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SOLUTIONS OF $p(x)$-LAPLACIAN EQUATIONS WITH CRITICAL EXPONENT AND PERTURBATIONS IN $\mathbb{R}^N$

XIA ZHANG, YONGQIANG FU

Abstract. Based on the theory of variable exponent Sobolev spaces, we study a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$ involving the critical exponent. Firstly, we modify the principle of concentration compactness in $W^{1,p(x)}(\mathbb{R}^N)$ and obtain a new type of Sobolev inequalities involving the atoms. Then, by using variational method, we obtain the existence of weak solutions when the perturbation is small enough.

1. Introduction

We study the solutions to the problem

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=|u|^{p^*(x)-2}u+h(x),\quad x\in\mathbb{R}^N, \tag{1.1}$$

where p is Lipschitz continuous on $\mathbb{R}^N$ and satisfies

$$1<p_-\le p(x)\le p_+<N, \tag{1.2}$$

$0\le h(\not\equiv 0)\in L^{p'(x)}(\mathbb{R}^N)$.

We will study (1.1) in the frame of variable exponent function spaces, the definitions of which will be given in section 2.

We say that $u\in W^{1,p(x)}(\mathbb{R}^N)$ is a weak solution of problem (1.1), if for any $v\in W^{1,p(x)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\big(|\nabla u|^{p(x)-2}\nabla u\nabla v+|u|^{p(x)-2}uv-|u|^{p^*(x)-2}uv-h(x)v\big)\,dx=0.$$

We can verify that the weak solution for (1.1) coincide with the critical point of the energy functional on $W^{1,p(x)}(\mathbb{R}^N)$:

$$\varphi(u)=\int_{\mathbb{R}^N}\Big(\frac{|\nabla u|^{p(x)}+|u|^{p(x)}}{p(x)}-\frac{|u|^{p^*(x)}}{p^*(x)}-h(x)u\Big)\,dx.$$

If $h(x)\equiv 0$, it is easy to verify that $u=0$ is a trivial solution to (1.1). The existence of nontrivial weak solutions for a class of $p(x)$-Laplacian equations without perturbations was studied in [3, 10, 12, 19] via variational methods. They verified

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the Palais-Smale conditions for the energy functional φ and obtained critical points for φ. Moreover, they obtained weak solutions for the $p(x)$-Laplacian equations.

In [12], we study the following type of $p(x)$-Laplacian equations with critical exponent:

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)+\lambda|u|^{p(x)-2}u=f(x,u)+h(x)|u|^{p^*(x)-2}u,\quad x\in\mathbb{R}^N. \quad (1.3)$$

The difficulty is due to the loss of compactness for the embedding $W^{1,p(x)}(\mathbb{R}^N)\hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$. To prove the Palais-Smale condition for the corresponding energy functional, we assume that the coefficient $h(x)$ of critical part satisfies $h(0)=h(\infty)=0$. Then, based on the principle of concentration compactness on $W^{1,p(x)}(\mathbb{R}^N)$ and symmetric critical point theorem, we obtain infinitely many radial weak solutions for (1.3).

When $p(x)$ is constant, equations with critical growth have been studied extensively, see for example [2, 5, 14, 21, 22]. The aim of this paper is to use variational method to show that (1.1) has at least one weak solution if $p(x)$ is function and $h(x)\not\equiv 0$. Here the difficulty is also caused by the loss of the compactness for the embedding $W^{1,p(x)}(\mathbb{R}^N)\hookrightarrow L^{p^*(x)}(\mathbb{R}^N)$. In this paper, by using Ekeland's variational principle [9], we obtain a Palais-Smale sequence if $\|h\|_{p'(x)}$ is sufficient small. We do not expect to prove the Palais-Smale condition for φ and will not make similar assumptions as in [12]. However, based on the principle of concentration compactness on variable exponent Sobolev space established in [12], we prove that the weak limit of Palais-Smale sequence is a weak solution for (1.1) (see Theorem 3.3). In order to obtain the main result, we also give a kind of modified Sobolev inequalities involving the atoms in the concentration-compactness principle (see Theorem 2.7) .

2. Preliminaries

In the studies of nonlinear problems with variable exponential growth, see for example [1, 3, 4, 6, 10, 15, 16, 20], variable exponent spaces play an important role. Since they were thoroughly studied by Kováčik and Rákosník [13], variable exponent spaces have been used to model various phenomena. In [17], Růžička presented the mathematical theory for the application of variable exponent Sobolev spaces in electro-rheological fluids. As another application, Chen, Levine and Rao [7] suggested a model for image restoration based on a variable exponent Laplacian.

For the convenience of the reader, we recall some definitions and basic properties of variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, where $\Omega\subset\mathbb{R}^N$ is a domain. For a deeper treatment on these spaces, we refer to [8].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p:\Omega\to[1,\infty]$, we denote

$$\rho_{p(x)}(u)=\int_{\Omega\setminus\Omega_\infty}|u|^{p(x)}\,dx+\sup_{x\in\Omega_\infty}|u(x)|,$$

where $\Omega_\infty=\{x\in\Omega:p(x)=\infty\}$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions u such that $\rho_{p(x)}(tu)<\infty$, for some $t>0$. $L^{p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{p(x)}=\inf\{\lambda>0:\rho_{p(x)}(\frac{u}{\lambda})\le 1\}.$$

For any $p \in \mathbf{P}(\Omega)$, we define the conjugate function $p'(x)$ as

$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 = \{x \in \Omega : p(x) = 1\}, \\ 1, & x \in \Omega_\infty, \\ \frac{p(x)}{p(x)-1}, & x \in \Omega \setminus (\Omega_1 \cup \Omega_\infty). \end{cases}$$

Theorem 2.1. *Let $p \in \mathbf{P}(\Omega)$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$,*

$$\int_\Omega |uv|\, dx \le 2\|u\|_{p(x)} \|v\|_{p'(x)}.$$

For any $p \in \mathbf{P}(\Omega)$, we denote

$$p_+ = \sup_{x\in\Omega} p(x), \quad p_- = \inf_{x\in\Omega} p(x)$$

and denote by $p_1 \ll p_2$ the fact that $\inf_{x\in\Omega} (p_2(x) - p_1(x)) > 0$.

