|_{1,p(\cdot)}>1\Rightarrow \|u\|_{1,p(\cdot)}^{p_-}\le\rho_{1,p(\cdot)}(u)\le\|u\|_{1,p(\cdot)}^{p_+}$;
(ii) $\|u\|_{p(\cdot)}<1\Rightarrow \|u\|_{1,p(\cdot)}^{p_+}\le\rho_{1,p(\cdot)}(u)\le\|u\|_{1,p(\cdot)}^{p_-}$;
(iii) $\|u\|_{1,p(\cdot)}<1$ *(respectively* $=1;>1$*)* $\Leftrightarrow \rho_{1,p(\cdot)}(u)<1$ *(respectively* $=1;>1$*);*

Given two bounded measurable functions $p(\cdot), q(\cdot) : \Omega \to \mathbb{R}$, we write

$$q(\cdot) \ll p(\cdot) \quad \text{if } \operatorname{ess\,inf}_{x\in\Omega}(p(x) - q(x)) > 0.$$

For more details about Lebesgue and Sobolev spaces with variable exponent, we refer to [10, 24, 25, 30, 32, 36] and the references therein.

3. Existence and uniqueness of weak solutions

In this part, we study the existence and uniqueness of a weak solution of (1.1) for the right-hand side $f \in L^\infty(\Omega)$. The concept of uniqueness is the same as in [2].

Definition 3.1. A weak solution of (1.1) is a measurable function such that

$$u \in W^{1,p(\cdot)}(\Omega), \quad b(u) \in L^\infty(\Omega)$$

and

$$\int_\Omega a(x, \nabla u).\nabla\varphi \, dx + \int_\Omega b(u)\varphi \, dx = \int_\Omega f\varphi \, dx, \quad \forall \varphi \in W^{1,p(\cdot)}(\Omega). \tag{3.1}$$

The main result of this part is the following.

Theorem 3.2. *Assume that* (1.8)–(1.12) *hold true and* $f \in L^\infty(\Omega)$. *Then there exists a unique weak solution of* (1.1).

Proof. (Part 1: Existence). For $k > 0$, we consider the following approximated problem.

$$\begin{gathered} T_k(b(u_k)) - \operatorname{div} a(x, \nabla u_k) = f \quad \text{in } \Omega \\ a(x, \nabla u_k).\eta = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

where $T_k(s) := \max\{-k, \min\{k, s\}\}$ is the truncation of T_k, for any $k > 0$. Note that as $T_k(b(u_k)) \in L^\infty(\Omega)$, by [21, Theorem 3.1], there exists $u_k \in W^{1,p(\cdot)}(\Omega)$ which is a weak solution of (3.2). We now show that $|b(u_k)| \leq \|f\|_\infty$ for all $k > 0$. We recall that for any $\epsilon > 0$,

$$H_\epsilon(s) = \min(\frac{s^+}{\epsilon}, 1),$$

$$\operatorname{sign}_0^+(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s \leq 0 \end{cases}$$

and if γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \gamma(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We take $\varphi = H_\epsilon(u_k - M)$ as a test function in (3.1) for the weak solution u_k and $M > 0$ a constant to be chosen later. We have

$$\int_\Omega a(x, \nabla u_k).\nabla H_\epsilon(u_k - M) \, dx + \int_\Omega T_k(b(u_k))H_\epsilon(u_k - M) \, dx = \int_\Omega f H_\epsilon(u_k - M)dx. \tag{3.3}$$

Let us denote $J = \int_\Omega a(x, \nabla u_k).\nabla H_\epsilon(u_k - M) \, dx$. We deduce that

$$J = \frac{1}{\epsilon} \int_{\{0<u_k-M<\epsilon\}} a(x, \nabla u_k).\nabla(u_k - M) \, dx \geq 0,$$

Then, according to (3.3), we obtain

$$\int_\Omega T_k(b(u_k))H_\epsilon(u_k - M)\,dx \le \int_\Omega fH_\epsilon(u_k - M)\,dx,$$

which is equivalent to saying

$$\int_\Omega (T_k(b(u_k)) - T_k(b(M)))H_\epsilon(u_k - M)\,dx \le \int_\Omega (f - T_k(b(M)))H_\epsilon(u_k - M)\,dx. \quad (3.4)$$

We now let ϵ approach 0 in the above inequality,

$$\int_\Omega (T_k(b(u_k)) - T_k(b(M)))^+\,dx \le \int_\Omega (f - T_k(b(M)))\,\mathrm{sign}_0^+(u_k - M)\,dx. \quad (3.5)$$

Choosing now $M = b_0^{-1}(\|f\|_\infty)$ in (3.5) (since b is surjective) to obtain

$$\int_\Omega (T_k(b(u_k)) - T_k(\|f\|_\infty))^+\,dx \le \int_\Omega (f - T_k(\|f\|_\infty))\,\mathrm{sign}_0^+(u_k - b_0^{-1}(\|f\|_\infty))\,dx. \quad (3.6)$$

Hence for all $k > \|f\|_\infty$, we have

$$\int_\Omega (T_k(b(u_k)) - T_k(\|f\|_\infty))^+\,dx \le \int_\Omega (f - \|f\|_\infty)\,\mathrm{sign}_0^+(u_k - b_0^{-1}(\|f\|_\infty))\,dx \le 0.$$

Then for all $k > \|f\|_\infty$, $(T_k(b(u_k)) - \|f\|_\infty)^+ = 0$ a.e. in Ω which is equivalent to saying

$$T_k(b(u_k)) \le \|f\|_\infty \quad \text{for all } k > \|f\|_\infty. \quad (3.7)$$

It remains to prove that $T_k(b(u_k)) \ge -\|f\|_\infty$ a.e. in Ω for all $k > \|f\|_\infty$. □

