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ON PACARD'S REGULARITY FOR THE EQUATION $-\Delta u = u^p$

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Abstract. It is shown that the singular set for a positive solution of the PDE $-\Delta u = u^p$ has Hausdorff dimension less than or equal to $n-2p'$, as conjectured by Pacard [12] in 1993.

1. Results

This note concerns the open question mentioned by Pacard in[12], especially its regularity criterion for positive weak solutions to $-\Delta u = u^p$ in a domain $\Omega \subset \mathbb{R}^n$, $p \geq n/(n-2)$, $n \geq 3$. By this we shall mean: $u \in L^p_{\rm loc}(\Omega)$ and

$$-\int \Delta\phi \cdot u\, dx = \int u^p \phi\, dx \tag{1.1}$$

for all $\phi \in C_0^\infty(\Omega)$. The main question here is to describe the size of the set $\mathrm{Sing}(u) \subset \Omega$ where a solution u becomes $+\infty$ and such that $u \in C^\infty(\Omega \setminus \mathrm{Sing}(u))$. Examples where such a set exists includes the simple case $u(x) = c_0|\bar{x}|^{-2/(p-1)}$, $x = (\bar{x}, \hat{x})$, $\bar{x} \in \mathbb{R}^{n-d}$, $\hat{x} \in \mathbb{R}^d$, a solution in the ball $B(0,R)$, centered at zero of radius R, and some constant c_0. Here $\mathrm{Sing}(u) = \mathbb{R}^d \cap B(0,R)$ and necessarily $d < n - 2p'$, $p' = p/(p-1)$. Note that when $p = n/(n-2)$, it is well known that (1.1) can have isolated singularities (here $d = 0$; see [8]). Furthermore, $n - 2p' = 0$ when $p = n/(n-2)$, because then $p' = n/2$. The case $p = (n+2)/(n-2)$, the "Yamabe case," has been also well studied in the literature; see [14]. And several authors have constructed solutions to (1.1) with a prescribed singular set $\mathrm{Sing}(u)$; e.g. [13], [6], [10]. But in all cases, it appears that solutions u to (1.1) behave like

$$u(y) \sim \mathrm{dist}(y, \mathrm{Sing}(u))^{-2/(p-1)} \tag{1.2}$$

as $y \to \mathrm{Sing}(u)$ in Ω.

The Pacard conjecture is that the Hausdorff dimension of $\mathrm{Sing}(u)$ is always $\leq n-2p'$, which certainly appears to be the case in all the examples considered. Pacard proves this, in [12], under an additional hypothesis, his hypothesis "H". However, it soon becomes clear that hypothesis H is much too strong, for it precludes isolated singularities when $p = n/(n-2)$, and for that matter any singularities when $n/(n-2) \leq p < (n/(n-2)) + \varepsilon$, for some $\varepsilon > 0$.

Thus the purpose of this note is to prove:

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Theorem 1.1. *Let u be a positive weak solution of* (1.1)*, then there exists an open set $\Omega' \subset \Omega$ such that $u \in C^\infty(\Omega')$ and $C_{2,p'}(\Omega \setminus \Omega') = 0$.*

The presentation of this note follows closely that of [12], so it is recommended that the reader have a copy of [12] at hand while reading the present note.

Here $C_{\alpha,p}(\cdot)$ is the capacity set function associated with the Sobolev space $W^{\alpha,p}(\Omega)$, α = positive integer. Also, one recalls from [2] that any set of $C_{2,p'}$-capacity zero has Hausdorff dimension $\le n-2p'$. Furthermore, it is not surprising that $\mathrm{Sing}(u) = \Omega\backslash\Omega'$ is of $C_{2,p'}$-capacity zero, given that this condition characterizes removable sets for equation (1.1); see [4].

For $p' < n/2$, we can use the standard definition of $C_{2,p'}$ using Riesz potentials on $\mathbb{R}^n$ especially when $\partial\Omega$ = boundary of Ω is smooth. For any compact $K \subset \mathbb{R}^n$

$$C_{2,p'}(K) = \inf\{\|f\|_{L^{p'}}^{p'} : f \ge 0, I_2 f \ge 1 \text{ on } K\}.$$

Here

$$I_2 f(x) = \int_{R^n} |x-y|^{2-n} f(y)\,dy.$$

Notice this definition easily implies

$$C_{2,p'}(\{x\colon I_2 f \ge \lambda\}) \le \frac{1}{\lambda^p} \cdot \|f\|_{L^{p'}}^{p'}. \tag{1.3}$$

The proof of our Theorem constitutes the main body of this note, 1-6. In 7, 8 and 9, we include further speculations.