Theorem 2.2. *Let $p \in \mathbf{P}(\Omega)$ with $p_+ < \infty$. For any $u \in L^{p(x)}(\Omega)$, we have*

1. *if $\|u\|_{p(x)} \ge 1$, then $\|u\|_{p(x)}^{p_-} \le \int_\Omega |u|^{p(x)}\, dx \le \|u\|_{p(x)}^{p_+}$;*
2. *if $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p_+} \le \int_\Omega |u|^{p(x)}\, dx \le \|u\|_{p(x)}^{p_-}$.*

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is the class of all functions $u \in L^{p(x)}(\Omega)$ such that $|\nabla u| \in L^{p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ is a Banach space equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

By $W_0^{1,p(x)}(\Omega)$ we denote the subspace of $W^{1,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$. Under the condition $1 \le p_- \le p(x) \le p_+ < \infty$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive. And we denote the dual space of $W_0^{1,p(x)}(\Omega)$ by $W^{-1,p'(x)}(\Omega)$.

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$|||u||| = \inf\{t > 0 : \int_\Omega \frac{|u|^{p(x)} + |\nabla u|^{p(x)}}{t^{p(x)}}\, dx \le 1\},$$

then $|||\cdot|||$ and $\|\cdot\|_{1,p(x)}$ are equivalent norms on $W^{1,p(x)}(\Omega)$. In fact, we have

$$\frac{1}{2}\|u\|_{1,p(x)} \le |||u||| \le 2\|u\|_{1,p(x)}.$$

Theorem 2.3. *For any $u \in W^{1,p(x)}(\Omega)$, we have*

1. *if $|||u||| \ge 1$, then $|||u|||^{p_-} \le \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \le |||u|||^{p_+}$;*
2. *if $|||u||| < 1$, then $|||u|||^{p_+} \le \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx \le |||u|||^{p_-}$.*

Theorem 2.4. *Let Ω be a bounded domain with the cone property. If $p \in C(\bar\Omega)$ satisfying* (1.2) *and q is a measurable function defined on Ω with*

$$p(x) \le q(x) \ll p^*(x) \triangleq \frac{Np(x)}{N - p(x)} \quad a.e.\ x \in \Omega,$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Theorem 2.5. *Let Ω be a domain with the cone property. If p is Lipschitz continuous and satisfies* (1.2)*, q is a measurable function defined on Ω with*

$$p(x) \le q(x) \le p^*(x) \quad a.e.\ x \in \Omega,$$

then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In the proof of main results in Section 3, we will use the following principle of concentration compactness in $W^{1,p(x)}(\mathbb{R}^N)$ established in [12].

Theorem 2.6. *Let $\{u_n\} \subset W^{1,p(x)}(\mathbb{R}^N)$ with $\|\|u_n\|\| \le 1$ such that*

$$u_n \to u \quad \text{weakly in } W^{1,p(x)}(\mathbb{R}^N),$$
$$|\nabla u_n|^{p(x)} + |u_n|^{p(x)} \to \mu \quad \text{weak-}* \text{ in } M(\mathbb{R}^N),$$
$$|u_n|^{p^*(x)} \to \nu \quad \text{weak-}* \text{ in } M(\mathbb{R}^N),$$

as $n \to \infty$. Denote

$$C^* = \sup\{\int_{\mathbb{R}^N} |u|^{p^*(x)}\,dx : \|\|u\|\| \le 1, u \in W^{1,p(x)}(\mathbb{R}^N)\}.$$

Then the limit measures are of the form

$$\mu = |\nabla u|^{p(x)} + |u|^{p(x)} + \sum_{j\in J} \mu_j\,\delta_{x_j} + \widetilde{\mu}, \quad \mu(\mathbb{R}^N) \le 1,$$
$$\nu = |u|^{p^*(x)} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu(\mathbb{R}^N) \le C^*,$$

where J is a countable set, $\{\mu_j\}, \{\nu_j\} \subset [0,\infty)$, $\{x_j\} \subset \mathbb{R}^N$, $\widetilde{\mu} \in M(\mathbb{R}^N)$ is a non-atomic nonnegative measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\begin{gathered} \nu(\mathbb{R}^N) \le 2^{(p_+ + p_+^*)/p_-} C^* \max\{\mu(\mathbb{R}^N)^{p_+^*/p_-}, \mu(\mathbb{R}^N)^{p_-^*/p_+}\}, \\ \nu_j \le C^* \max\{\mu_j^{\frac{p_+^*}{p_-}}, \mu_j^{p_-^*/p_+}\}, \end{gathered} \tag{2.1}$$

where $p_+^* = \sup_{x\in\mathbb{R}^N} p^*(x)$, $p_-^* = \inf_{x\in\mathbb{R}^N} p^*(x)$.

To obtain the main result, we prove the following modified version of Theorem 2.6 in which we give a new form of the inequality (2.1).

Theorem 2.7. *Under the hypotheses of Theorem 2.6, for any $j \in J$, the atom x_j satisfies:*

$$\nu_j \le C^* \mu_j^{\frac{p^*(x_j)}{p(x_j)}}, \tag{2.2}$$

where J and x_j are as in Theorem 2.6.

Firstly, we give two lemmas.