Let us remark that as u_k is a weak solution of (3.2), then $(-u_k)$ is a weak solution to the following problem

$$\begin{gathered} T_k(\tilde{b}(u_k)) - \mathrm{div}\,\tilde{a}(x, \nabla u_k) = \tilde{f} \quad \text{in } \Omega \\ \tilde{a}(x, \nabla u_k).\eta = 0 \quad \text{on } \partial\Omega, \end{gathered} \quad (3.8)$$

where $\tilde{a}(x,\xi) = -a(x,-\xi)$, $\tilde{b}(s) = -b(-s)$ and $\tilde{f} = -f$. According to (3.7), we deduce that

$$T_k(-b(u_k)) \le \|f\|_\infty \quad \text{a.e. in } \Omega \text{ for all } k > \|f\|_\infty.$$

Therefore, we obtain

$$T_k(b(u_k)) \ge -\|f\|_\infty \quad \forall k > \|f\|_\infty. \quad (3.9)$$

It follows from (3.7) and (3.9) that for all $k > \|f\|_\infty$, $|T_k(b(u_k))| \le \|f\|_\infty$ which implies

$$|b(u_k)| \le \|f\|_\infty \quad \text{a.e. in } \Omega. \quad (3.10)$$

We now fix $k = \|f\|_\infty + 1$ in (3.2) to end the proof of the existence result.

Part 2: Uniqueness. Let u_1 and u_2 be two weak solutions of (1.1). Let us take $\varphi = T_k(u_1 - u_2)$ as a test function in (3.1) for u_1 and also for u_2, to get

$$\int_\Omega a(x,\nabla u_1).\nabla T_k(u_1 - u_2)\,dx + \int_\Omega b(u_1)T_k(u_1 - u_2)\,dx = \int_\Omega fT_k(u_1 - u_2)\,dx,$$

and

$$\int_\Omega a(x,\nabla u_2).\nabla T_k(u_1 - u_2)\,dx + \int_\Omega b(u_2)T_k(u_1 - u_2)\,dx = \int_\Omega fT_k(u_1 - u_2)\,dx.$$

Adding the two preceding relations, we obtain

$$\int_\Omega (a(x,\nabla u_1) - a(x,\nabla u_2)).\nabla T_k(u_1-u_2)\,dx + \int_\Omega (b(u_1)-b(u_2))T_k(u_1-u_2)\,dx = 0. \tag{3.11}$$

From (3.11) we deduce that

$$\int_\Omega (a(x,\nabla u_1) - a(x,\nabla u_2)).\nabla T_k(u_1-u_2)\,dx = 0, \tag{3.12}$$

$$\int_\Omega (b(u_1)-b(u_2))T_k(u_1-u_2)\,dx = 0. \tag{3.13}$$

Thanks to (3.12) and inequality (1.11), we obtain

$$u_1 - u_2 = c \quad \text{a.e. in } \Omega \tag{3.14}$$

and the relation (3.13) gives

$$\lim_{k\to 0}\int_\Omega (b(u_1)-b(u_2))\frac{1}{k}T_k(u_1-u_2)\,dx = \int_\Omega |b(u_1)-b(u_2)|\,dx = 0.$$

Finally, we obtain

$$\begin{gathered} u_1 - u_2 = c \quad \text{a.e. in } \Omega \\ \text{and } b(u_1) = b(u_2). \end{gathered} \tag{3.15}$$

4. Entropy solutions

In this section, we study the existence and uniqueness of entropy solutions to problem (1.1) when the right-hand side $f \in L^1(\Omega)$. We first recall some notations. Set

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{measurable such that } T_k(u) \in W^{1,p(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

As in [6] (see also [1]), we can prove the following result.

Proposition 4.1. *Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(\cdot)}(\Omega)$ then $v \in (L^{p(\cdot)}(\Omega))^N$ and $v = \nabla u$ in the usual sense.*

We define $\mathcal{T}^{1,p(\cdot)}_{\mathcal{H}}(\Omega)$ as the set of functions $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$ such that there exists a sequence $(u_n)_n \subset W^{1,p(\cdot)}(\Omega)$ satisfying the following conditions:

(C1) $u_n \to u$ a.e. in Ω.
(C2) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $L^1(\Omega)$ for any $k > 0$.

The symbol $\mathcal{H}$ in the notation is related to the fact that we consider here Homogeneous Neumann Boundary condition. For the Nonhomogeneous Neumann Boundary condition, we need to add the definition of the set in the following boundary condition, to give meaning to the solution at the boundary.

(C3) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. in $\partial\Omega$.

In this case, the set will be $\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ where tr is related to the trace of an element $u \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$ (see [3, 6]).

We can now introduce the notion of an entropy solution of (1.1).

Definition 4.2. A measurable function u is an entropy solution to problem (1.1) if $u \in \mathcal{T}_{\mathcal{H}}^{1,p(\cdot)}(\Omega)$, $b(u) \in L^1(\Omega)$ and for every $k > 0$,

$$\int_\Omega a(x, \nabla u).\nabla T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx \leq \int_\Omega f(x)T_k(u-\varphi)dx, \quad (4.1)$$

for all $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Our main result in this section is the following.

Theorem 4.3. *Assume* (1.8)-(1.12) *and* $f \in L^1(\Omega)$. *Then there exists a unique entropy solution* u *to* (1.1).

To prove the above theorem, we need the following propositions among which, some can be proved following [7] with necessary changes in detail. But those which are new will be proved.