1. If $u = u(x)$ is a positive weak solution to (1.1), then u belongs to the Morrey space $L^{p,2p'}(\Omega)$.

Proof. (This result is due to Pacard [11], and it has also been observed by Brezis.) The Morrey space in question — here we extend functions outside Ω by zero — is those $f \in L^p_{\rm loc}(\mathbb{R}^n)$ such that

$$\Big(\sup_{x\in\mathbb{R}^n,\, r>0} r^{\lambda-n} \int_{B(x,r)} |f(y)|^p\,dy\Big)^{1/p} \equiv \|f\|_{L^{p,\lambda}} < \infty,$$

for $1 \le p < \infty$, $0 < \lambda \le n$. Again, recall that we will only be dealing with the case $p' < n/2$. The case $p' = n/2$ can be handled using the usual modifications; see [2].

So now set $\phi(x) = \eta\left(\frac{x-x_0}{r}\right)^\sigma$, $\eta \in C_0^\infty(B(0,1))$ for $\sigma > 2p'$. Then

$$\int u^p \eta^\sigma \le \frac{C}{r^2}\Big(\int u^p \eta^\sigma\Big)^{1/p} \cdot r^{n/p'} \tag{1.4}$$

by Hölder's inequality. The result follows. □

2. A modified Pacard Lemma [12]:

Lemma 1.2. *Let u be a positive weak solution of* (1.1)*, then there are constants c_p such that for $x \in \Omega$ and r small*

$$\fint_{B(x,r)} u^p \le c_p\Big\{\Big(\fint_{B(x,2r)} u^{p-1}\Big)^{p'} + \fint_{B(x,2r)} u(y)^p \Big(\int_{B(y,2r)} |y-z|^{2-n} u(z)^{p-1}\,dz\Big)dy\Big\} \tag{1.5}$$

for $p \ge 2$, and

$$\fint_{B(x,r)} u^p \le c_p\Big\{\Big(\fint_{B(x,2r)} u\Big)^{p} + \fint_{B(x,2r)} u(y)^p \Big(\int_{B(y,2r)} |y-z|^{2-n} u(z)^{p-1}\,dz\Big)dy\Big\} \tag{1.6}$$

for $1 < p < 2$.

Here, the integrals with a bar denote integral averages.

Proof. (Outline from [12].) Inequalities (1.5) and (1.6) follow from the following inequality for positive weak solutions to (1.1); see [12] or [9]:

$$u(y) \leq \fint_{B(y,r)} u + \frac{r^n}{n(n-2)} \fint_{B(y,r)} |y-z|^{2-n} u(z)^p \, dz. \tag{1.7}$$

To get our result, simply multiply (1.7) through by u^{p-1} and integrate over a ball centered at x of radius r. □

This Lemma is important for at least two reasons:
(a) If the quantity

$$\int_{B(y,R)} |y-z|^{2-n} u(z)^{p-1} \, dz \tag{1.8}$$

can be made uniformly small for R small and all y in some neighborhood of $x \in \Omega$, then (1.5) or (1.6) can be used to engage the theory of reverse Hölder inequalities; see [7] or [5]. In each case, one can then deduce that $u \in L^q$ in that neighborhood of x, where $q > p$. This, it turns out, is the crucial step in proving C^∞-regularity in that neighborhood. We return to this below in section 6.

(b) It is less than intuitive that the potential $I_2 u^{p-1}$ (or some part of it) should play a significant role here in describing the pointwise behavior of u near $\mathrm{Sing}(u)$ in Ω. One expects $u = I_2 u^p$ to be of some service here but not $I_2 u^{p-1}$. Notice that the section 1 result plus the embeddings of Morrey spaces under the Riesz potential operator I_2 imply that $I_2 u^{p-1} \in BMO$, the John-Nirenberg space of functions of bounded mean oscillations; see [2] or [1]. This fact alone suggests that $\exp(c \cdot I_2 u^{p-1})$ might be of interest here. We speculate further on this in section 8.