Lemma 2.8. *Let $x \in \mathbb{R}^N$. For any $\delta > 0$, there exists $k(\delta) > 0$ independent of x such that for $0 < r < R$ with $\frac{r}{R} \le k(\delta)$, there is a cut-off function η_R^r with $\eta_R^r \equiv 1$ in $B_r(x)$, $\eta_R^r \equiv 0$ outside $B_R(x)$, and for any $u \in W^{1,p(x)}(\mathbb{R}^N)$,*

$$\begin{aligned} &\int_{B_R(x)} (|\nabla(\eta_R^r u)|^{p(x)} + |\eta_R^r u|^{p(x)})\,dx \\ &\le \int_{B_R(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \delta \max\{\|\|u\|\|^{p_+}, \|\|u\|^{p_-}\}. \end{aligned}$$

The above lemma is obtained by a similar discussion to the one in [11, Lemma 3.1].

Lemma 2.9. *Let $x \in \mathbb{R}^N$, $\delta > 0$ and $\frac{r}{R} < k(\delta)$, where $k(\delta)$ is from Lemma 2.8. Then for any $u \in W^{1,p(x)}(\mathbb{R}^N)$, we have*

$$
\begin{aligned}
&\int_{B_r(x)} |u|^{p^*(x)}\,dx \\
&\le C^* \max\Big\{\Big(\int_{B_R(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \delta \max\{\||u\||^{p_+}, \||u\||^{p_-}\}\Big)^{p^*_{x,R,+}/p_{x,R,-}}, \\
&\quad \Big(\int_{B_R(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \delta \max\{\||u\||^{p_+}, \||u\||^{p_-}\}\Big)^{p^*_{x,R,-}/p_{x,R,+}}\Big\},
\end{aligned}
$$

where

$$
\begin{gathered}
p_{x,R,-} \triangleq \inf_{y \in B_R(x)} p(y), \quad p_{x,R,+} \triangleq \sup_{y \in B_R(x)} p(y), \\
p^*_{x,R,-} \triangleq \inf_{y \in B_R(x)} p^*(y), \quad p^*_{x,R,+} \triangleq \sup_{y \in B_R(x)} p^*(y).
\end{gathered}
$$

Proof. Using the cut-off function η_R^r in Lemma 2.8 and the definition of C^*, we obtain

$$
\begin{aligned}
\int_{B_r(x)} |u|^{p^*(x)}\,dx &\le \int_{B_R(x)} |u\eta_R^r|^{p^*(x)}\,dx \\
&\le C^* \max\{\||u\eta_R^r\||^{p^*_{x,R,+}}, \||u\eta_R^r\||^{p^*_{x,R,-}}\} \\
&\le C^* \max\Big\{\Big(\int_{B_R(x)} (|\nabla(u\eta_R^r)|^{p(x)} + |u\eta_R^r|^{p(x)})\,dx\Big)^{p^*_{x,R,+}/p_{x,R,-}}, \\
&\quad \Big(\int_{B_R(x)} (|\nabla(u\eta_R^r)|^{p(x)} + |u\eta_R^r|^{p(x)})\,dx\Big)^{p^*_{x,R,-}/p_{x,R,+}}\Big\}.
\end{aligned}
$$

Then, by Lemma 2.8, we obtain the result. □

Proof of Theorem 2.7. Let $x_0 \in \mathbb{R}^N$. By Lemma 2.9, for any $\delta > 0$, there exists $k(\delta) > 0$ such that for $0 < r < R$ with $r/R \le k(\delta)$,

$$
\begin{aligned}
&\int_{B_r(x_0)} |u_n|^{p^*(x)}\,dx \\
&\le C^* \max\Big\{\Big(\int_{B_R(x_0)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx \\
&\quad + \delta \max\{\||u_n\||^{p_+}, \||u_n\||^{p_-}\}\Big)^{p^*_{x_0,R,+}/p_{x_0,R,-}}, \\
&\Big(\int_{B_R(x_0)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx + \delta \max\{\||u_n\||^{p_+}, \||u_n\||^{p_-}\}\Big)^{p^*_{x_0,R,-}/p_{x_0,R,+}}\Big\}.
\end{aligned}
$$

For any $0 < r' < r$, $R' > R$. Let $\eta_1 \in C_0^\infty(B_r(x_0))$ such that $0 \le \eta_1 \le 1$; $\eta_1 \equiv 1$ in $B_{r'}(x_0)$, $\eta_2 \in C_0^\infty(B_{R'}(x_0))$ such that $0 \le \eta_2 \le 1$; $\eta_2 \equiv 1$ in $B_R(x_0)$. We obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u_n|^{p^*(x)}\eta_1\,dx \\
&\le \int_{B_r(x_0)} |u_n|^{p^*(x)}\,dx \\
&\le C^* \max\Big\{\Big(\int_{B_R(x_0)} (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\,dx + \delta\Big)^{p^*_{x_0,R,+}/p_{x_0,R,-}},
\end{aligned}
$$

$$\Big(\int_{B_R(x_0)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})\,dx+\delta\Big)^{p^*_{x_0,R,-}/p_{x_0,R,+}}\Big\}.$$

