Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 122, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE SYMMETRIC SOLUTIONS FOR A SINGULAR SEMILINEAR ELLIPTIC PROBLEM WITH CRITICAL EXPONENT

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Abstract. Let be Γ a closed subgroup of $O(N)$. We consider the semilinear elliptic problem

$$-\Delta u - \frac{b(x)}{|x|^2}u - a(x)u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 4$. We establish the multiplicity of symmetric positive solutions, nodal solutions, and solutions which are Γ invariant but are not $\widetilde{\Gamma}$ invariant, where $\Gamma \subset \widetilde{\Gamma} \subset O(N)$.

1. Introduction

We consider the singular semilinear elliptic problem with critical nonlinearity,

$$\begin{gathered} -\Delta u - b(x)\frac{u}{|x|^2} - a(x)u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 4)$ is a smooth bounded domain, $0 \in \Omega$, $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, and f, a, b are continuous real function defined on $\mathbb{R}^N$, $f > 0$ on $\overline{\Omega}$, $0 < b(x) < \overline{\mu} := (\frac{N-2}{2})^2$ for all $x \in \overline{\Omega}$, and $0 < \max_{\overline{\Omega}} a(x) < \lambda_{1,b}$ where $\lambda_{1,b}$ is the first Dirichlet eigenvalue of $-\Delta - \frac{b_0}{|x|^2}$ on Ω with $b_0 := \max_{\overline{\Omega}} b(x)$.

Some previous works about this problem, are as follows:

When $a(x) = \lambda$, $b(x) = 0$ and $f(x) = 1$, problem (1.1) has been studied by many authors [2, 15, 13, 6, 5]. In [3] the authors proved for $b(x) = 0$ a multiplicity sign changing result where a and f are continuous functions. Jannelli [12] investigate the problem with $b(x) = \mu \in [0, \overline{\mu} - 1]$, $f(x) = 1$ and $a(x) = \lambda \in (0, \lambda_1)$ where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω and got the existence of nontrivial positive solution. Cao and Peng [4] proved the existence of a pair of sign changing solutions for $N \geq 7$, $b(x) = \mu \in [0, \overline{\mu} - 4]$, $a(x) = \lambda \in (0, \lambda_1)$, and $f(x) = 1$. For $a(x) = \lambda$ and $b(x) = \mu$, Han and Liu [11] proved the existence of one non trivial solution.

2000 *Mathematics Subject Classification.* 35J75, 35J57, 35J60.
Key words and phrases. Critical exponent; singular problem; symmetric solutions.

Submitted March 27, 2012. Published July 20, 2012.
This work was presented in the II Joint meeting RSME-SMM, 2012, Málaga, España.

Guo and Niu [9] proved the existence of a symmetric nodal solution and a positive solution for $a(x) = \lambda \in (0, \lambda_1)$, where λ_1 is the first Dirichlet eigenvalue of $-\Delta - \frac{\mu}{|x|^2}$ on Ω, with $b(x) = \mu$, Ω and f invariant under a subgroup of $O(N)$, this result was generalized by Guo, Niu, Cui [10] changing the term $a(x)u$ by a function depend on x and u, both proofs was based on previous work by Smets [16].

2. Statement of results

We write again the partial differential equations to consider

$$\begin{gathered} -\Delta u - b(x)\frac{u}{|x|^2} - a(x)u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(\gamma x) = u(x) \quad \forall x \in \Omega,\ \gamma \in \Gamma. \end{gathered} \tag{2.1}$$

In this problem the symmetries are given by Γ a closed subgroup of orthogonal transformation $O(N)$. We suppose Ω a Γ-invariant smooth bounded domain in $\mathbb{R}^N$ such that $0 \in \Omega$, and $N \geq 4$. The critical Sobolev exponent is given by $2^* := \frac{2N}{N-2}$. The functions a, b and f are Γ-invariant continuous real valued defined on $\mathbb{R}^N$, with the following additional hypothesis, $0 < a(x) < \lambda_{1,b}$, where $\lambda_{1,b}$ is the first Dirichlet eigenvalue of $-\Delta - \frac{b_0}{|x|^2}$, where $b_0 = \max_{\bar\Omega} b(x)$ and $0 < b(x) < \overline{\mu} := (\frac{N-2}{2})^2$. We note that $\lambda_{1,b}$ depends of the domain of $-\Delta - \frac{b_0}{|x|^2}$.

Let $\Gamma x := \{\gamma x : \gamma \in \Gamma\}$ be the Γ-orbit of a point $x \in \mathbb{R}^N$, and $\#\Gamma x$ its cardinality, and denote by $X/\Gamma := \{\Gamma x : x \in X\}$ the Γ-orbit space of $X \subset \mathbb{R}^N$ with the quotient topology.

Let us recall that the least energy solutions of

$$\begin{gathered} -\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \to 0 \quad \text{as } |x| \to \infty \end{gathered} \tag{2.2}$$

are the instantons given by Aubin and Talenti (see [1, 18].)

$$U_0^{\varepsilon,y}(x) := C(N)\Big(\frac{\varepsilon}{\varepsilon^2 + |x-y|^2}\Big)^{(N-2)/2}, \tag{2.3}$$

where $C(N) = (N(N-2))^{(N-2)/4}$. Is well known that if the domain is not $\mathbb{R}^N$, there is no minimal energy solutions of (2.2). These solutions are minimizers for

$$S := \min_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*} dx\big)^{2/2^*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Similarly, for $0 < b(0) < \overline{\mu}$, the critical problem

$$\begin{gathered} -\Delta u - b(0)\frac{u}{|x|^2} = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N \\ u \to 0 \quad \text{as } |x| \to \infty, \end{gathered} \tag{2.4}$$

was studied by Terracini [20] and gives the solutions

$$U_{b(0)}(x) := C_{b(0)}(N)\Big(\frac{\varepsilon}{\varepsilon^2|x|^{(\sqrt{\overline{\mu}}-\sqrt{\overline{\mu}-b(0)})/\sqrt{\overline{\mu}}}+|x|^{(\sqrt{\overline{\mu}}+\sqrt{\overline{\mu}-b(0)})/\sqrt{\overline{\mu}}}}\Big)^{(N-2)/2},$$

where $\varepsilon > 0$ and $C_{b(0)}(N) = (\frac{4N(\overline{\mu}-b(0))}{N-2})^{(N-2)/4}$. In this case the solutions are minimizers for

$$S_{b(0)} := \min_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 - b(0)\frac{u^2}{|x|^2})dx}{\big(\int_{\mathbb{R}^N}|u|^{2^*}dx\big)^{2/2^*}}.$$