Proposition 4.4. *Assume* (1.8)-(1.12), $f \in L^1(\Omega)$ *and* $q(\cdot) : \Omega \to [1, +\infty)$ *a measurable function. Let* u *be an entropy solution of* (1.1). *If there exists a positive constant* M *such that*

$$\int_{\{|u|>k\}} k^{q(x)}dx \leq M \quad \text{for all } k > 0 \quad (4.2)$$

then

$$\int_{\{|\nabla u|^{\alpha(\cdot)}>k\}} k^{q(x)}dx \leq C\|f\|_1 + M \quad \text{for all } k > 0,$$

where $\alpha(\cdot) = p(\cdot)/(q(\cdot)+1)$ *and* C *is a positive constant.*

Proposition 4.5. *Assume that* (1.8)-(1.12) *hold and* $f \in L^1(\Omega)$. *Let* u *be an entropy solution of* (1.1). *Then*

$$\int_\Omega |\nabla T_k(u)|^{p(x)}dx \leq C'k\|f\|_1 \quad \text{for all } k > 0 \quad (4.3)$$

and

$$\|b(u)\|_1 \leq C'' \operatorname{meas}(\Omega)\|f\|_1, \quad (4.4)$$

where C' *and* C'' *are positive constants.*

Proposition 4.6. *Assume that* (1.8)-(1.12) *hold and* $f \in L^1(\Omega)$. *Let* u *be an entropy solution of* (1.1). *Then*

$$\int_{\{|u|\leq k\}} |\nabla T_k(u)|^{p_-}dx \leq C'''(k+1) \quad \text{for all } k > 0, \quad (4.5)$$

where C''' *is a positive constant.*

Proposition 4.7. *Assume that* (1.8)-(1.12) *hold true and* $f \in L^1(\Omega)$. *Let* u *be an entropy solution of* (1.1). *Then*

$$\operatorname{meas}\{|u| > h\} \leq \frac{\|f\|_1}{\min(b(h), |b(-h)|)} \quad \text{for all } h \text{ large enough} \quad (4.6)$$

and

$$\operatorname{meas}\{|\nabla u| > h\} \leq \frac{const(\|f\|_1, p_-)}{h^{p_- -1}} \quad \text{for all } h \geq 1. \quad (4.7)$$

Proof. We first prove (4.6). Indeed, by (4.4) (see [7, proof of (4.4)], we have

$$\int_{\{|u|>h\}} |b(u)|dx \le \|f\|_1.$$

From this inequality, we deduce that

$$\min(b(h), |b(-h)|) \int_{\{|u|>h\}} dx \le \|f\|_1.$$

The proof of (4.7) is similar to that of [7, Proposition 4.8]. □

We remark that since b is continuous and surjective, by (4.6), we deduce that

$$\text{meas}\{|u| > h\} \to 0 \quad \text{as } h \to +\infty.$$

4.1. **Proof of Theorem 4.3. Uniqueness of entropy solution.** Let $h > 0$ and u_1, u_2 be two entropy solutions of (1.1). We write the entropy inequality (4.1) corresponding to the solution u_1 with $T_h(u_2)$ as a test function and to the solution u_2 with $T_h(u_1)$ as a test function. Upon addition, we obtain

$$\begin{aligned}
&\int_{\{|u_1-T_h(u_2)|\le k\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx \\
&+ \int_{\{|u_2-T_h(u_1)|\le k\}} a(x,\nabla u_2).\nabla(u_2 - T_h(u_1))dx \\
&+ \int_\Omega b(u_1)T_k(u_1 - T_h(u_2))dx + \int_\Omega b(u_2)T_k(u_2 - T_h(u_1))dx \\
&\le \int_\Omega f(x)\Big(T_k(u_1 - T_h(u_2)) + T_k(u_2 - T_h(u_1))\Big)dx.
\end{aligned} \tag{4.8}$$

Now define

$$E_1 := \{|u_1 - u_2| \le k, |u_2| \le h\}, \quad E_2 := E_1 \cap \{|u_1| \le h\}, \quad E_3 := E_1 \cap \{|u_1| > h\}.$$

We start with the first integral in (4.8). By (1.12), we have

$$\begin{aligned}
&\int_{\{|u_1-T_h(u_2)|\le k\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx \\
&= \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx \\
&\quad + \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|> h\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx \\
&= \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1 - u_2)dx \\
&\quad + \int_{\{|u_1-h\text{sign}(u_2)|\le k\}\cap\{|u_2|> h\}} a(x,\nabla u_1).\nabla u_1 dx \\
&\ge \int_{\{|u_1-T_h(u_2)|\le k\}\cap\{|u_2|\le h\}} a(x,\nabla u_1).\nabla(u_1 - u_2)dx \\
&= \int_{E_1} a(x,\nabla u_1).\nabla(u_1 - u_2)dx \\
&= \int_{E_2} a(x,\nabla u_1).\nabla(u_1 - u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla(u_1 - u_2)dx
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&= \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx + \int_{E_3} a(x,\nabla u_1).\nabla u_1 dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2 dx \\
&\geq \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx - \int_{E_3} a(x,\nabla u_1).\nabla u_2 dx.
\end{aligned}
$$

Using (1.10) and (2.1), we estimate the last integral in (4.9) as follows.

$$
\begin{aligned}
&|\int_{E_3} a(x,\nabla u_1).\nabla u_2 dx| \\
&\leq C_1 \int_{E_3} (j(x) + |\nabla u_1|^{p(x)-1})|\nabla u_2| dx \\
&\leq C_1 \Big(|j|_{p'(\cdot)} + ||\nabla u_1|^{p(x)-1}|_{p'(\cdot),\{h<|u_1|\leq h+k\}}\Big)|\nabla u_2|_{p(\cdot),\{h-k<|u_2|\leq h\}},
\end{aligned} \tag{4.10}
$$

where $\big||\nabla u_1|^{p(x)-1}\big|_{p'(\cdot),\{h<|u_1|\leq h+k\}} = \big\||\nabla u_1|^{p(x)-1}\big\|_{L^{p'(\cdot)}(\{h<|u_1|\leq h+k\})}$.