Letting $n\to\infty$, we obtain

$$\begin{aligned}
&\nu(\bar B_{r'}(x_0))\\
&\le \int_{\mathbb{R}^N}\eta_1\,d\nu\\
&\le C^*\max\Big\{\Big(\int_{\mathbb{R}^N}\eta_2\,d\mu+\delta\Big)^{p^*_{x_0,R,+}/p_{x_0,R,-}},\Big(\int_{\mathbb{R}^N}\eta_2\,d\mu+\delta\Big)^{p^*_{x_0,R,-}/p_{x_0,R,+}}\Big\}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\nu(\{x_0\})\\
&\le \nu(\bar B_{r'}(x_0))\\
&\le C^*\max\Big\{\Big(\mu(\bar B_{R'}(x_0))+\delta\Big)^{p^*_{x_0,R,+}/p_{x_0,R,-}},\Big(\mu(\bar B_{R'}(x_0))+\delta\Big)^{p^*_{x_0,R,-}/p_{x_0,R,+}}\Big\},
\end{aligned}$$

where $\bar B_{R'}(x_0)$ is the closure of $B_{R'}(x_0)$. Let $\delta\to 0$, $R'\to 0$. Thus we have

$$\begin{aligned}
\nu(\{x_0\})&\le C^*\max\Big\{\mu(\{x_0\})^{p^*(x_0)/p(x_0)},\mu(\{x_0\})^{p^*(x_0)/p(x_0)}\Big\}\\
&=C^*\mu(\{x_0\})^{p^*(x_0)/p(x_0)}.
\end{aligned}$$

Then, for any $j\in J$, the atom x_j satisfies $\nu_j\le C^*\mu_j^{p^*(x_j)/p(x_j)}$. The proof is complete. □

3. Main Results

In this section, we prove that (1.1) has at least one nontrivial weak solution $u_0\in W^{1,p(x)}(\mathbb{R}^N)$. First, we prove the following preliminary result which will show that the weak limit of Palais-Smale sequence of φ is a weak solution for (1.1) (see Theorem 3.3).

Throughout this paper, we denote by C universal positive constants unless otherwise specified.

Theorem 3.1. *Let $\{u_n\}$ be a sequence in $W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n\to u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$ and $\varphi'(u_n)\to 0$ in $W^{-1,p'(x)}(\mathbb{R}^N)$, as $n\to\infty$. Then $\nabla u_n\to\nabla u$ a.e. in $\mathbb{R}^N$, as $n\to\infty$. Moreover, $\varphi'(u)=0$.*

Proof. Since $u_n\to u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$, passing to a subsequence, still denoted by $\{u_n\}$, we may assume that there exist $\mu,\nu\in M(\mathbb{R}^N)$ such that $|\nabla u_n|^{p(x)}+|u_n|^{p(x)}\to\mu$ and $|u_n|^{p^*(x)}\to\nu$ weakly-$*$ in $M(\mathbb{R}^N)$, where $M(\mathbb{R}^N)$ is the space of finite nonnegative Borel measures on $\mathbb{R}^N$. By Theorems 2.6 and 2.7, there exist some countable set J, $\{\mu_j\},\{\nu_j\}\subset(0,\infty)$ and $\{x_j\}\subset\mathbb{R}^N$ such that

$$\mu=|\nabla u|^{p(x)}+|u|^{p(x)}+\sum_{j\in J}\mu_j\,\delta_{x_j}+\widetilde{\mu}, \tag{3.1}$$

$$\nu=|u|^{p^*(x)}+\sum_{j\in J}\nu_j\,\delta_{x_j}, \tag{3.2}$$

$$\nu_j\le C^*\mu_j^{p^*(x_j)/p(x_j)}, \tag{3.3}$$

where

$$C^* = \sup\Big\{ \int_{\mathbb{R}^N} |u|^{p^*(x)}\,dx : \||u\|| \le 1, u \in W^{1,p(x)}(\mathbb{R}^N)\Big\},$$

where $\widetilde{\mu} \in M(\mathbb{R}^N)$ is a nonatomic positive measure, δ_{x_j} is the Dirac measure at x_j.

In the following, we prove that J is a finite set or empty. In fact, for any $\varepsilon > 0$, let $\phi \in C_0^\infty(B_{2\varepsilon}(0))$ such that $0 \le \phi \le 1$, $|\nabla\phi| \le \frac{2}{\varepsilon}$; $\phi \equiv 1$ on $B_\varepsilon(0)$. For any $j \in J$, $\{\phi(\cdot - x_j)u_n\}$ is bounded on $W^{1,p(x)}(\mathbb{R}^N)$. Then we have $\langle\varphi'(u_n),\, \phi(\cdot - x_j)u_n\rangle \to 0$, as $n \to \infty$. Note that

$$\begin{aligned}
&\langle\varphi'(u_n), \phi(\cdot - x_j)u_n\rangle\\
&= \int_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla(u_n\phi(x-x_j)) + |u_n|^{p(x)}\phi(x-x_j) - |u_n|^{p^*(x)}\phi(x-x_j)\\
&\quad - h(x)u_n\phi(x-x_j)\big)\,dx\\
&= \int_{\mathbb{R}^N} \big((|\nabla u_n|^{p(x)} + |u_n|^{p(x)})\phi(x-x_j) + |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi(x-x_j)\cdot u_n\\
&\quad - |u_n|^{p^*(x)}\phi(x-x_j) - h(x)u_n\phi(x-x_j)\big)\,dx.
\end{aligned}$$

As $u_n \to u$ in $L^{p(x)}(B_{2\varepsilon}(x_j))$ and $h \in L^{p'(x)}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N} h(x)u_n\phi(x-x_j)\,dx \to \int_{\mathbb{R}^N} h(x)u\phi(x-x_j)\,dx,$$

as $n \to \infty$. Using (3.1) and (3.2) we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi(x-x_j)\cdot u_n\,dx\\
&= \int_{\mathbb{R}^N} -\phi(x-x_j)\,d\mu + \int_{\mathbb{R}^N} h(x)u\phi(x-x_j)\,dx + \int_{\mathbb{R}^N} \phi(x-x_j)\,d\nu.
\end{aligned} \tag{3.4}$$