In the following we denote by

$$M := \{y\in\overline{\Omega} : \frac{\#\Gamma y}{f(y)^{(N-2)/2}} = \min_{x\in\overline{\Omega}}\frac{\#\Gamma x}{f(x)^{(N-2)/2}}\}.$$

We shall assume that f, a, and b satisfy:

(F1) $f(x) > 0$ for all $x\in\overline{\Omega}$ and $f(0) = 1$.
(F2) f is locally flat at M; that is, there exist $r > 0$, $\nu > N$ and $A > 0$ such that

$$|f(x) - f(y)| \le A|x-y|^{\nu} \quad \text{if } y\in M \text{ and } |x-y| < r.$$

(B1) $0 < b(x) < \overline{\mu}$ for all $x\in\overline{\Omega}$, We denote by $b_0 := \max_{\overline{\Omega}} b(x)$.
(A1) If $a_0 := \max_{\overline{\Omega}} a(x)$ it must hold $0 < a_0 < \lambda_{1,b}$, where $\lambda_{1,b}$ denote the first eigenvalue of $-\Delta - \frac{b_0}{|x|^2}$.
(A2) $a(x) > 0$ for all $x\in M$.

With the above conditions we define

$$\langle u,v\rangle_{a,b} := \int_\Omega\Big(\nabla u\cdot\nabla v - b(x)\frac{uv}{|x|^2} - a(x)uv\Big)dx$$

which is an inner product in $H_0^1(\Omega)$ and its induced norm is

$$\|u\|_{a,b} := \sqrt{\langle u,u\rangle_{a,b}} = \Big(\int_\Omega(|\nabla u|^2 - b(x)\frac{u^2}{|x|^2} - a(x)u^2)dx\Big)^{1/2}.$$

Using the Hardy inequality,

$$\int_\Omega \frac{u^2}{|x|^2}dx \le \frac{1}{\overline{\mu}}\int_\Omega|\nabla u|^2dx, \quad \forall u\in H_0^1(\Omega), \tag{2.5}$$

we will prove the equivalence of the norms $\|u\|_{a,b}$ and $\|u\| := \|u\|_{0,0}$ in $H_0^1(\Omega)$. Since $\lambda_{1,b}$ is the first eigenvalue of $-\Delta - \frac{b_0}{|x|^2}$ on $H_0^1(\Omega)$,

$$\int_\Omega a_0|u|^2dx \le \frac{a_0}{\lambda_{1,b}}\int_\Omega\Big(|\nabla u|^2 - b_0\frac{u^2}{|x|^2}\Big)dx. \tag{2.6}$$

Therefore,

$$\begin{aligned}\|u\|_{a,b}^2 &:= \int_\Omega \Big(|\nabla u|^2 - b(x)\frac{u^2}{|x|^2} - a(x)|u|^2|Big)dx \\ &\geq \int_\Omega \Big(|\nabla u|^2 - b_0\frac{u^2}{|x|^2}\Big)dx - \frac{a_0}{\lambda_{1,b}}\int_\Omega (|\nabla u|^2 - b_0\frac{u^2}{|x|^2}) \\ &\geq (1-\frac{a_0}{\lambda_{1,b}})\int_\Omega \Big(|\nabla u|^2 - b_0\frac{u^2}{|x|^2}\Big)dx, \quad \text{and by (2.5)} \\ &\geq (1-\frac{a_0}{\lambda_{1,b}})(1-\frac{b_0}{\overline{\mu}})\int_\Omega |\nabla u|^2 dx \\ &= (1-\frac{a_0}{\lambda_{1,b}})(1-\frac{b_0}{\overline{\mu}})\|u\|^2.\end{aligned} \tag{2.7}$$

The other inequality holds since $0 < a_0 < \lambda_{1,b}$, implies $a_1 = \min_{\overline{\Omega}} a(x) \leq a_0 < \lambda_{1,b} < \lambda_1$ where λ_1 denote the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$; therefore,

$$\begin{aligned}\|u\|_{a,b}^2 &\leq \int_\Omega (|\nabla u|^2 - a(x)|u|^2)dx \\ &\leq \int_\Omega |\nabla u|^2 dx - \frac{a_1}{\lambda_1}\int_\Omega |\nabla u|^2 dx, \\ &\leq (1-\frac{a_1}{\lambda_1})\int_\Omega |\nabla u|^2 dx\,.\end{aligned}$$

If $f \in C(\overline{\Omega})$ and (F1) is satisfied, then the norms

$$|u|_{2^*} := (\int_\Omega |u|^{2^*}dx)^{1/2^*}, \quad \text{and} \quad |u|_{f,2^*} := (\int_\Omega f(x)|u|^{2^*}dx)^{1/2^*}$$

are equivalent. We denote

$$\ell_f^\Gamma := \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)S.$$

We will use the following non existence assumption.

(A3) The problem

$$\begin{gathered}-\Delta u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(\gamma x) = u(x) \quad \forall x \in \Omega,\ \gamma \in \Gamma\end{gathered} \tag{2.8}$$

does not have a positive solution u which satisfies $\|u\|^2 \leq \ell_f^\Gamma$.

If Ω is a smooth starshaped domain is well known that (A3) is satisfied [19].

2.1. **Multiplicity of positive solutions.** Our next result generalizes the work of Guo and Niu [9] for problem (2.1) and establishes a relationship between the topology of the domain and the multiplicity of positive solutions. For $\delta > 0$ let

$$M_\delta^- := \{y \in M : \mathsf{dist}(y, \partial\Omega) \geq \delta\},\ B_\delta(M) := \{z \in \mathbb{R}^N : \mathsf{dist}(z, M) \leq \delta\}. \tag{2.9}$$

Theorem 2.1. *Let $N \geq 4$, (A1), (A2), (B1), (F1), (F2), (A3) and $\ell_f^\Gamma \leq S_{b(0)}^{N/2}$ hold. Given $\delta, \delta' > 0$ there exist $\lambda^* \in (0, \lambda_{1,b})$, $\mu^* \in (0, \overline{\mu})$ such that for all $a(x) \in (0, \lambda^*)$, $b(x) \in (0, \mu^*)$ $\forall x \in \Omega$ the problem (2.1) has at least*

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$$

positive solutions which satisfy

$$\ell_f^\Gamma - \delta' \le \|u\|_{a,b}^2 < \ell_f^\Gamma.$$

2.2. **Multiplicity of nodal solutions.** Let G be a closed subgroup of $O(N)$ for which Ω and $f : \mathbb{R}^N \to \mathbb{R}$ are G-invariant. We denote by Γ the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2 := \{-1, 1\}$.