Notice that $u(x) = cI_2 u^p(x)$ in Ω for some constant c, hence

$$I_2 u^{p-1} = cI_2 (I_2 u^p)^{p-1}.$$

This is precisely the classical non-linear potential from [2]; i.e., for (α, p): $I_\alpha (I_\alpha \mu)^{p'-1}$, when $\alpha = 2$, and p' is replaced by p, and the measure $d\mu = u^p \, dx$.

3. $I_2 u^{p-1}(x) < \infty$ implies

$$\lim_{r \to 0} r^{2p'-n} \int_{B(x,r)} u(y)^p \, dy = 0. \tag{1.9}$$

Proof. This follows from a fundamental estimate from Nonlinear Potential Theory; see [2] or [3]. The estimate is for the so-called "nonlinear potentials" associated with the capacities $C_{2,p'}$:

$$I_2 (I_2 u^p)^{p-1}(x) \geq c \cdot W_{2,p'}^{u^p \, dy}(x), \tag{1.10}$$

where the W-potential here is the associated Wolff potential

$$W_{\alpha,p}^{\mu}(x) \equiv \int_0^\infty [r^{\alpha p - n} \mu(B(x,r))]^{p'-1} \frac{dr}{r},$$

for $0 < \alpha < n$, $1 < p < n/\alpha$, and $\mu =$ non-negative Borel measure on $\mathbb{R}^n$. In (1.10), $d\mu = u^p \, dy$. Our result follows since both $r^{2p'-n}$ and $\int_{B(x,r)} u^p$ are monotone functions of r. It should perhaps be added here that the reverse inequality to (1.10) may fail for $p > 2(n-1)/(n-2)$; see [3]. □

4. $\xi_u(x) =$ the jump discontinuity of $I_2 u^{p-1}$ at x when $I_2 u^{p-1}(x) < \infty$.

Proof. Here we compute

$$\overline{\lim_{y\to x}} I_2 u^{p-1}(y) = \xi_u(x) + I_2 u^{p-1}(x)$$

where

$$\xi_u(x) \equiv \overline{\lim_{y\to x}} (n-2) \int_0^{|x-y|} r^{2-n} \Big(\int_{B(y,r)} u^{p-1} \Big) \frac{dr}{r}. \tag{1.11}$$

Notice that $\xi_u(x) = 0$, when u is continuous at x. In fact, Fubini's theorem gives

$$I_2 u^{p-1}(y) = (n-2) \int_0^{\infty} r^{2-n} \Big(\int_{B(y,r)} u^{p-1} \Big) \frac{dr}{r}. \tag{1.12}$$

And writing (1.12) as $\big(\int_0^{|x-y|} \dots + \int_{|x-y|}^{\infty} \big)(n-2)$, we easily see that the last integral tends to $I_2 u^{p-1}(x)$ as $y \to x$ since $B(y,r) \subset B(x,2r)$ and $I_2 u^{p-1}(x) < \infty$ allows us to use dominated convergence. Hence the result follows. Note that we also have

$$\xi_u(x) = \overline{\lim_{y\to x}} \int_{|y-z|<|x-y|} |y-z|^{2-n} u(z)^{p-1}\, dz \tag{1.13}$$

since

$$\lim_{r\to 0} r^{2-n} \int_{B(x,r)} u(y)^{p-1}\, dy = 0$$

follows from $I_2 u^{p-1}(x) < \infty$. □

Thus the jump discontinuity $\xi_u(x)$ is generally ≥ 0 for $x \in \operatorname{Sing}(u)$. But notice that $\xi_\varphi(x) = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^n)$.

5. $C_{2,p'}(\operatorname{Sing}(u)) = 0$.