It is easy to verify that $\|\nabla\phi(x-x_j)\cdot u_n\|_{p(x)} \to \|\nabla\phi(x-x_j)\cdot u\|_{p(x)}$, as $n \to \infty$. Then

$$\begin{aligned}
&\lim_{n\to\infty} \Big|\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla\phi(x-x_j)\cdot u_n\,dx\Big|\\
&\le \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-1}|\nabla\phi(x-x_j)\cdot u_n|\,dx\\
&\le \limsup_{n\to\infty} 2\||\nabla u_n\|^{p(x)-1}|_{p'(x)} \cdot \|\nabla\phi(x-x_j)\cdot u_n\|_{p(x)} \le C\|\nabla\phi(x-x_j)\cdot u\|_{p(x)}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\phi(x-x_j)\cdot u|^{p(x)}\,dx\\
&= \int_{B_{2\varepsilon}(x_j)} |\nabla\phi(x-x_j)\cdot u|^{p(x)}\,dx\\
&\le 2\||\nabla\phi(x-x_j)|^{p(x)}\|_{(\frac{p^*(x)}{p(x)})', B_{2\varepsilon}(x_j)} \cdot \||u|^{p(x)}\|_{\frac{p^*(x)}{p(x)}, B_{2\varepsilon}(x_j)}
\end{aligned}$$

and

$$\begin{aligned}
\int_{B_{2\varepsilon}(x_j)} (|\nabla\phi(x-x_j)|^{p(x)})^{(\frac{p^*(x)}{p(x)})'}\,dx &= \int_{B_{2\varepsilon}(x_j)} |\nabla\phi|^N\,dx \le (\frac{2}{\varepsilon})^N meas(B_{2\varepsilon}(x_j))\\
&= \frac{4^N}{N}\omega_N,
\end{aligned}$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. As $\int_{B_{2\varepsilon}(x_j)}(|u|^{p(x)})^{\frac{p^*(x)}{p(x)}}\,dx \to 0$, as $\varepsilon \to 0$, we obtain $\|\nabla\phi(x-x_j)\cdot u\|_{p(x)} \to 0$, which implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi(x-x_j)\cdot u_n\,dx \to 0,$$

as $\varepsilon \to 0$. Similarly, we can also get

$$|\int_{\mathbb{R}^N}h(x)u\phi(x-x_j)\,dx| \le \int_{B_{2\varepsilon}(x_j)}|h(x)u|\,dx \to 0,$$

as $\varepsilon \to 0$.

Thus, it follows from (3.4) that $0 = -\mu(\{x_j\}) + \nu(\{x_j\})$; i.e., $\mu_j = \nu_j$ for any $j \in J$. Using (3.3) we obtain

$$\nu_j \le C^*\mu_j^{p^*(x_j)/p(x_j)},$$

which implies that $\nu_j \ge (C^*)^{\frac{p(x_j)}{p(x_j)-p^*(x_j)}} \ge \min\{(C^*)^{-\frac{p_-}{(p^*-p)_+}}, (C^*)^{-\frac{p_+}{(p^*-p)_-}}\}$ for any $j \in J$. As ν is finite, J must be a finite set or empty.

Next, we prove that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$, as $n \to \infty$.

(1) If J is a finite nonempty set, say $J = \{1, 2, \dots, m\}$. Let $d = \min\{d(x_i, x_j) : i, j \in J \text{ with } i \ne j\}$. There exists $R_0 > 0$ such that $B_d(x_j) \subset B_{R_0}$ for any $j \in J$. Take $0 < \varepsilon < \frac{d}{4}$, $B_{2\varepsilon}(x_i) \cap B_{2\varepsilon}(x_j) = \emptyset$ for any $i, j \in J$ with $i \ne j$. Denote $\Omega_{R,\varepsilon} = \{x \in B_R : d(x, x_j) > 2\varepsilon \text{ for any } j \in J\}$.

In the following, we will verify that for any $R > R_0$,

$$\int_{\Omega_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)\,dx \to 0, \quad \text{as } n \to \infty.$$

Let $\psi \in C_0^\infty(B_{2R})$ such that $0 \le \psi \le 1$; $\psi \equiv 1$ on B_R. Define

$$\psi_\varepsilon(x) = \psi(x) - \sum_{j=1}^m \phi(x-x_j).$$

We derive that $\psi_\varepsilon \in C_0^\infty(B_{2R})$ such that $0 \le \psi_\varepsilon \le 1$; $\psi_\varepsilon \equiv 0$ on $\cup_{j=1}^m B_\varepsilon(x_j)$ and $\psi_\varepsilon \equiv 1$ on $(\mathbb{R}^N \setminus \cup_{j=1}^m B_{2\varepsilon}(x_j)) \cap B_R$. Thus

$$\begin{aligned}
0 &\le \int_{\Omega_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)\,dx \\
&\le \int_{B_{2R}}(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_n - \nabla u)\psi_\varepsilon\,dx \\
&= \langle\varphi'(u_n), u_n\psi_\varepsilon\rangle - \langle\varphi'(u_n), u\psi_\varepsilon\rangle - \int_{B_{2R}}|\nabla u|^{p(x)-2}\nabla u(\nabla u_n - \nabla u)\psi_\varepsilon\,dx \\
&\quad - \int_{B_{2R}}\big(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u_n + |u_n|^{p(x)}\psi_\varepsilon - |u_n|^{p^*(x)}\psi_\varepsilon - h(x)u_n\psi_\varepsilon\big)\,dx \\
&\quad + \int_{B_{2R}}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u + |u_n|^{p(x)-2}u_nu\psi_\varepsilon \\
&\quad - |u_n|^{p^*(x)-2}u_nu\psi_\varepsilon - h(x)u\psi_\varepsilon\Big)\,dx.
\end{aligned}$$