A real valued function u defined in Ω will be called τ-equivariant if

$$u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G.$$

In this section we study the problem

$$\begin{gathered} -\Delta u - b(x)\frac{u}{|x|^2} - a(x)u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G \end{gathered} \tag{2.10}$$

If $g \in \Gamma$ then all τ-equivariant functions u satisfy $u(gx) = u(x)$ for all $x \in \Omega$; i.e., are Γ-invariant. If u is a τ-equivariant function and $g \in \tau^{-1}(-1)$ then $u(gx) = -u(x)$ for all $x \in \Omega$. Thus all non trivial τ-equivariant solution of (2.10) change sign.

Definition 2.2. A subset X of $\mathbb{R}^N$ is Γ-connected if it is a Γ-invariant subset X of $\mathbb{R}^N$ and if cannot be written as the union of two disjoint open Γ-invariant subsets. A real valued function $u : \Omega \to \mathbb{R}$ is $(\Gamma, 2)$-nodal if the sets

$$\{x \in \Omega : u(x) > 0\} \quad \text{and} \quad \{x \in \Omega : u(x) < 0\}$$

are nonempty and Γ-connected.

For each G-invariant subset X of $\mathbb{R}^N$, we define

$$X^\tau := \{x \in X : Gx = \Gamma x\}.$$

Let $\delta > 0$, define

$$M_{\tau,\delta}^- := \{y \in M : \operatorname{dist}(y, \partial\Omega \cup \Omega^\tau) \ge \delta\},$$

and $B_\delta(M)$ as in (2.9).

The next theorem is a multiplicity result for τ-equivariant $(\Gamma, 2)$-nodal solutions for (2.1).

Theorem 2.3. *Let $N \ge 4$, (A1), (A2), (B1), (F1), (F2), (A3), and $\ell_f^\Gamma \le S_{b(0)}^{N/2}$ hold. If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$ defined on a closed subgroup G of $O(N)$ for which Ω and the functions a, b, f are G-invariant. Given $\delta, \delta' > 0$ there exists $\lambda^* \in (0, \lambda_{1,b})$, $\mu^* \in (0, \overline{\mu})$ such that for all $a(x) \in (0, \lambda^*)$, $b(x) \in (0, \mu^*)$ for all $x \in \Omega$ problem (2.1) has at least*

$$\operatorname{cat}_{(B_\delta(M) \setminus B_\delta(M)^\tau)/G}(M_{\tau,\delta}^-/G)$$

pairs $\pm u$ of τ-equivariants $(\Gamma, 2)$-nodal solutions which satisfy

$$2\ell_f^\Gamma - \delta' \le \|u\|_{a,b}^2 < 2\ell_f^\Gamma.$$

2.3. **Non symmetric properties for solutions.** Let $\Gamma \subset \widetilde{\Gamma} \subset O(N)$. Next we give sufficient conditions for the existence of many solutions which are Γ-invariant but are not $\widetilde{\Gamma}$-invariant.

Theorem 2.4. *Let $N \geq 4$, (A1), (A2), (B1), (F1), (F2), (A3), and $\ell_f^\Gamma \leq S_{b(0)}^{N/2}$ hold. Let $\widetilde{\Gamma}$ be a closed subgroup of $O(N)$ containing Γ, for which Ω and the functions a, b, f are $\widetilde{\Gamma}$-invariant and*

$$\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}} < \min_{x\in\overline{\Omega}} \frac{\#\widetilde{\Gamma} x}{f(x)^{(N-2)/2}}.$$

Given $\delta, \delta' > 0$ there exist $\lambda^* \in (0, \lambda_{1,b})$, $\mu^* \in (0, \overline{\mu})$ such that for all $a(x) \in (0, \lambda^*)$, $b(x) \in (0, \mu^*)$ for all $x \in \Omega$ problem (2.1) has at least

$$\operatorname{cat}_{B_\delta(M)/\Gamma}(M_\delta^-/\Gamma)$$

positive solutions which are not $\widetilde{\Gamma}$-invariant and satisfy

$$2\ell_f^\Gamma - \delta' \leq \|u\|_{a,b}^2 < 2\ell_f^\Gamma.$$

3. The variational problem

To generalize the notation we introduce a homomorphism $\tau : G \to \mathbb{Z}/2$ defined on a closed subgroup G of $O(N)$. Recall the problem (2.10),

$$\begin{gathered} -\Delta u - b(x)\frac{u}{|x|^2} - a(x)u = f(x)|u|^{2^*-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ u(gx) = \tau(g)u(x) \quad \forall x \in \Omega,\ g \in G, \end{gathered}$$

where Ω is a G-invariant bounded smooth subset of $\mathbb{R}^N$, and a, b, and f are a G-invariant continuous functions which satisfy (A1), (A2), (B1), (F1) and (F2).

Let $\Gamma := \ker \tau$. If τ is not an epimorphism then the problems (2.10) and (2.1) coincide. In the other case we obtain solutions for the problem (2.10) and in particular are sign changing solutions of (2.1).

The homomorphism τ induces the natural action of G on $H_0^1(\Omega)$ given by

$$(gu)(x) := \tau(g)u(g^{-1}x).$$

Due the symmetries, the solutions are in the fixed point space of the action or the space of τ-equivariant functions

$$\begin{aligned} H_0^1(\Omega)^\tau &:= \{u \in H_0^1(\Omega) : gu = u \ \forall g \in G\} \\ &= \{u \in H_0^1(\Omega) : u(gx) = \tau(g)u(x)\ g \in G,\ \forall x \in \Omega\}. \end{aligned}$$

The fixed point space of the restriction of this action to Γ

$$H_0^1(\Omega)^\Gamma = \{u \in H_0^1(\Omega) : u(gx) = u(x)\ \forall g \in \Gamma,\ \forall x \in \Omega\}$$

are the Γ-invariant functions of $H_0^1(\Omega)$. The norms $\|\cdot\|_{a,b}$, $\|\cdot\|$ on $H_0^1(\Omega)$ and $|\cdot|_{2^*}$, $|\cdot|_{f,2^*}$ on $L^{2^*}(\Omega)$ are G-invariant with respect to the action induced by τ; therefore the functional

$$\begin{aligned} E_{a,b,f}(u) &:= \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 - a(x)\frac{u^2}{|x|^2} - b(x)|u|^2\Big)dx - \frac{1}{2^*}\int_\Omega f(x)|u|^{2^*}dx \\ &= \frac{1}{2}\|u\|_{a,b}^2 - \frac{1}{2^*}|u|_{f,2^*}^{2^*} \end{aligned}$$

is G-invariant, with derivative

$$DE_{a,b,f}(u)v = \int_\Omega \Big(\nabla u\cdot\nabla v - b(x)\frac{uv}{|x|^2} - a(x)uv\Big)dx - \int_\Omega f(x)|u|^{2^*-2}uv\,dx.$$

