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EXISTENCE OF POSITIVE SOLUTIONS TO THREE-POINT ϕ-LAPLACIAN BVPS VIA HOMOTOPIC DEFORMATIONS

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Abstract. Under suitable conditions and via homotopic deformation, we provide existence results for a positive solution to the three-point ϕ-Laplacian boundary-value problem

$$\begin{gathered} -(a\phi(u'))'(x) = b(x)f(x,u(x)), \quad x \in (0,1), \\ u(0) = \alpha u(\eta), \quad u'(1) = 0, \end{gathered}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, b does not vanish identically, and f is continuous.

1. Introduction

We are interested in the existence of a positive solution to the three-point boundary-value problem

$$\begin{gathered} -(a\phi(u'))'(x) = b(x)f(x,u(x)), \quad x \in (0,1), \\ u(0) = \alpha u(\eta), \quad u'(1) = 0, \end{gathered} \tag{1.1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, $\alpha, \eta \in [0,1)$, $a, b \in C([0,1], [0,+\infty))$, $a > 0$ in $[0,1]$, b does not vanish identically, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous.

Because of their physical applications, the study of ϕ-Laplacian second-order differential equations subject to various boundary conditions have received a great deal of attention during the latter two decades; see [1]-[13], [15]-[18] and references therein. The differential operator in all of the cited papers, corresponds to the case where a is identically equal to 1. When seeking a positive solution when the nonlinearity positivity is guaranteed, authors are frequently led to using Krasnoselskii's compression and expansion of a cone principal to prove existence of a fixed point for some completely continuous operator $T : K \to K$ where K is a cone in some functional Banach space. For example, if we want use Krasnoselskii's theorem on norm compression and expansion of a cone, we may look for $0 < R_1, R_2$ such that $\|Tu\| \le \|u\|$ for all $u \in K \cap \partial B(0,R_1)$ and $\|Tu\| \ge \|u\|$ for all $u \in K \cap \partial B(0,R_2)$, where $B(0,R)$ denotes the open ball centered at 0 and having radius R. The realization of the second inequality often requires a special cone left invariant by T; see the cone considered in [1] and [2] where a is identically equal to 1 and the cone

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K_p considered in Section 2 for the case $\phi = \phi_p$. But a such cone does not exist for general ϕ and a. To overcome this difficulty we use an homotopy deformation on the differential operator in (1.1), and we obtain existence results.

In this article, ψ is the inverse function of ϕ, and for $p > 1$, $\phi_p(x) = |x|^{p-2}x$ and $\psi_p = \phi_p^{-1}$.

We will use the following lemmas concerning computations of the fixed point index, i, for a compact map $A : B(0,R) \cap K \to K$ where K is a cone in a Banach space E.

Lemma 1.1. *If $\|Ax\| < \|x\|$ for all $x \in \partial B(0,R) \cap K$, then*
$$i(A, B(0,R) \cap K, K) = 1\,.$$

Lemma 1.2. *If $\|Ax\| > \|x\|$ for all $x \in \partial B(0,R) \cap K$, then*
$$i(A, B(0,R) \cap K, K) = 0\,.$$

An elaborate presentation of the fixed point index theory can be found in [14]. In what follows, we let E be the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm, for $u \in E$, $\|u\| = \sup\{|u(t)| : t \in [0,1]\}$. K is the normal cone of nonnegative functions in E, $K = \{u \in E : u(t) \geq 0, t \in [0,1]\}$.

2. Related lemmas

Let $N : E \to E$ be defined for $u \in E$ by
$$Nu(x) = \frac{\alpha}{1-\alpha}\int_0^\eta \psi\Big(\frac{1}{a(t)}\int_t^1 b(s)\phi(u(s))ds\Big)dt + \int_0^x \psi\Big(\frac{1}{a(t)}\int_t^1 b(s)\phi(u(s))ds\Big)dt,$$
$F : K \to K$, the Nemitski operator defined for $u \in K$ by $Fu(x) = \psi(f(x, u(x)))$, and $T = NF$.

When $\phi = \phi_p$ with $p > 1$, ψ, N and T are denoted, respectively, ψ_p, N_p and T_p.