Since u_1 is an entropy solution of (1.1), by taking $\varphi = T_h(u_1)$ in the entropy inequality (4.1), and using (1.12), we obtain

$$
\int_{\{h<|u_1|\leq h+k\}} |\nabla u_1|^{p(x)} dx \leq Ck\|f\|_1.
$$

So by Lemma 2.1,

$$
\big||\nabla u_1|^{p(x)-1}\big|_{p'(\cdot),\{h<|u_1|\leq h+k\}} \leq C' < +\infty,
$$

where C' is a constant which does not depend on h. Therefore,

$$
C_1(|j|_{p'(\cdot)} + ||\nabla u_1|^{p(x)-1}|_{p'(\cdot),\{h<|u_1|\leq h+k\}}) \leq C_1\Big(|j|_{p'(\cdot)} + C'\Big) < +\infty.
$$

Since u_2 is an entropy solution to problem (1.1), by taking $\varphi = T_h(u_2)$ in the entropy inequality (4.1) and using (1.12), we obtain

$$
\int_{\{h<|u_2|\leq h+k\}} |\nabla u_2|^{p(x)} dx \leq Ck \int_{\{|u_2|>h\}} |f| dx.
$$

Using inequality (4.6), we have $\operatorname{meas}\{|u_2| > h\} \to 0$ as $h \to +\infty$. As $f \in L^1(\Omega)$ we obtain

$$
Ck \int_{\{|u_2|>h\}} |f| dx \to 0 \quad \text{as } h \to +\infty \text{ for any fixed number } k > 0.
$$

From the above convergence we deduce that

$$
\lim_{h\to+\infty} \int_{\{h<|u_2|\leq h+k\}} |\nabla u_2|^{p(x)} dx = 0, \quad \text{for any fixed number } k > 0.
$$

Hence

$$
\lim_{h\to+\infty} \int_{\{h-k<|u_2|\leq h\}} |\nabla u_2|^{p(x)} dx = \lim_{l\to+\infty} \int_{\{l<|u_2|\leq l+k\}} |\nabla u_2|^{p(x)} dx = 0,
$$

for any fixed $k > 0$ with $l = h - k$. So by Lemma 2.1, $|\nabla u_2|_{p(\cdot),\{h-k<|u_2|\leq h\}} \to 0$ as $h \to +\infty$, for any fixed number $k > 0$. Therefore, from (4.9) and (4.10), we obtain

$$
\int_{\{|u_1-T_h(u_2)|\leq k\}} a(x,\nabla u_1).\nabla(u_1 - T_h(u_2))dx \geq I_h + \int_{E_2} a(x,\nabla u_1).\nabla(u_1-u_2)dx, \tag{4.11}
$$

where I_h converges to zero as $h \to +\infty$.

We may adopt the same procedure for study the second term in (4.8) to obtain

$$\int_{\{|u_2-T_h(u_1)|\leq k\}} a(x,\nabla u_2).\nabla(u_2-T_h(u_1))dx \geq J_h - \int_{E_2} a(x,\nabla u_2).\nabla(u_1-u_2)dx, \tag{4.12}$$

where J_h converges to zero as $h\to+\infty$. Now for all $h,k>0$, set

$$K_h = \int_\Omega b(u_1)T_k(u_1-T_h(u_2))dx + \int_\Omega b(u_2)T_k(u_2-T_h(u_1))dx.$$

We have

$$b(u_1)T_k(u_1-T_h(u_2)) \to b(u_1)T_k(u_1-u_2) \quad \text{a.e. in } \Omega \text{ as } h\to+\infty$$

and

$$|b(u_1)T_k(u_1-T_h(u_2))| \leq k|b(u_1)| \in L^1(\Omega).$$

Then by Lebesgue Theorem, we deduce that

$$\lim_{h\to+\infty}\int_\Omega b(u_1)T_k(u_1-T_h(u_2))dx = \int_\Omega b(u_1)T_k(u_1-u_2)dx. \tag{4.13}$$

Similarly, we have

$$\lim_{h\to+\infty}\int_\Omega b(u_2)T_k(u_2-T_h(u_1))dx = \int_\Omega b(u_2)T_k(u_2-u_1)dx. \tag{4.14}$$

Using (4.13) and (4.14), we obtain

$$\lim_{h\to+\infty} K_h = \int_\Omega (b(u_1)-b(u_2))T_k(u_1-u_2)dx. \tag{4.15}$$

We next examine the right-hand side of (4.8). For all $k>0$,

$$f(x)\Big(T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))\Big) \to f(x)\Big(T_k(u_1-u_2)+T_k(u_2-u_1)\Big) = 0$$

a.e. in Ω as $h\to+\infty$ and

$$|f(x)\Big(T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))\Big)| \leq 2k|f(x)| \in L^1(\Omega).$$

Lebesgue Theorem allows us to write

$$\lim_{h\to+\infty}\int_\Omega f(x)\Big(T_k(u_1-T_h(u_2))+T_k(u_2-T_h(u_1))\Big)dx = 0. \tag{4.16}$$

Using (4.11), (4.12), (4.15) and (4.16), we obtain

$$\begin{aligned} &\int_{\{|u_1-u_2|\leq k\}} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\Big(\nabla u_1-\nabla u_2\Big)dx \\ &+\int_\Omega (b(u_1)-b(u_2))T_k(u_1-u_2)dx \leq 0. \end{aligned} \tag{4.17}$$

Therefore,

$$\int_\Omega (b(u_1)-b(u_2))T_k(u_1-u_2)dx = 0, \tag{4.18}$$

from which we deduce that

$$\lim_{k\to 0}\int_\Omega (b(u_1)-b(u_2))\frac{1}{k}T_k(u_1-u_2)\,dx = \int_\Omega |b(u_1)-b(u_2)|\,dx = 0. \tag{4.19}$$

It also follows from (4.17) that

$$\int_{\{|u_1-u_2|\leq k\}} \Big(a(x,\nabla u_1)-a(x,\nabla u_2)\Big).\Big(\nabla u_1-\nabla u_2\Big)dx = 0. \tag{4.20}$$