Proof. Here we set

$$\operatorname{Sing}_\lambda(u) = \{x \in \Omega \colon \xi_u(x) \geq \lambda\}. \tag{1.14}$$

And for $\operatorname{Sing}(u)$ needed in our Theorem, we take λ in (1.14) to be $1/(4c_p)$, c_p the constant in the Pacard Lemma (section 2). Now if $x \in \operatorname{Sing}(u)$, then for any $y \in N(x) \cap \operatorname{Sing}(u)$, $N(x) =$ some neighborhood of x,

$$\lambda \leq \xi_u(x) \leq c I_2(|u^{p-1} - \varphi|)(y) + \lambda/2,$$

hence

$$C_{2,p'}(N(x) \cap [I_2(|u^{p-1} - \varphi|) > \lambda/2]) \leq \big(\frac{2}{\lambda}\big)^{p'} \|u^{p-1} - \varphi\|^{p'}_{L^{p'}(\Omega)}.$$

So taking φ to be an $L^{p'}$ smooth approximation to u^{p-1} yields $C_{2,p'}(N(x) \cap \operatorname{Sing}(u)) = 0$ and the final result follows due to the countable subadditivity of $C_{2,p'}$; see [2]. □

6. Deducing $u \in C^\infty(\Omega \setminus \operatorname{Sing}(u))$. (Here we follow the path forged by Pacard [12].)

Proof. The reason for our choice of $\lambda = 1/(4c_p)$ above now becomes clear: for $x \in \Omega - \operatorname{Sing}_\lambda(u)$, (1.8) then does not exceed $1/(2c_p)$ for some $R > 0$ and all y in a neighborhood of x. This together with the modified Pacard Lemma yields that $u \in L^q$ in that neighborhood of x, for some $q > p$ by the reverse Hölder inequality theory mentioned earlier. We are now in position to use Lemmas 4 and 5 from [12]. Using (1.9), we have:

there exists constant $\theta \in (0,1)$ such that

$$\frac{1}{(\theta R)^{n-2p'}} \int_{B(x,\theta R)} u^p \le \frac{1}{2} \frac{1}{R^{n-2p'}} \int_{B(x,R)} u^p. \tag{1.15}$$

Iterating (1.15) yields: for such x as above

$$\frac{1}{(\theta^k R_1)^{n-2p'}} \int_{B(x,\theta^k R_1)} u^p \le 2^{-k} \frac{1}{R_1^{n-2p'}} \int_{B(x,R_1)} u^p \tag{1.16}$$

for all $k \in \mathbb{Z}^+$. Now one can choose a $\mu < 2p'$ such that $\theta^{2p'-\mu} > 1/2$ and derive that in fact in this neighborhood of x that $u \in L^{p,\mu}$ (note that the notation here differs from that in [12], a fact we prefer). And now, as in [12], we can easily get $u \in C^\infty$ in this neighborhood since $\mu < 2p'$. □

7. We mention a simple regularity criterion that can be used, for example, to get $u \in C^\infty$ in all of Ω: if $u \in L^{n(p-1)/2,\lambda}(\Omega)$ for some $\lambda < n$, then, in fact, $u \in C^\infty(\Omega)$. This might be stated as a corollary to the main theorem, for one immediately sees that this condition implies that $\xi_u(x) = 0$ for all $x \in \Omega$; i.e., $I_2 u^{p-1}$ is continuous on Ω and our theory implies then that $u \in C^\infty(\Omega)$. Notice that this condition also implies that there are no bounded point discontinuities for u in Ω (a fact well known), but this then confirms that indeed $\operatorname{Sing}(u)$ is made up of points where $u(y) \to +\infty$ as $y \to \operatorname{Sing}(u)$. And that agrees, of course, with (1.2).

8. A conjecture seems to now be in order: there is a function $\beta(x) > 0$ such that for all $x \in [I_2 u^{p-1} = +\infty]$

$$u(y) \sim \exp\big(\beta(x) I_2 u^{p-1}(y)\big) \tag{1.17}$$

as $y \to x \in \operatorname{Sing}(u)$. Since $I_2 u^{p-1} = I_2 (I_2 u^p)^{p-1}$ and the equivalence of this nonlinear potential with the Wolff potential, at least for $p < \frac{1}{2}(\frac{n-1}{n-2})$, we expect $\beta(x)$ to be something like

$$\frac{2}{p-1} \frac{1}{D(x)^{p-1}} \tag{1.18}$$

where $D(x) = \underline{\lim}_{r\to 0} r^{2p'-n} \int_{B(x,r)} u^p$, $x \in [I_2 u^{p-1} = +\infty]$, by comparing this with the examples where (1.2) holds.

9. A further conjecture is that one can prove our Theorem for $-\Delta$ replaced by the differential operator $L = -\sum_{i,j} (a_{ij} u_{x_i})_{x_j} + cu$ studied in [4].

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