Note that

$$|\int_{B_{2R}}(|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u_n - |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u)\,dx|$$

$$
\begin{aligned}
&\le C\int_{B_{2R}} |\nabla u_n|^{p(x)-1}|u_n-u|\,dx\\
&\le C\||\nabla u_n|^{p(x)-1}\|_{p'(x)}\|u_n-u\|_{p(x),B_{2R}},
\end{aligned}
$$

which implies

$$
\int_{B_{2R}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u_n\,dx-\int_{B_{2R}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla\psi_\varepsilon\cdot u\,dx\to 0,
$$

as $n\to\infty$. Similarly, we obtain

$$
\int_{B_{2R}} |u_n|^{p(x)}\psi_\varepsilon\,dx-\int_{B_{2R}} |u_n|^{p(x)-2}u_nu\psi_\varepsilon\,dx\to 0,
$$

and

$$
\int_{B_{2R}} h(x)u_n\psi_\varepsilon\,dx-\int_{B_{2R}} h(x)u\psi_\varepsilon\,dx\to 0.
$$

As $u_n\to u$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$. Using Theorem 2.4 we obtain $u_n\to u$ in $L^{p(x)}(B_{2R})$, for any $R>0$. Passing to a subsequence, still denoted by $\{u_n\}$, a diagonal process enables us to assume that $u_n\to u$ a.e. in $\mathbb{R}^N$, as $n\to\infty$. Thus $|u_n\psi_\varepsilon|^{p^*(x)}\to|u\psi_\varepsilon|^{p^*(x)}$ a.e. in $\mathbb{R}^N$. As $|u_n-u|^{p^*(x)}\le 2^{p^*_+}(|u_n|^{p^*(x)}+|u|^{p^*(x)})$, by Fatou's Lemma, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} 2^{p^*_++1}|u\psi_\varepsilon|^{p^*(x)}\,dx\\
&=\int_{\mathbb{R}^N}\liminf_{n\to\infty}(2^{p^*_+}|u_n\psi_\varepsilon|^{p^*(x)}+2^{p^*_+}|u\psi_\varepsilon|^{p^*(x)}-|u_n\psi_\varepsilon-u\psi_\varepsilon|^{p^*(x)})\,dx\\
&\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}(2^{p^*_+}|u_n\psi_\varepsilon|^{p^*(x)}+2^{p^*_+}|u\psi_\varepsilon|^{p^*(x)}-|u_n\psi_\varepsilon-u\psi_\varepsilon|^{p^*(x)})\,dx\\
&=\int_{\mathbb{R}^N} 2^{p^*_++1}|u\psi_\varepsilon|^{p^*(x)}\,dx-\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n\psi_\varepsilon-u\psi_\varepsilon|^{p^*(x)}\,dx.
\end{aligned}
$$

Using (3.2), we have $\int_{\mathbb{R}^N}|u_n|^{p^*(x)}|\psi_\varepsilon|^{p^*(x)}\,dx\to\int_{\mathbb{R}^N}|u|^{p^*(x)}|\psi_\varepsilon|^{p^*(x)}\,dx$, thus

$$
\int_{\mathbb{R}^N}|u_n\psi_\varepsilon-u\psi_\varepsilon|^{p^*(x)}\,dx\to 0,
$$

as $n\to\infty$. Moreover, we derive

$$
\int_{B_{2R}} |u_n|^{p^*(x)}\psi_\varepsilon\,dx-\int_{B_{2R}} |u_n|^{p^*(x)-2}u_nu\psi_\varepsilon\,dx\to 0.
$$

Then

$$
\int_{\Omega_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\to 0.
$$

As in the proof of [6, Theorem 3.1], $\Omega_{R,\varepsilon}$ is divided into two parts:

$$
\Omega^1_{R,\varepsilon}=\{x\in\Omega_{R,\varepsilon}:p(x)<2\},\quad \Omega^2_{R,\varepsilon}=\{x\in\Omega_{R,\varepsilon}:p(x)\ge 2\}.
$$

On $\Omega^1_{R,\varepsilon}$, we obtain

$$
\begin{aligned}
&\int_{\Omega^1_{R,\varepsilon}}|\nabla u_n-\nabla u|^{p(x)}\,dx\\
&\le C\int_{\Omega^1_{R,\varepsilon}}\big((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\big)^{\frac{p(x)}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \big(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\big)^{\frac{2-p(x)}{2}}\,dx\\
&\le C\|\big((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\big)^{\frac{p(x)}{2}}\|_{\frac{2}{p(x)},\Omega^1_{R,\varepsilon}}\\
&\times \|(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)})^{\frac{2-p(x)}{2}}\|_{\frac{2}{2-p(x)},\Omega^1_{R,\varepsilon}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_{\Omega^1_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\\
&\le \int_{\Omega_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx,
\end{aligned}
$$

which implies

$$
\|\big((|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\big)^{p(x)/2}\|_{2/p(x),\Omega^1_{R,\varepsilon}}\to 0.
$$

As $\{u_n\}$ is bounded in $W^{1,p(x)}(\mathbb{R}^N)$, we obtain $\int_{\Omega^1_{R,\varepsilon}}|\nabla u_n-\nabla u|^{p(x)}\,dx\to 0$, as $n\to\infty$.

On $\Omega^2_{R,\varepsilon}$, we obtain

$$
\begin{aligned}
&\int_{\Omega^2_{R,\varepsilon}}|\nabla u_n-\nabla u|^{p(x)}\,dx\\
&\le C\int_{\Omega^2_{R,\varepsilon}}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\to 0,
\end{aligned}
$$

as $n\to\infty$. Thus, we obtain

$$
\int_{\Omega_{R,\varepsilon}}|\nabla u_n-\nabla u|^{p(x)}\,dx\to 0
$$

for any $R>R_0$, $0<2\varepsilon<\frac{d}{2}$. Moreover, up to a subsequence, we assume that $\nabla u_n\to\nabla u$ a.e. in $\mathbb{R}^N$.