Hence, from (4.19) and (4.20), we obtain

$$u_1 - u_2 = c \quad \text{a.e. in } \Omega.$$
$$\text{and } b(u_1) = b(u_2).$$

Existence of entropy solution. Let $f_n = T_n(f)$; then $\{f_n\}_{n=1}^{+\infty}$ is a sequence of bounded functions which strongly converges to $f \in L^1(\Omega)$ and is such that

$$\|f_n\|_1 \leq \|f\|_1, \quad \text{for all } n \in \mathbb{N}. \tag{4.21}$$

We consider the problem

$$\begin{gathered} -\operatorname{div} a(x, \nabla u_n) + b(u_n) = f_n \quad \text{in } \Omega, \\ a(x, \nabla u_n).\eta = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.22}$$

It follows from Theorem 3.2 that there exists a unique function $u_n \in W^{1,p(\cdot)}(\Omega)$ such that

$$\int_\Omega a(x, \nabla u_n).\nabla\varphi dx + \int_\Omega b(u_n)\varphi dx = \int_\Omega f_n\varphi dx \tag{4.23}$$

for all $\varphi \in W^{1,p(\cdot)}(\Omega)$. Our aim is to prove that these approximated solutions u_n tend, as n goes to infinity, to a measurable function u which is an entropy solution to the limit problem (1.1). To start with, we prove the following lemma.

Lemma 4.8. *For any $k > 0$,*

$$\|T_k(u_n)\|_{1,p(\cdot)} \leq 1 + C,$$

where $C = C(C_3, k, f, p_-, p_+, \operatorname{meas}(\Omega))$ is a positive constant.

Proof. By taking $\varphi = T_k(u_n)$ in (4.23), we obtain

$$\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n)dx + \int_\Omega b(u_n)T_k(u_n)dx = \int_\Omega f_nT_k(u_n)dx.$$

Since all the terms in the left-hand side of equality above are nonnegative and

$$\int_\Omega f_nT_k(u_n)dx \leq k\|f_n\|_1 \leq k\|f\|_1,$$

by using (1.12) we obtain

$$\int_\Omega |\nabla T_k(u_n)|^{p(x)}dx \leq Ck\|f\|_1. \tag{4.24}$$

We also have that

$$\int_\Omega |T_k(u_n)|^{p(x)}dx = \int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p(x)}dx + \int_{\{|u_n|>k\}} |T_k(u_n)|^{p(x)}dx.$$

Furthermore,

$$\int_{\{|u_n|>k\}} |T_k(u_n)|^{p(x)}dx = \int_{\{|u_n|>k\}} k^{p(x)}dx \leq \begin{cases} k^{p_+} \operatorname{meas}(\Omega) & \text{if } k \geq 1, \\ \operatorname{meas}(\Omega) & \text{if } k < 1 \end{cases}$$

and

$$\int_{\{|u_n|\leq k\}} |T_k(u_n)|^{p(x)}dx \leq \int_{\{|u_n|\leq k\}} k^{p(x)}dx \leq \begin{cases} k^{p_+} \operatorname{meas}(\Omega) & \text{if } k \geq 1, \\ \operatorname{meas}(\Omega) & \text{if } k < 1. \end{cases}$$

This allows us to write

$$\int_\Omega |T_k(u_n)|^{p(x)}dx \leq 2(1 + k^{p_+}) \operatorname{meas}(\Omega). \tag{4.25}$$

Hence, adding (4.24) and (4.25) yields

$$\rho_{1,p(\cdot)}(T_k(u_n)) \le Ck\|f\|_1 + (1+k^{p_+})\,\mathrm{meas}(\Omega) = C(C_3,k,f,p_+,\mathrm{meas}(\Omega)). \quad (4.26)$$

For $\|T_k(u_n)\|_{1,p(\cdot)} \ge 1$, we have

$$\|T_k(u_n)\|_{1,p(\cdot)}^{p_-} \le \rho_{1,p(\cdot)}(T_k(u_n)) \le C(C_3,k,f,p_+,\mathrm{meas}(\Omega)),$$

which is equivalent to

$$\|T_k(u_n)\|_{1,p(\cdot)} \le \Big(C(C_3,k,f,p_+,\mathrm{meas}(\Omega))\Big)^{1/p_-} = C(C_3,k,f,p_-,p_+,\mathrm{meas}(\Omega)).$$

The above inequality gives

$$\|T_k(u_n)\|_{1,p(\cdot)} \le 1 + C(C_3,k,f,p_-,p_+,\mathrm{meas}(\Omega)).$$

The proof is complete. □

From Lemma 4.8 we deduce that for any $k>0$, the sequence $\{T_k(u_n)\}_{n=1}^{+\infty}$ is uniformly bounded in $W^{1,p(\cdot)}(\Omega)$ and so in $W^{1,p_-}(\Omega)$. Then, up to a subsequence we can assume that for any $k>0$, $T_k(u_n)$ converges weakly to σ_k in $W^{1,p_-}(\Omega)$, and so $T_k(u_n)$ strongly converges to σ_k in $L^{p_-}(\Omega)$.