By the principle of symmetric criticality [14], the critical points of its restriction to $H_0^1(\Omega)^\tau$ are the solutions of (2.10), and all non trivial solutions lie on the Nehari manifold

$$\begin{aligned}\mathcal{N}^\tau_{a,b,f} &:= \{u\in H_0^1(\Omega)^\tau : u\neq 0, DE_{a,b,f}(u)u = 0\}\\ &= \{u\in H_0^1(\Omega)^\tau : u\neq 0, \|u\|^2_{a,b} = |u|^{2^*}_{f,2^*}\}.\end{aligned}$$

which is of class C^2 and radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\tau$ by the radial projection

$$\pi_{a,b,f} : H_0^1(\Omega)^\tau\setminus\{0\}\to\mathcal{N}^\tau_{a,b,f}\quad \pi_{a,b,f}(u) := (\frac{\|u\|^2_{a,b}}{|u|^{2^*}_{f,2^*}})^{(N-2)/4}u.$$

Therefore, the nontrivial solutions of (2.10) are precisely the critical points of the restriction of $E_{a,b,f}$ to $\mathcal{N}^\tau_{a,b,f}$. If $\tau\equiv 1$ we write $\mathcal{N}^\Gamma_{a,b,f}$.

An easy computation gives

$$E_{a,b,f}(u) = \frac{1}{N}\|u\|^2_{a,b} = \frac{1}{N}|u|^{2^*}_{f,2^*}\quad \forall u\in\mathcal{N}^\tau_{a,b,f} \tag{3.1}$$

and

$$E_{a,b,f}(\pi_{a,b,f}(u)) = \frac{1}{N}(\frac{\|u\|^2_{a,b}}{|u|^2_{f,2^*}})^{N/2}\quad \forall u\in H_0^1(\Omega)^\tau\backslash\{0\}.$$

We define

$$\begin{aligned}m(a,b,f) &:= \inf_{\mathcal{N}_{a,b,f}} E_{a,b,f}(u) = \inf_{\mathcal{N}_{a,b,f}}\frac{1}{N}\|u\|^2_{a,b}\\ &= \inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{1}{N}(\frac{\|u\|^2_{a,b}}{|u|^2_{f,2^*}})^{N/2}.\end{aligned}$$

In the restrictions for the Nehari manifolds we denote by

$$m^\Gamma(a,b,f) := \inf_{\mathcal{N}^\Gamma_{a,b,f}} E_{a,b,f},\quad m^\tau(a,b,f) := \inf_{\mathcal{N}^\tau_{a,b,f}} E_{a,b,f}.$$

3.1. **Estimates for the infimum.** From the definition of Nehari Manifold and (3.1) we obtain that $m^\Gamma(a,b,f)>0$.

Proposition 3.1. *Let $a(x)\le a'(x)<\lambda_{1,b}$, $b(x)\le b'(x)<\bar\mu$, for all $x\in\bar\Omega$, and $f:\mathbb{R}^N\to\mathbb{R}$, with the conditions above. Then*

$$m(a',b',f)\le m(a,b,f),\quad m^\Sigma(a',b',f)\le m^\Sigma(a,b,f),$$

with $\Sigma=\Gamma$ or $\Sigma=\tau$.

Proof. By definition of $\|\cdot\|_{a,b}$ we obtain $\|u\|^2_{a',b'}\le\|u\|^2_{a,b}$. Let $u\in H_0^1(\Omega)\setminus\{0\}$, then

$$\begin{aligned}m(a',b',f) &\le E_{a',b',f}(\pi_{a',b',f}(u))\\ &= \frac{1}{N}\Big(\frac{\|u\|^2_{a',b'}}{|u|^2_{f,2^*}}\Big)^{N/2}\end{aligned}$$

$$\leq \frac{1}{N}\Big(\frac{\|u\|_{a,b}^2}{|u|_{f,2^*}^2}\Big)^{N/2} = E_{a,b,f}(\pi_{a,b,f}(u)),$$

and from this inequality, the conclusion follows. □

We denote by $\lambda_{1,b}$ the first Dirichlet eigenvalue of $-\Delta - \frac{b_0}{|x|^2}$ in $H_0^1(\Omega)$.

Lemma 3.2. *With the conditions (a_1) and (b), for $u \in H_0^1(\Omega)^\tau$, we obtain*

$$E_{0,0,f}(\pi_{0,0,f}(u)) \leq \Big(\frac{\bar\mu}{\bar\mu - b_0}\Big)^{N/2}\big(\frac{\lambda_{1,b}}{\lambda_{1,b}-a_0}\big)^{N/2} E_{a,b,f}(\pi_{a,b,f}(u)).$$

Proof. Since

$$E_{a,b,f}(\pi_{a,b,f}(u)) = \frac{1}{N}\Big(\frac{\|u\|_{a,b}^2}{|u|_{f,2^*}^2}\Big)^{N/2} = \frac{1}{N}\Big(\frac{\|u\|_{a,b}^N}{|u|_{f,2^*}^N}\Big),$$

by (2.7) we have

$$\big(1-\frac{a_0}{\lambda_{1,b}}\big)^{N/2}\big(1-\frac{b_0}{\bar\mu}\big)^{N/2}\|u\|^N \leq \|u\|_{a,b}^N$$

then

$$\big(1-\frac{a_0}{\lambda_{1,b}}\big)^{N/2}\big(1-\frac{b_0}{\bar\mu}\big)^{N/2}\frac{1}{N}\frac{\|u\|^N}{|u|_{f,2^*}^N} \leq E_{a,b,f}(\pi_{a,b,f}(u))$$

so

$$E_{0,0,f}(\pi_{0,0,f}(u)) \leq \big(\frac{\bar\mu}{\bar\mu - b_0}\big)^{N/2}\big(\frac{\lambda_{1,b}}{\lambda_{1,b}-a_0}\big)^{N/2} E_{a,b,f}(\pi_{a,b,f}(u)),$$

which completes the proof. □

Corollary 3.3. $m^\tau(0,0,f) \leq (\frac{\bar\mu}{\bar\mu-b_0})^{N/2}(\frac{\lambda_{1,b}}{\lambda_{1,b}-a_0})^{N/2} m^\tau(a,b,f)$.