It is easy to see that N is completely continuous (by the Ascoli-Arzela theorem), that F is bounded (maps bounded sets into bounded sets), and that u is a positive solution to (1.1) if and only if u is a nontrivial fixed point to the completely continuous operator $T = NF$.

For $p > 1$, the set $K_p = \{u \in K : u(x) \geq \rho_p(x)\|u\| \text{ in } [0,1]\}$ is a cone in E where
$$\rho_p(x) = \frac{1}{\overline{\rho}}\int_0^x \frac{dt}{\psi_p(a(t))}, \quad \overline{\rho} = \int_0^1 \frac{dt}{\psi_p(a(t))}.$$

Lemma 2.1. *For all $p > 1$, $T_p(K) \subset K_p$.*

Proof. Let $u \in K$, $v = T_p u$ and set $w = v - \rho_p\|v\|$. We have that v is nondecreasing on $[0,1]$ and $\|v\| = v(1)$. Indeed, from $(a\phi_p(u'))' = -b(t)f(t,u(t)) \leq 0$, we deduce that $a\phi_p(u')$ is non-increasing in $[0,1]$. Furthermore, it follows from $u'(1) = 0$ that $u' \geq 0$ in $[0,1]$ and u is nondecreasing on $[0,1]$, which leads in turn to $v(x) \geq v(0)$ on $[0,1]$. Assume that $v(0) < 0$. Then we get from $v(0) = \alpha v(\eta)$ that $\alpha \neq 0$ and $v(\eta) = \frac{1}{\alpha}v(0) < v(0)$, which contradicts v is nondecreasing. So, $v(x) \geq v(0) \geq 0$.

Now assume that for some $t_0 \in (0,1)$, $w(t_0) < 0$ and let $t_* \in (0,1)$ be such that
$$w(t_*) = \min_{t\in[0,1]} w(t), \quad w'(t_*) = 0.$$
In this case, there exists $t_1, t_2 \in (0,1)$ such that
$$t_1 < t_* < t_2, \quad w'(t_1) < w'(t_*) = 0 < w'(t_2);$$

that is,

$$v'(t_1) - \rho_p'(t_1)\|v\| < 0 < v'(t_2) - \rho_p'(t_2)\|v\|.$$

Since for all x, y, with $x \neq y$,

$$(\phi_p(x) - \phi_p(y))(x - y) > 0,$$

we obtain

$$a(t_1)(\phi_p(v'(t_1)) - \phi_p(\rho_p'(t_1)\|v\|)) < 0 < a(t_2)(\phi_p(v'(t_2)) - \phi_p(\rho_p'(t_2))\|v\|),$$

which contradicts $(a(\phi_p(v') - \phi_p(\rho_p')\|v\|))'(t) = -b(t)f(t, u(t)) \leq 0$. This completes the proof. □

The proof of the next lemma is immediate, and so we omit it.

Lemma 2.2. *For $p > 1$, let*

$$\begin{aligned} c(p) = \frac{\alpha}{1-\alpha} \int_0^\eta \psi_p\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi_p(\rho_p(s))ds\Big)dt \\ + \int_0^1 \psi_p\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi_p(\rho_p(s))ds\Big)dt. \end{aligned}$$

Then for all $u \in K_p$, $\|N_p u\| \geq c(p)\|u\|$.

In the remainder of this section, we will present two results providing fixed point index calculations in the case where $\phi = \phi_p$. These are needed for the proofs of the main results of this paper. Set for $p > 1$

$$\gamma(p) = \int_{\frac{1}{2}}^1 \psi_p\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi_p(\rho_p(s))ds\Big)dt.$$

Lemma 2.3. *Assume that $\phi = \phi_p$ with $p > 1$ and*

$$\liminf_{x\to\infty} \Big(\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}\Big) = l_\infty \quad \textit{with} \quad l_\infty\phi_p(\gamma(p)) > 1.$$

Then there exists $R_\infty(p) > 0$ such that $i(T_p, B(0, R)\cap K, K) = 0$ for all $R \geq R_\infty(p)$.