Proposition 4.9. . *Assume that* (1.8)-(1.12) *hold and $u_n \in W^{1,p(\cdot)}(\Omega)$ is the solution of* (4.22). *Then the sequence $\{u_n\}_{n=1}^{+\infty}$ is Cauchy in measure. In particular, there exists a measurable function u and a subsequence still denoted $\{u_n\}_{n=1}^{+\infty}$ such that $u_n \to u$ in measure.*

Proof. Let $s>0$ and $k>0$ be fixed. Define

$$E_n := \{|u_n|>k\}, \quad E_m := \{|u_m|>k\}, \quad E_{n,m} := \{|T_k(u_n)-T_k(u_m)|>s\}.$$

Note that

$$\{|u_n-u_m|>s\} \subset E_n \cup E_m \cup E_{n,m}$$

and hence

$$\mathrm{meas}\{|u_n-u_m|>s\} \le \mathrm{meas}(E_n)+\mathrm{meas}(E_m)+\mathrm{meas}(E_{n,m}). \quad (4.27)$$

Let $\epsilon>0$. Using Proposition 4.7, we choose $k=k(\epsilon)$ such that

$$\mathrm{meas}(E_n) \le \epsilon/3 \quad \text{and} \quad \mathrm{meas}(E_m) \le \epsilon/3. \quad (4.28)$$

Since $T_k(u_n)$ converges strongly in $L^{p_-}(\Omega)$, then it is a Cauchy sequence in $L^{p_-}(\Omega)$. Thus

$$\mathrm{meas}(E_{n,m}) \le \frac{1}{s^{p_-}} \int_\Omega |T_k(u_n)-T_k(u_m)|^{p_-}\,dx \le \frac{\epsilon}{3}, \quad (4.29)$$

for all $n,m \ge n_0(s,\epsilon)$. Finally, from (4.27), (4.28) and (4.29), we obtain

$$\mathrm{meas}\{|u_n-u_m|>s\} \le \epsilon \quad \text{for all } n,m \ge n_0(s,\epsilon). \quad (4.30)$$

Relations (4.30) imply that the sequence $\{u_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in measure and the proof is complete. □

Note that as $u_n \to u$ in measure, up to a subsequence, we can assume that $u_n \to u$ a. e. in Ω. In the sequel, we need the following two technical lemmas (see [18, 30]).

Lemma 4.10. *Let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(\cdot)}(\Omega)$ for some $1 \ll p(\cdot) \in L^\infty(\Omega)$, then $v_n \to v$ strongly in $L^1(\Omega)$.*

The second technical lemma is a well known result in the measure theory [18].

Lemma 4.11. *Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) < +\infty$. Consider a measurable function $\gamma : X \to [0, +\infty]$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0.$$

Then, for every $\epsilon > 0$, there exists $\delta > 0$, such that

$$\mu(A) < \epsilon, \quad \text{for all } A \in \mathcal{M} \text{ with } \int_A \gamma d\mu < \delta.$$

We are ready for proving that the function u in the Proposition 4.9 is an entropy solution of (1.1). Let $\varphi \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. For any $k > 0$, choose $T_k(u_n - \varphi)$ as a test function in (4.23). We obtain

$$\begin{aligned}
&\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - \varphi)dx + \int_\Omega b(u_n)T_k(u_n - \varphi)dx \\
&= \int_\Omega f_n(x)T_k(u_n - \varphi)dx.
\end{aligned} \tag{4.31}$$

The following proposition is useful to pass to the limit in the first term of (4.31).

Proposition 4.12. *Assume that (1.8)–(1.12) hold and $u_n \in W^{1,p(\cdot)}(\Omega)$ is the weak solution to (4.22). Then*

(i) *∇u_n converges in measure to the weak gradient of u;*
(ii) *For all $k > 0$, $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in $(L^1(\Omega))^N$.*
(iii) *For all $t > 0$, $a(x, \nabla T_t(u_n))$ converges strongly to $a(x, \nabla T_t(u))$ in $(L^1(\Omega))^N$ and weakly in $(L^{p'(\cdot)}(\Omega))^N$.*

Proof. (i) We claim that the sequence $\{\nabla u_n\}_{n=1}^{+\infty}$ is Cauchy in measure. Indeed, let $s > 0$ and consider

$$A_{n,m} := \{|\nabla u_n| > h\} \cup \{|\nabla u_m| > h\}, \quad B_{n,m} := \{|u_n - u_m| > k\}$$

and

$$C_{n,m} := \{|\nabla u_n| \le h, |\nabla u_m| \le h, \, |u_n - u_m| \le k, \, |\nabla u_n - \nabla u_m| > s\},$$

where h and k will be chosen later. Note that

$$|\nabla u_n - \nabla u_m| > s\} \subset A_{n,m} \cup B_{n,m} \cup C_{n,m}. \tag{4.32}$$

Let $\epsilon > 0$. By Proposition 4.7 (relation (4.7)), we may choose $h = h(\epsilon)$ large enough such that

$$\text{meas}(A_{n,m}) \le \epsilon/3, \tag{4.33}$$

for all $n, m \ge 0$. On the other hand, by Proposition 4.9,

$$\text{meas}(B_{n,m}) \le \epsilon/3, \tag{4.34}$$

for all $n, m \ge n_0(k, \epsilon)$. Moreover, since $a(x, \xi)$ is continuous with respect to ξ for a.e. $x \in \Omega$, by assumption (1.11) there exists a real valued function $\gamma : \Omega \to [0, +\infty]$ such that $\text{meas}(\{x \in \Omega : \gamma(x) = 0\}) = 0$ and

$$(a(x, \xi) - a(x, \xi')).(\xi - \xi') \ge \gamma(x), \tag{4.35}$$

for all $\xi, \xi' \in \mathbb{R}^N$ such that $|\xi| \le h$, $|\xi'| \le h$, $|\xi - \xi'| \ge s$, for a.e. $x \in \Omega$. Let $\delta = \delta(\epsilon)$ be given by Lemma 4.11, replacing ϵ and A by $\epsilon/3$ and $C_{n,m}$ respectively.