(2) If J is empty. Let $\psi\in C_0^\infty(B_{2R})$ such that $0\le\psi\le 1$; $\psi\equiv 1$ in B_R, we obtain

$$
\begin{aligned}
0&\le\int_{B_R}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\\
&\le\int_{B_{2R}}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\psi\,dx.
\end{aligned}
$$

Similarly to (1), we obtain

$$
\int_{B_R}(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)(\nabla u_n-\nabla u)\,dx\to 0,
$$

as $n\to\infty$, which implies

$$
\int_{B_R}|\nabla u_n-\nabla u|^{p(x)}\,dx\to 0,
$$

for any $R>0$. Thus, we may assume that $\nabla u_n\to\nabla u$ a.e. in $\mathbb{R}^N$.

As $\{|\nabla u_n|^{p(x)-2}\nabla u_n\}$ is bounded in $(L^{p'(x)}(\mathbb{R}^N))^N$ and $|\nabla u_n|^{p(x)-2}\nabla u_n$ converges to $|\nabla u|^{p(x)-2}\nabla u$ a.e. in $\mathbb{R}^N$, we obtain

$$
|\nabla u_n|^{p(x)-2}\nabla u_n\to|\nabla u|^{p(x)-2}\nabla u\quad\text{weakly in }(L^{p'(x)}(\mathbb{R}^N))^N.
$$

Similarly, we obtain

$$|u_n|^{p(x)-2}u_n \to |u|^{p(x)-2}u \quad \text{weakly in } L^{p'(x)}(\mathbb{R}^N)$$

and

$$|u_n|^{p^*(x)-2}u_n \to |u|^{p^*(x)-2}u \quad \text{weakly in } L^{(p^*(x))'}(\mathbb{R}^N).$$

Thus, for any $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{gathered}
\int_{\mathbb{R}^N} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v \to \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx,\\
\int_{\mathbb{R}^N} |u_n|^{p(x)-2}u_n v \to \int_{\mathbb{R}^N} |u|^{p(x)-2}uv\, dx,\\
\int_{\mathbb{R}^N} |u_n|^{p^*(x)-2}u_n v \to \int_{\mathbb{R}^N} |u|^{p^*(x)-2}uv\, dx.
\end{gathered}$$

Note that

$$\langle \varphi'(u_n), v\rangle = \int_{\mathbb{R}^N} \big(|\nabla u_n|^{p(x)-2}\nabla u_n \nabla v + |u_n|^{p(x)-2}u_n v - |u_n|^{p^*(x)-2}u_n v - h(x)v\big)\, dx$$

and $\varphi'(u_n) \to 0$ in $W^{-1,p'(x)}(\mathbb{R}^N)$, as $n \to \infty$, we obtain

$$\begin{aligned}
\langle \varphi'(u), v\rangle &= \int_{\mathbb{R}^N} \big(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv - |u|^{p^*(x)-2}uv - h(x)v\big)\, dx\\
&= 0.
\end{aligned} \tag{3.5}$$

As p is Lipschitz continuous on $\mathbb{R}^N$, it follows that p satisfies the weak Lipschitz condition [18]. Thus, $C_0^\infty(\mathbb{R}^N)$ is dense on $W^{1,p(x)}(\mathbb{R}^N)$. Using (3.5), we obtain

$$\langle \varphi'(u), v\rangle = 0,$$

for any $v \in W^{1,p(x)}(\mathbb{R}^N)$; i.e. $\varphi'(u) = 0$. □

We remark that in the proof of Theorem 3.1, we use the inequality (2.2) in Theorem 2.7. As $p(x) \ll p^*(x)$, $p^*(x) - p(x) \geq (p^* - p)_- > 0$ for any $x \in \mathbb{R}^N$. Then, we avoided the assumption $p^*_- > p_+$ and obtained that the set of atoms J is empty or finite.

Next, using Theorem 3.1 we prove that there exists a critical point for φ. The following result of the variational functional φ is required by using Ekeland's variational principle.

Lemma 3.2. *There exist $\rho_0 > 0$, $h_0 > 0$ such that if $\|h\|_{p'(x)} \leq h_0$, we have $\varphi(u) > 0$ for any $u \in \{u \in W^{1,p(x)}(\mathbb{R}^N) : |||u||| = \rho_0\}$.*

Proof. For any $u \in W^{1,p(x)}(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
\varphi(u) &\geq \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{p_+} - \frac{|u|^{p^*(x)}}{(p^*)_-} - h(x)u\Big)\, dx\\
&= \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_+} - h(x)u\Big)\, dx\\
&\quad + \int_{\mathbb{R}^N} \Big(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_+} - \frac{|u|^{p^*(x)}}{(p^*)_-}\Big)\, dx.
\end{aligned}$$

As $p(x) \ll p^*(x)$ and $p(x)$ are Lipschitz continuous on $\mathbb{R}^N$, as in the proof of [6, Theorem 3.1], there exists a sequence of disjoint open N-cubes $\{Q_i\}_{i=1}^\infty$ with side $r > 0$ such that $\mathbb{R}^N = \cup_{i=1}^\infty \overline{Q_i}$,

$$p_{i+} \triangleq \sup_{x\in Q_i} p(x) < p^*_{i-} \triangleq \inf_{x\in Q_i} p^*(x),$$

and $p^*_{i-} - p_{i+} > \gamma \triangleq \frac{1}{2}\inf_{x\in\mathbb{R}^N}(p^*(x) - p(x))$, for $i = 1, 2, \dots$.