For the proof of the next lemma we refer the reader to [3].

Lemma 3.4. *If $\Omega\cap M \neq \emptyset$ then: (a) $m^\Gamma(0,0,f) \leq \frac{1}{N}\ell_f^\Gamma$. (b) If there exists $y \in \Omega \cap M$ with $\Gamma x \neq Gy$, then $m^\tau(0,0,f) \leq \frac{2}{N}\ell_f^\Gamma$.*

3.2. **A compactness result.**

Definition 3.5. *A sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying*

$$E_{a,b,f}(u_n) \to c \quad \text{and} \quad \nabla E_{a,b,f}(u_n) \to 0.$$

is called a Palais-Smale sequence for $E_{a,b,f}$ at c. We say that $E_{a,b,f}$ satisfies the Palais-Smale condition $(PS)_c$ if every Palais-Smale sequence for $E_{a,b,f}$ at c has a convergent subsequence. If $\{u_n\} \subset H_0^1(\Omega)^\tau$ then $\{u_n\}$ is a τ-equivariant Palais-Smale sequence and $E_{a,b,f}$ satisfies the τ-equivariant Palais-Smale condition, $(PS)_c^\tau$. If $\tau \equiv 1$ $\{u_n\}$ is a Γ-invariant Palais-Smale sequence and $E_{a,b,f}$ satisfies the Γ-invariant Palais-Smale condition $(PS)_c^\Gamma$.

To describe the τ-equivariant Palais-Smale sequence for $E_{a,b,f}$ we use the next theorem proved by Guo and Niu [9]. which is based on results of Struwe [17].

Theorem 3.6. *Let (u_n) be a τ-equivariant Palais-Smale sequence in $H_0^1(\Omega)^\tau$ for $E_{a,b,f}$ at $c \geq 0$. Then there exist a solution u of (2.10), m, $l \in \mathbb{N}$; a closed subgroup G^i of finite index in G, sequences $\{y_n^i\} \subset \Omega$, $\{r_n^i\} \subset (0,\infty)$, a solution $\widehat{u}_0^i$ of (2.2) for $i = 1,\dots,m$; and $\{R_n^j\} \subset (0,\infty)$, a solution $\widehat{u}_b^j$ of (2.4) for $j = 1,\dots,l$. Such that*

(i) $G_{y_n^i} = G^i$,
(ii) $(r_n^i)^{-1}\operatorname{dist}(y_n^i, \partial\Omega) \to \infty$, $y_n^i \to y^i$, *if* $n \to \infty$, *for* $i = 1, \dots, m$,
(iii) $(r_n^i)^{-1}|gy_n^i - g'y_n^i| \to \infty$, *if* $n \to \infty$, *and* $[g] \neq [g'] \in G/G^i$ *for* $i = 1, \dots, m$,
(iv) $\widehat{u}_0^i(gx) = \tau(g)\widehat{u}_0^i(x)$, $\forall x \in \mathbb{R}^N$ *and* $g \in G^i$,
(v) $\widehat{u}_b^j(gx) = \tau(g)\widehat{u}_b^j(x)$, *for all* $x \in \mathbb{R}^N$ *and* $g \in G$, $R_n^j \to 0$ *for* $j = 1, \dots, l$,
(vi)
$$\begin{aligned} u_n(x) = u(x) + \sum_{i=1}^m \sum_{[g]\in G/G^i} (r_n^i)^{(2-N)/2} f(y^i)^{(2-N)/4} \\ \times \tau(g)\widehat{u}_0^i\big(g^{-1}\big(\frac{x - gy_n^i}{r_n^i}\big)\big) + \sum_{j=1}^l (R_n^j)^{\frac{2-N}{2}} \widehat{u}_b^j\big(\frac{x}{R_n^j}\big) + o(1) \end{aligned}$$
(vii) $E_{a,b,f}(u_n) \to E_{a,b,f}(u) + \sum_{i=1}^m \big(\frac{\#(G/G^i)}{f(y^i)^{(N-2)/2}}\big) E^\infty_{0,0,1}(\widehat{u}_0^i) + \sum_{j=1}^l E^\infty_{0,b(0),1}(\widehat{u}_b^j)$, *as* $n \to \infty$

Corollary 3.7. $E_{a,b,f}$ *satisfies* $(PS)_c^\tau$ *at every value*
$$c < \min\big\{\#(G/\Gamma)\frac{\ell_f^\Gamma}{N}, \frac{\#(G/\Gamma)}{N} S_{b(0)}^{N/2}\big\}.$$

Proof. From the inequality of the value c and the part (vii) of the theorem, we obtain that m and l are equal to zero. The convergence follows from (vi). $\square$

4. The bariorbit map

In the following we suppose the condition $\ell_f^\Gamma \leq S_{b(0)}^{N/2}$ hold and we will assume the next nonexistence condition.

(NE) The infimum of $E_{0,0,f}$ is not achieved in $\mathcal{N}_{0,0,f}^\Gamma$.

With these conditions, Corollary 3.7 and Lemma 3.4 imply that
$$m^\Gamma(0,0,f) := \inf_{\mathcal{N}_{0,0,f}^\Gamma} E_{0,0,f} = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{1}{N} S^{N/2}. \tag{4.1}$$
Let
$$M := \{y \in \overline{\Omega} : \frac{\#\Gamma y}{f(y)^{(N-2)/2}} = \min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\}.$$
For every $y \in \mathbb{R}^N$, $\gamma \in \Gamma$, the isotropy subgroups satisfy $\Gamma_{\gamma y} = \gamma\Gamma_y\gamma^{-1}$. Therefore the set of isotropy subgroups of Γ-invariant subsets consists of complete conjugacy classes. We choose $\Gamma_i \subset \Gamma$, $i = 1, .., m$, one in each conjugacy class of an isotropy subgroup of M. Set
$$\begin{gathered} M^i := \{y \in M : \Gamma_y = \Gamma_i\} = \{y \in M : \gamma y = y \ \forall \gamma \in \Gamma_i\}, \\ \Gamma M^i := \{\gamma y : \gamma \in \Gamma,\ y \in M^i\} = \{y \in M : (\Gamma_y) = (\Gamma_i)\}. \end{gathered}$$
By definition of M it follows that f is constant on each ΓM^i, then we can define
$$f_i := f(\Gamma M^i) \in \mathbb{R}.$$
The compactness of M allows us to fix $\delta_0 > 0$ such that
$$\begin{gathered} |y - \gamma y| \geq 3\delta_0 \quad \forall y \in M,\ \gamma \in \Gamma \text{ if } \gamma y \neq y, \\ \operatorname{dist}(\Gamma M^i, \Gamma M^j) \geq 3 \quad \forall i, j = 1, \dots, m \text{ if } i \neq j, \end{gathered} \tag{4.2}$$