As u_n is a weak solution of (4.22), using $T_k(u_n - u_m)$ as a test function in (4.23), we obtain

$$\begin{aligned} &\int_\Omega a(x, \nabla u_n).\nabla T_k(u_n - u_m)dx + \int_\Omega b(u_n)T_k(u_n - u_m)dx \\ &= \int_\Omega f_n T_k(u_n - u_m)dx \le k\|f\|_1. \end{aligned}$$

Similarly for u_m, we have

$$\begin{aligned} &\int_\Omega a(x, \nabla u_m).\nabla T_k(u_m - u_n)dx + \int_\Omega b(u_m)T_k(u_m - u_n)dx \\ &= \int_\Omega f_m T_k(u_m - u_n)dx \le k\|f\|_1. \end{aligned}$$

Adding these two inequalities yields

$$\begin{aligned} &\int_{\{|u_n - u_m| \le k\}} (a(x, \nabla u_n) - a(x, \nabla u_m)).(\nabla u_n - \nabla u_m)dx \\ &+ \int_\Omega \Big(b(u_n) - b(u_m)\Big)T_k(u_n - u_m)dx \le 2k\|f\|_1. \end{aligned}$$

Since the second term of the above inequality is nonnegative, by using (4.35) we obtain

$$\int_{C_{n,m}} \gamma(x)dx \le \int_{C_{n,m}} (a(x, \nabla u_n) - a(x, \nabla u_m)).(\nabla u_n - \nabla u_m)dx \le 2k\|f\|_1 < \delta,$$

where $k = \delta/4\|f\|_1$. From Lemma 4.11, it follows that

$$\text{meas}(C_{n,m}) \le \epsilon/3. \tag{4.36}$$

Thus, using (4.32), (4.33), (4.34) and (4.36), we obtain

$$\text{meas}(\{|\nabla u_n - \nabla u_m| > s\}) \le \epsilon, \quad \text{for all } n, m \ge n_0(s, \epsilon) \tag{4.37}$$

and then the claim is proved. Consequently, $\{\nabla u_n\}_{n=1}^{+\infty}$ converges in measure to some measurable function v. To complete the proof of (i), we need the following lemma.

Lemma 4.13. (a) *For a.e.* $t \in \mathbb{R}$, $\nabla T_t(u_n)$ *converges in measure to* $v\chi_{\{|u|<t\}}$*;*
(b) *for a.e.* $t \in \mathbb{R}$, $\nabla T_t(u) = v\chi_{\{|u|<t\}}$*;*
(c) $\nabla T_t(u) = v\chi_{\{|u|<t\}}$ *holds for all* $t \in \mathbb{R}$.

Proof. Proof of part (a). We know that $\nabla u_n \to v$ in measure. Thus, $\chi_{\{|u|<t\}}\nabla u_n \to \chi_{\{|u|<t\}}v$ in measure. Now, let us show that $(\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}})\nabla u_n \to 0$ in measure. For that, it is sufficient to show that $(\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}) \to 0$ in measure. Now, for all $\delta > 0$,

$$\begin{aligned} &\{|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\|\nabla u_n| > \delta\} \\ &\subset \{|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}| \ne 0\} \\ &\subset \{|u| = t\} \cup \{u_n < t < u\} \cup \{u < t < u_n\} \cup \{u_n < -t < u\} \cup \{u < -t < u_n\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\text{meas}\{|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}\|\nabla u_n| > \delta\} \\ &\le \text{meas}\{|u| = t\} + \text{meas}\{u_n < t < u\} + \text{meas}\{u < t < u_n\} \\ &\quad + \text{meas}\{u_n < -t < u\} + \text{meas}\{u < -t < u_n\}. \end{aligned} \tag{4.38}$$

Note that $\text{meas}\{|u| = t\} \le \text{meas}\{t-h < u < t+h\} + \text{meas}\{-t-h < u < -t+h\} \to 0$ as $h \to 0$ for a.e. t, since u is a fixed function. Next,

$$\text{meas}\{u_n < t < u\} \le \text{meas}\{t < u < t+h\} + \text{meas}\{|u - u_n| > h\}$$

for all $h > 0$. Due to Proposition 4.9, for all fixed $h > 0$, we have $\text{meas}\{|u - u_n| > h\} \to 0$ as $n \to +\infty$. Since $\text{meas}\{t < u < t+h\} \to 0$ as $h \to 0$, for all $\epsilon > 0$, one can find N such that for all $n > N$, $\text{meas}\{u_n < t < u\} < \epsilon/2 + \epsilon/2 = \epsilon$ by choosing h and then N. Each of the other terms in the right-hand side of (4.38) can be treated in the same way as for $\text{meas}\{u_n < t < u\}$. Thus, $\text{meas}\{|\chi_{\{|u_n|<t\}} - \chi_{\{|u|<t\}}||\nabla u_n| > \delta\} \to 0$ as $n \to +\infty$. Finally, since $\nabla T_t(u_n) = \nabla u_n \chi_{\{|u_n|<t\}}$, the claim (a) follows.

Proof of part (b). Let ψ_t be the weak $W^{1,p(\cdot)}$-limit of $T_t(u_n)$, then it is also the strong L^1-limit of $T_t(u_n)$. But, as T_t is a Lipschitz function, the convergence in measure of u_n to u implies the convergence in measure of $T_t(u_n)$ to $T_t(u)$. Thus, by the uniqueness of the limit in measure, ψ_t is identified with $T_t(u)$, we conclude that $\nabla T_t(u_n) \to \nabla T_t(u)$ weakly in $L^{p(\cdot)}(\Omega)$.

The previous convergence also ensures that $\nabla T_t(u_n)$ converges to $\nabla T_t(u)$ weakly in $L^1(\Omega)$. On the other hand, by (a), $\nabla T_t(u_n)$ converges to $v\chi_{\{|u|<t\}}$ in measure. By Lemma 4.10, since $\nabla T_t(u_n)$ is uniformly bounded in $L^{p_-}(\Omega)$, the convergence is actually strong in $L^1(\Omega)$; thus it is also weak in $L^1(\Omega)$. By the uniqueness of a weak L^1-limit, $v\chi_{\{|u|<t\}}$ coincides with $\nabla T_t(u)$.