By [8, Corollary 8.3.2], there exists $r_0 = r_0(r, N, p_+, p_-) > 1$ independent of $i \in \mathbb{N}$ such that for any $v \in W^{1,p(x)}(Q_i)$, $\|v\|_{p^*(x)} \le r_0 \|\|v\|\|$. Then, for any $u \in W^{1,p(x)}(\mathbb{R}^N)$, we obtain $\|u\|_{p^*(x),Q_i} \le r_0\|\|u\|\|_{Q_i}$.

If $\|\|u\|\| \le r_0^{-1}$, then $\|\|u\|\|_{Q_i} \le \|\|u\|\| \le r_0^{-1}$, for any $i \in \mathbb{N}$. Thus, $\|u\|_{p^*(x),Q_i} \le 1$. Using Theorems 2.2 and 2.3 we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N}\Big(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_+} - \frac{|u|^{p^*(x)}}{(p^*)_-}\Big)\,dx &= \sum_{i=1}^\infty \int_{Q_i}\Big(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{2p_+} - \frac{|u|^{p^*(x)}}{(p^*)_-}\Big)\,dx \\
&\ge \sum_{i=1}^\infty \Big(\frac{\|\|u\|\|_{Q_i}^{p_{i+}}}{2p_+} - \frac{r_0^{p^*_{i-}}}{(p^*)_-}\|\|u\|\|_{Q_i}^{(p^*)_{i-}}\Big) \\
&\ge \sum_{i=1}^\infty \frac{\|\|u\|\|_{Q_i}^{p_{i+}}}{2p_+}\Big(1 - \frac{2p_+}{(p^*)_-} r_0^{p^*_{i-}}\|\|u\|\|_{Q_i}^{\gamma}\Big).
\end{aligned}$$

Denote $\rho_0 = \min\{r_0^{-1}, (\frac{2p_+}{(p^*)_-} r_0^{p^*_{i-}})^{-1/\gamma}\}$. If $\|\|u\|\| \le \rho_0$, then

$$\int_{\mathbb{R}^N}\Big(\frac{|\nabla u|^{p(x)}| + |u|^{p(x)}}{2} - |u|^{p^*(x)}\Big)\,dx \ge 0.$$

We obtain

$$\varphi(u) \ge \frac{\|\|u\|\|^{p_+}}{2p_+} - 2\|h\|_{p'(x)}\|u\|_{p(x)} \ge \frac{\|\|u\|\|^{p_+}}{2p_+} - C\|h\|_{p'(x)}\|\|u\|\|. \tag{3.6}$$

Thus, it suffices to take $\|h\|_{p'(x)}$ small enough. □

Then, using Ekeland's variational principle and Lemma 3.2, we obtain a Palais-Smale sequence for φ. Based on Theorem 3.1, we have the following result, which shows that φ has a critical if $\|h\|_{p'(x)}$ is small. Moreover, we obtain a nontrivial weak solution for (1.1).

Theorem 3.3. *If $\|h\|_{p'(x)} \le h_0$, there exists $u_0 \in \{u \in W^{1,p(x)}(\mathbb{R}^N) : \|\|u\|\| \le \rho_0\}$ such that u_0 is a weak solution of* (1.1)*, where ρ_0, h_0 are from Lemma 3.2.*

Proof. Denote

$$c_1 = \inf\{\varphi(u) : u \in W^{1,p(x)}(\mathbb{R}^N) \text{ with } \|\|u\|\| \le \rho_0\}.$$

It follows from (3.6) that $c_1 > -\infty$. Note that $h(x) \ge 0$ and $h(x) \not\equiv 0$, there exists $v \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h(x)v\,dx > 0$. Take $0 < s < 1$, we obtain

$$\begin{aligned}
\varphi(sv) &= \int_{\mathbb{R}^N}\big(\frac{|\nabla sv|^{p(x)} + |sv|^{p(x)}}{p(x)} - \frac{|sv|^{p^*(x)}}{p^*(x)} - h(x)sv\big)\,dx \\
&\le s^{p_-}\int_{\mathbb{R}^N}\frac{|\nabla v|^{p(x)} + |v|^{p(x)}}{p(x)}\,dx - s\int_{\mathbb{R}^N} h(x)v\,dx.
\end{aligned}$$

As $p_- > 1$, we have $\|\|sv\|\| < \rho_0$ and $\varphi(sv) < 0$, when s is sufficiently small. Thus $c_1 < 0$.

By Ekeland's variational principle, there exists $\{u_n\} \subset \{u \in W^{1,p(x)}(\mathbb{R}^N) : \|\|u\|\| \le \rho_0\}$ such that $\varphi(u_n) \to c_1$ and

$$\varphi(w) \ge \varphi(u_n) - \frac{1}{n}\|\|w - u_n\|\|, \tag{3.7}$$

for any $w \in W^{1,p(x)}(\mathbb{R}^N)$ with $\|\|w\|\| \le \rho_0$.

Since $c_1 < 0$, we assume that $\varphi(u_n) < 0$. It follows from Lemma 3.2 that $\|\|u_n\|\| < \rho_0$. Using (3.7), we obtain $\varphi'(u_n) \to 0$ in $W^{-1,p'(x)}(\mathbb{R}^N)$, as $n \to \infty$. As $\{u_n\}$ is bounded in $W^{1,p(x)}(\mathbb{R}^N)$, we assume that $u_n \to u_0$ weakly in $W^{1,p(x)}(\mathbb{R}^N)$, then $\|\|u_0\|\| \le \rho_0$. By Theorem 3.1, we obtain $\varphi'(u_0) = 0$. □

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Xia Zhang
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China.
Department of Mathematics, Pohang University of Science and Technology, Pohang, Korea
E-mail address: piecesummer1984@163.com

Yongqiang Fu
Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
E-mail address: fuyqhagd@yahoo.cn