and such that the isotropy subgroup of each point in $M^i_{\delta_0} := \{z \in \mathbb{R}^N : \gamma z = z \forall \gamma \in \Gamma_i, \operatorname{dist}(z, M^i) \le \delta_0\}$ is precisely Γ_i. Define

$$W_{\varepsilon,z} := \sum_{[g]\in\Gamma/\Gamma_i} f_i^{\frac{2-N}{4}} U_{\varepsilon,gz} \quad \text{if } z \in M^i_{\delta_0},$$

where $U_{\varepsilon,y} := U_0^{\varepsilon,y}$ is defined by (2.3). For each $\delta \in (0, \delta_0)$ define

$$\begin{gathered} M_\delta := M^1_\delta \cup \dots \cup M^m_\delta, \\ B_\delta := \{(\varepsilon, z) : \varepsilon \in (0, \delta),\ z \in M_\delta\}, \\ \Theta_\delta := \{\pm W_{\varepsilon,z} : (\varepsilon, z) \in B_\delta\}, \qquad \Theta_0 := \Theta_{\delta_0}. \end{gathered}$$

We mention the next result proved in [3] about the construction of bariorbit maps.

Proposition 4.1. *Let $\delta \in (0, \delta_0)$, and assume that* (NE) *holds. There exists $\eta > m^\Gamma(0, 0, f)$ with following properties: For each $u \in \mathcal{N}^\Gamma_{0,0,f}$ such that $E_{0,0,f}(u) \le \eta$ we have*

$$\inf_{W\in\Theta_0} \|u - W\| < \sqrt{\frac{1}{2} N m^\Gamma(0, 0, f)},$$

and there exist precisely one $\nu \in \{-1, 1\}$, one $\varepsilon \in (0, \delta_0)$ and one Γ-orbit $\Gamma z \subset M_{\delta_0}$ such that

$$\|u - \nu W_{\varepsilon,z}\| = \inf_{W\in\Theta_0} \|u - W\|.$$

Moreover $(\varepsilon, z) \in B_\delta$

4.1. **Definition of the bariorbit map.** Fix $\delta \in (0, \delta_0)$ and choose $\eta > m^\Gamma(0, 0, f)$ as in Proposition 4.1. Define

$$\begin{gathered} E^\eta_{0,0,f} := \{u \in H^1_0(\Omega) : E_{0,0,f}(u) \le \eta\}, \\ B_\delta(M) := \{z \in \mathbb{R}^N : \operatorname{dist}(z, M) \le \delta\}, \end{gathered}$$

and the space of Γ-orbits of $B_\delta(M)$ by $B_\delta(M)/\Gamma$. From Proposition 4.1 we have the following definition.

Definition 4.2. The *bariorbit map* $\beta^\Gamma : \mathcal{N}^\Gamma_{0,0,f} \cap E^\eta_{0,0,f} \to B_\delta(M)/\Gamma$ is defined by

$$\beta^\Gamma(u) = \Gamma y \overset{def}{\Longleftrightarrow} \|u \pm W_{\varepsilon,y}\| = \min_{W\in\Theta_0} \|u - W\|.$$

This map is continuous and $\mathbb{Z}/2$-invariant by the compactness of M_δ. If Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, choose $g_\tau \in \tau^{-1}(-1)$. Let $u \in \mathcal{N}^\tau_{0,0,f}$ then u changes sign and $u^-(x) = -u^+(g_\tau^{-1}x)$. Therefore, $\|u^+\|^2 = \|u^-\|^2$ and $|u^+|^{2^*}_{f,2^*} = |u^-|^{2^*}_{f,2^*}$. So

$$u \in \mathcal{N}^\tau_{0,0,f} \Rightarrow u^\pm \in \mathcal{N}^\Gamma_{0,0,f} \text{ and } E_{0,0,f}(u) = 2E_{0,0,f}(u^\pm). \tag{4.3}$$

Lemma 4.3. *$E_{0,0,f}$ does not achieve its infimum at $\mathcal{N}^\tau_{0,0,f}$, moreover*

$$m^\tau(0, 0, f) := \inf_{\mathcal{N}^\tau_{0,0,f}} E_{0,0,f} = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) \frac{2}{N} S^{N/2} = 2m^\Gamma(0, 0, f).$$

Proof. Suppose that there exists $u \in \mathcal{N}^\tau_{0,0,f}$ such that $E_{0,0,f}(u) = m^\tau(0, 0, f)$. Then $u^+ \in \mathcal{N}^\Gamma_{0,0,f}$ and by Lemma 3.4,

$$m^\tau(0, 0, f) \le \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big) \frac{2}{N} S^{N/2}.$$

Hence

$$m^\Gamma(0,0,f) \le E_{0,0,f}(u^+) = \frac{1}{2} m^\tau(0,0,f) \le \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{\frac{N-2}{2}}}\Big)\frac{1}{N} S^{N/2} = m^\Gamma(0,0,f).$$

Thus u^+ is a minimum of $E_{0,0,f}$ on $\mathcal{N}^\Gamma_{0,0,f}$, which contradicts (NE). The corollary 3.7 implies

$$m^\tau(0,0,f) = \Big(\min_{x\in\overline{\Omega}} \frac{\#\Gamma x}{f(x)^{(N-2)/2}}\Big)\frac{2}{N} S^{N/2}.$$

□

The property (4.3) implies $u^\pm \in \mathcal{N}^\Gamma_{0,0,f} \cap E^\eta_{0,0,f}$ for all $u \in \mathcal{N}^\tau_{0,0,f} \cap E^{2\eta}_{0,0,f}$, so

$$\|u^+ - \nu W_{\varepsilon,y}\| = \min_{W\in\Theta_0} \|u^+ - W\| \Leftrightarrow \|u^- + \nu W_{\varepsilon,g_\tau y}\| = \min_{W\in\Theta_0} \|u^- - W\|. \quad (4.4)$$

Therefore,

$$\beta^\Gamma(u^+) = \Gamma y \Longleftrightarrow \beta^\Gamma(u^-) = \Gamma(g_\tau y), \quad (4.5)$$

and

$$\beta^\Gamma(u^+) \ne \beta^\Gamma(u^-) \quad \forall u \in \mathcal{N}^\tau_{0,0,f} \cap E^{2\eta}_{0,0,f}. \quad (4.6)$$

Set $B_\delta(M)^\tau := \{z \in B_\delta(M) : Gz = \Gamma z\}$.