Proof of part (c). Let $0 < t < s$, and s be such that $v\chi_{\{|u|<s\}}$ coincides with $\nabla T_s(u)$. Then

$$\nabla T_t(u) = \nabla T_t(T_s(u)) = \nabla T_s(u)\chi_{\{|T_s(u)|<t\}} = v\chi_{\{|u|<s\}}\chi_{\{|u|<t\}} = v\chi_{\{|u|<t\}}.$$

Now, we complete the proof of (i), by combining Lemma 4.13-(c) and Proposition 4.1.

(ii) Let $s > 0$, $k > 0$ and consider

$$F_{n,m} = \{|\nabla u_n - \nabla u_m| > s, |u_n| \le k, |u_m| \le k\},$$
$$G_{n,m} = \{|\nabla u_m| > s, |u_n| > k, |u_m| \le k\},$$
$$H_{n,m} = \{|\nabla u_n| > s, |u_m| > k, |u_n| \le k\}, \quad I_{n,m} = \{0 > s, |u_m| > k, |u_n| > k\}.$$

Note that

$$\{|\nabla T_k(u_n) - \nabla T_k(u_m)| > s\} \subset F_{n,m} \cup G_{n,m} \cup H_{n,m} \cup I_{n,m}. \tag{4.39}$$

Let $\epsilon > 0$. By Proposition 4.7, we may choose $k(\epsilon)$ such that

$$\text{meas}(G_{n,m}) \le \frac{\epsilon}{4}, \ \text{meas}(H_{n,m}) \le \frac{\epsilon}{4} \text{ and } \text{meas}(I_{n,m}) \le \frac{\epsilon}{4}. \tag{4.40}$$

Therefore, using (4.37), (4.39) and (4.40), we obtain

$$\text{meas}(\{|\nabla T_k(u_n) - \nabla T_k(u_m)| > s\}) \le \epsilon, \quad \text{for all } n, m \ge n_1(s, \epsilon). \tag{4.41}$$

Consequently, $\nabla T_k(u_n)$ converges in measure to $\nabla T_k(u)$. Then, using lemmas 4.8 and 4.10, (ii) follows.

(iii) By lemmas 4.10 and 4.13, for all $t > 0$, $a(x, \nabla T_t(u_n))$ converges strongly to $a(x, \nabla T_t(u))$ in $(L^1(\Omega))^N$, and $a(x, \nabla T_t(u_n))$ converges weakly to $\chi_t \in (L^{p'(\cdot)}(\Omega))^N$ in $(L^{p'(\cdot)}(\Omega))^N$. Since each of the convergence implies the weak L^1-convergence, χ_t can be identified with $a(x, \nabla T_t(u))$; thus, $a(x, \nabla T_t(u)) \in (L^{p'(\cdot)}(\Omega))^N$. The proof of (iii) is then complete. Thus the proof is complete. □

We are now able to pass to the limit in the identity (4.31). For the right-hand side, the convergence is obvious since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and a.e. in Ω.

For the second term of (4.31), we have

$$\int_\Omega b(u_n)T_k(u_n-\varphi)dx = \int_\Omega (b(u_n)-b(\varphi))T_k(u_n-\varphi)dx + \int_\Omega b(\varphi)T_k(u_n-\varphi)dx.$$

The quantity $(b(u_n)-b(\varphi))T_k(u_n-\varphi)$ is nonnegative and since for all $s\in\mathbb{R}$, $s\mapsto b(s)$ is continuous, we obtain

$$(b(u_n)-b(\varphi))T_k(u_n-\varphi) \to (b(u)-b(\varphi))T_k(u-\varphi) \quad \text{a.e. in } \Omega.$$

Then, it follows by Fatou's Lemma that

$$\liminf_{n\to+\infty} \int_\Omega (b(u_n)-b(\varphi))T_k(u_n-\varphi)dx \ge \int_\Omega (b(u)-b(\varphi))T_k(u-\varphi)dx.$$

We have $b(\varphi)\in L^1(\Omega)$. Since $T_k(u_n-\varphi)$ converges weakly-$*$ to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $b(\varphi)\in L^1(\Omega)$, it follows that

$$\lim_{n\to+\infty} \int_\Omega b(\varphi)T_k(u_n-\varphi)dx = \int_\Omega b(\varphi)T_k(u-\varphi)dx.$$

Next, we write the first term in (4.31) in the form

$$\int_{\{|u_n-\varphi|\le k\}} a(x,\nabla u_n).\nabla u_n dx - \int_{\{|u_n-\varphi|\le k\}} a(x,\nabla u_n).\nabla\varphi dx. \tag{4.42}$$

Set $l = k+\|\varphi\|_\infty$. The second integral in (4.42) is equal to

$$\int_{\{|u_n-\varphi|\le k\}} a(x,\nabla T_l(u_n)).\nabla\varphi dx.$$

Since $a(x,\nabla T_l(u_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$ (by (1.10) and (4.24)), by Proposition 4.12-(iii), it converges weakly to $a(x,\nabla T_l(u))$ in $(L^{p'(\cdot)}(\Omega))^N$. Therefore,

$$\lim_{n\to+\infty} \int_{\{|u_n-\varphi|\le k\}} a(x,\nabla T_l(u_n)).\nabla\varphi dx = \int_{\{|u-\varphi|\le k\}} a(x,\nabla T_l(u)).\nabla\varphi dx.$$

Moreover, $a(x,\nabla u_n).\nabla u_n$ is nonnegative and converges a.e. in Ω to $a(x,\nabla u).\nabla u$. Thanks to Fatou's Lemma, we obtain

$$\liminf_{n\to+\infty} \int_{\{|u_n-\varphi|\le k\}} a(x,\nabla u_n).\nabla u_n dx \ge \int_{\{|u-\varphi|\le k\}} a(x,\nabla u).\nabla u dx.$$

Gathering results, we obtain

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx \le \int_\Omega fT_k(u-\varphi)dx.$$

We conclude that u is an entropy solution of (1.1). □

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