Proposition 4.4. *The map*

$$\beta^\tau : \mathcal{N}^\tau_{0,0,f} \cap E^{2\eta}_{0,0,f} \to (B_\delta(M) \setminus B_\delta(M)^\tau)/\Gamma, \quad \beta^\tau(u) := \beta^\Gamma(u^+),$$

is well defined, continuous and $\mathbb{Z}/2$-equivariant; i.e., $\beta^\tau(-u) = \Gamma(g_\tau y)$ if and only if $\beta^\tau(u) = \Gamma y$.

Proof. If $u \in \mathcal{N}^\tau_{0,0,f} \cap E^{2\eta}_{0,0,f}$ and $\beta^\tau(u) = \Gamma y \in B_\delta(M)^\tau/\Gamma$ then $\beta^\Gamma(u^+) = \Gamma y = \Gamma(g_\tau y) = \beta^\Gamma(u^-)$, this is a contradiction to (4.6). We conclude that $\beta^\tau(u) \notin B_\delta(M)^\tau/\Gamma$. The continuity and $\mathbb{Z}/2$-equivariant properties follows by β^Γ ones. □

5. Multiplicity of solutions

5.1. **Lusternik-Schnirelmann theory.** An involution on a topological space X is a map $\varrho_X : X \to X$, such that $\varrho_X \circ \varrho_X = id_X$. Providing X with an involution amounts to defining an action of $\mathbb{Z}/2$ on X and viceversa. The trivial action is given by the identity $\varrho_X = id_X$, the action of $G/\Gamma \simeq \mathbb{Z}/2$ on the orbit space $\mathbb{R}^N/\Gamma$ where $G \subset O(N)$ and Γ is the kernel of an epimorphism $\tau : G \to \mathbb{Z}/2$, and the antipodal action $\varrho(u) = -u$ on $\mathcal{N}^\tau_{a,b,f}$. A map $f : X \to Y$ is called $\mathbb{Z}/2$-equivariant (or a $\mathbb{Z}/2$-map) if $\varrho_Y \circ f = f \circ \varrho_X$, and two $\mathbb{Z}/2$-maps, $f_0, f_1 : X \to Y$, are said to be $\mathbb{Z}/2$-homotopic if there exists a homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\varrho_X x, t) = \varrho_Y \Theta(x,t)$ for every $x \in X$, $t \in [0,1]$. A subset A of X is $\mathbb{Z}/2$-equivariant if $\varrho_X a \in A$ for every $a \in A$.

Definition 5.1. The $\mathbb{Z}/2$-category of a $\mathbb{Z}/2$-map $f : X \to Y$ is the smallest integer $k := \mathbb{Z}/2 - \mathrm{cat}(f)$ with following properties

- (i) There exists a cover of $X = X_1 \cup \ldots \cup X_k$ by k open $\mathbb{Z}/2$-invariant subsets,
- (ii) The restriction $f|_{X_i} : X_i \to Y$ is $\mathbb{Z}/2$-homotopic to the composition $\kappa_i \circ \alpha_i$ of a $\mathbb{Z}/2$-map $\alpha_i : X_i \to \{y_i, \varrho_Y y_i\}$, $y_i \in Y$, and the inclusion $\kappa_i : \{y_i, \varrho_Y y_i\} \hookrightarrow Y$.

If not such covering exists, we define $\mathbb{Z}/2 - \mathrm{cat}(f) := \infty$.

If A is a $\mathbb{Z}/2$-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion we write

$$\mathbb{Z}/2 - \operatorname{cat}_X(A) := \mathbb{Z}/2 - \operatorname{cat}(\iota), \quad \mathbb{Z}/2 - \operatorname{cat}_X(X) := \mathbb{Z}/2 - \operatorname{cat}(X).$$

Note that if $\varrho_x = \mathrm{id}_X$ then

$$\mathbb{Z}/2 - \operatorname{cat}_X(A) := \operatorname{cat}_X(A), \quad \mathbb{Z}/2 - \operatorname{cat}(X) := \operatorname{cat}(X),$$

are the usual Lusternik-Schnirelmann category (see [21, definition 5.4]).

Theorem 5.2. *Let $\phi : M \to \mathbb{R}$ be an even functional of class C^1, and M a submanifold of a Hilbert space of class C^2, symmetric with respect to the origin. If ϕ is bounded below and satisfies $(PS)_c$ for each $c \le d$, then ϕ has at least $\mathbb{Z}/2$-cat(ϕ^d) pairs critical points such that $\phi(u) \le d$ (see* [8]*).*

5.2. **Proof of Theorems.** We prove Theorem 2.3; the proof of Theorem 2.1 is analogous. Recall that if τ is the identity or an epimorphism then $\#(G/\Gamma)$ is 1 or 2.

Proof of Theorem 2.3. By Corollary 3.7, $E_{a,b,f}$ satisfies $(PS)^\tau_\theta$ for

$$\theta < \min\big\{\#(G/\Gamma)\frac{\ell^\Gamma_f}{N}, \frac{\#(G/\Gamma)}{N}S^{N/2}_{b(0)}\big\}.$$

By Lusternik-Schnirelmann theory $E_{a,b,f}$ has at least $\mathbb{Z}/2$-cat$(\mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f})$ pairs $\pm u$ of critical points in $\mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f}$. We are going to estimate this category for an appropriate value of θ.

Without lost of generality we can assume that $\delta \in (0, \delta_0)$, with δ_0 as in (4.2). Let $\eta > \frac{\ell^\Gamma_f}{N}$, $\mu^* \in (0, \overline{\mu})$ and $\lambda^* \in (0, \lambda_{1,b})$ such that

$$(\frac{\bar\mu}{\bar\mu - \mu^*})^{N/2}(\frac{\lambda_{1,b}}{\lambda_{1,b} - \lambda^*})^{N/2} = \min\{2, \frac{N\eta}{\#(G/\Gamma)\ell^\Gamma_f}, \frac{\ell^\Gamma_f}{\ell^\Gamma_f - \delta'}\}.$$

By Lemma 3.2, if $u \in \mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f}$, $b_0 \in (0, \mu^*)$, $a_0 \in (0, \lambda^*)$ we have

$$\begin{aligned} E_{0,0,f}(\pi_{0,0,f}(u)) &\le (\frac{\bar\mu}{\bar\mu - b_0})^{N/2}(\frac{\lambda_{1,b}}{\lambda_{1,b} - a_0})^{N/2} E_{a,b,f}(u) \\ &< (\frac{\bar\mu}{\bar\mu - b_0})^{N/2}(\frac{\lambda_{1,b}}{\lambda_{1,b} - a_0})^{N/2} \#(G/\Gamma)\frac{\ell^\Gamma_f}{N} \\ &\le \#(G/\Gamma)\eta. \end{aligned}$$

Let β^τ be the τ-bariorbit function, defined in Proposition 4.4. Hence the composition map

$$\beta^\tau \circ \pi_{0,0,f} : \mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f} \to (B_\delta(M) \setminus B_\delta(M)^\tau)/\Gamma,$$

is a well defined $\mathbb{Z}/2$-invariant continuous function.

Since $N \ge 4$, by [3, Lemma 3 and Proposition 3], using (F2) we can choose $\varepsilon > 0$ small enough and $\theta := \theta_\varepsilon < \#(G/\Gamma)\frac{\ell^\Gamma_f}{N}$ such that

$$E_{a,b,f}(\pi_{a,b,f}(w^\tau_{\varepsilon,y})) \le \theta < \#(G/\Gamma)\frac{\ell^\Gamma_f}{N}, \quad \forall y \in M^-_\delta,$$

where $w^\tau_{\varepsilon,y} = w^\Gamma_{\varepsilon,y} - w^\Gamma_{\varepsilon,g_\tau y}$, $\tau(g_\tau) = -1$, and

$$w^\Gamma_{\varepsilon,y}(x) = \sum_{[\gamma]\in\Gamma/\Gamma_y} f(y)^{\frac{2-N}{4}} U_{\varepsilon,\gamma y}(x)\varphi_{\gamma y}(x).$$

Thus the map $\alpha^\tau_\delta : M^-_{\tau,\delta}/\Gamma \to \mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f}$, defined by

$$\alpha^\tau_\delta(\Gamma y) := \pi_{a,b,f}(w^\tau_{\varepsilon,y})$$

is a well defined $\mathbb{Z}/2$-invariant continuous function. Moreover $\beta^\tau(\pi_{0,0,f}(\alpha^\tau_\delta(\Gamma y))) = \Gamma y$ for all $y \in M^-_{\tau,\delta}$. Therefore,

$$\mathbb{Z}/2 - \operatorname{cat}(\mathcal{N}^\tau_{a,b,f} \cap E^\theta_{a,b,f}) \geq \operatorname{cat}_{((B_\delta(M)\setminus B_\delta(M)^\tau)/\Gamma)}(M^-_{\tau,\delta}/\Gamma).$$

So (2.10) has at least

$$\operatorname{cat}_{((B_\delta(M)\setminus B_\delta(M)^\tau)/G)}(M^-_{\tau,\delta}/G)$$

pairs $\pm u$ solution which satisfy

$$E_{a,b,f}(u) < \#(G/\Gamma)\frac{\ell^\Gamma_f}{N}.$$

By the choice of λ^* and μ^* we have that

$$(\frac{\bar\mu}{\bar\mu-\mu^*})^{N/2}(\frac{\lambda_1}{\lambda_1-\lambda^*})^{N/2} \leq \frac{\ell^\Gamma_f}{\ell^\Gamma_f - \delta'},$$

then

$$\begin{aligned}
\#(G/\Gamma)\frac{\ell^\Gamma_f-\delta'}{N} &\leq (\frac{\bar\mu-b_0}{\bar\mu})^{N/2}(\frac{\lambda_1-a_0}{\lambda_1})^{N/2}\#(G/\Gamma)\frac{\ell^\Gamma_f}{N}\\
&\leq m^\tau(a,b,f) \leq E_{a,b,f}(u)\\
&= \frac{1}{N}\|u\|^2_{a,b} < \#(G/\Gamma)\frac{\ell^\Gamma_f}{N}
\end{aligned}$$

therefore, $\#(G/\Gamma)\ell^\Gamma_f - \delta'' \leq \|u\|^2_{a,b} < \#(G/\Gamma)\ell^\Gamma_f$. □

Proof of Theorem 2.4. By Theorem 2.1 there exist λ and μ sufficiently close to zero such that (2.1) has at least $\operatorname{cat}_{B_\delta(M)/\Gamma}(M^-_\delta/\Gamma)$ positive solutions with $E_{a,b,f}(u) < \frac{\ell^\Gamma_f}{N}$.

Observe that $\frac{\ell^\Gamma_f}{N} < m^{\widetilde\Gamma}(0,0,f)$. Indeed, if $m^{\widetilde\Gamma}(0,0,f)$ is not achieved then by the hypothesis $m^{\widetilde\Gamma}(0,0,f) = \frac{\ell^{\widetilde\Gamma}_f}{N} > \frac{\ell^\Gamma_f}{N}$. On the other hand if $u \in \mathcal{N}^{\widetilde\Gamma}_{0,0,f} \subset \mathcal{N}^\Gamma_{0,0,f}$ satisfies $E_{0,0,f}(u) = m^{\widetilde\Gamma}(0,0,f)$ we obtain

$$\frac{\ell^\Gamma_f}{N} = m^\Gamma(0,0,f) < m^{\widetilde\Gamma}(0,0,f) = E_{0,0,f}(u).$$

By Corollary 3.3, there exist $\widehat\lambda \in (0,\lambda_1)$ and $\widehat\mu \in (0,\bar\mu)$ such that for each $\lambda \in (0,\widehat\lambda)$ and $\mu \in (0,\widehat\mu)$ such that

$$\frac{\ell^\Gamma_f}{N} < m^{\tilde\Gamma}(0,0,f) \leq (\frac{\lambda_1}{\lambda_1-\lambda})^{N/2}(\frac{\overline\mu}{\overline\mu-\mu})^{N/2} m^{\tilde\Gamma}(a,b,f).$$

Then

$$E_{a,b,f}(u) < \frac{\ell^\Gamma_f}{N} < m^{\tilde\Gamma}(a,b,f).$$

Therefore u is not $\tilde\Gamma$-invariant solution. □

Acknowledgements. The authors wish to thank the Instituto de Matemáticas of the Universidad Nacional Autónoma de México for hospitality during 2012 which contributed essentially to this research.

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