Proof. It follows, from the permanence property of the fixed point index and Lemma 2.1, that

$$i(T_p, B(0, R) \cap K, K) = i(T_p, B(0, R) \cap K_p, K_p).$$

Let $\epsilon > 0$ be such that $(l_\infty + \epsilon)\phi_p(\gamma(p)) > 1$. We deduce from the definition of l_∞ that there exists $r_\infty(p) > 0$ such that

$$f(t, u) \geq (l_\infty + \epsilon)\phi_p(u) \quad \text{for all } (t, u) \in [0, 1] \times [r_\infty(p), +\infty).$$

Thus, we have for all $u \in K_p \cap B(0, r)$, with $r > R_\infty(p) = (r_\infty(p)/\rho_p(\frac{1}{2}))$,

$$\|Lu\| \geq Lu\Big(\frac{1}{2}\Big) \geq \int_0^{1/2} \psi_p\Big(\frac{1}{a(t)} \int_t^1 b(s)f(s, u(s))ds\Big)dt \geq \psi_p(l_\infty + \epsilon)\gamma(p)\|u\| \geq \|u\|$$

and by Lemma 1.2, $i(T_p, B(0, r) \cap K, K) = 0$. □

Lemma 2.4. *Assume that $\phi = \phi_p$ with $p > 1$, and*

$$\liminf_{x\to 0} \Big(\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}\Big) = l_0, \quad \textit{with } l_0\phi_p(\gamma(p)) > 1.$$

Then there exists $R_0 > 0$ such that $i(T_p, B(0, R) \cap K, K) = 0$, for all $R \leq R_0$.

Proof. Let $\epsilon > 0$ be such that $(l_0 + \epsilon) > \phi_p(\gamma(p))$. We deduce from the definition of l_0 that there exists $R_0(p) > 0$ such that

$$f(t,u) \geq (l_0 + \epsilon)\phi_p(u) \quad \text{for all } (t,u) \in [0,1] \times [0, R_0(p)].$$

As in the proof of Lemma 2.3, for all $u \in K_p \cap \partial B(0,r)$ with $0 < r < R_0(p)$, we have $\|Lu\| \geq \psi_p(l_0+\epsilon)\gamma(p)\|u\| \geq \|u\|$ and so $i(T_p, B(0,R) \cap K, K) = i(T_p, B(0,R) \cap K_p, K_p) = 0$. □

3. Main results

In this article, we assume that There exist $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$ such that

$$t^\beta \phi(x) \leq \phi(tx) \leq t^\alpha \phi(x) \quad \text{for all } x \geq 0,\ t \in (0,1). \tag{3.1}$$

We deduce immediately from (3.1)

$$t^{1/\alpha}\psi(x) \leq \psi(tx) \leq t^{1/\beta}\psi(x) \quad \text{for all } x \geq 0 \text{ and } t \in (0,1). \tag{3.2}$$

Let ψ^+, ψ^- be the functions defined on $[0,+\infty)$ by

$$\psi^+(x) = \begin{cases} x^{1/\beta} & \text{if } x \leq 1 \\ x^{1/\alpha} & \text{if } x \geq 1, \end{cases} \qquad \psi^-(x) = \begin{cases} x^{1/\alpha} & \text{if } x \leq 1 \\ x^{1/\beta} & \text{if } x \geq 1. \end{cases}$$

It follows from (3.2) that, for all $t \geq 0$ and $x \geq 0$,

$$\psi^-(t)\psi(x) \leq \psi(tx) \leq \psi^+(t)\psi(x). \tag{3.3}$$

Set

$$f^0 = \limsup_{u \to 0} \Big(\max_{t \in [0,1]} \frac{\psi(f(t,u))}{u} \Big), \quad f^\infty = \limsup_{u \to +\infty} \Big(\max_{t \in [0,1]} \frac{\psi(f(t,u))}{u} \Big),$$

$$\Gamma = \frac{\alpha}{1-\alpha} \int_0^\eta \psi^+\Big(\frac{1}{a(t)} \int_t^1 b(s)ds\Big)dt + \int_0^1 \psi^+\Big(\frac{1}{a(t)} \int_t^1 b(s)ds\Big)dt.$$

Theorem 3.1. *Assume that in addition to* (3.1), *the following conditions are satisfied:* $\Gamma f^0 < 1$, *there exists* $p > 1$ *such that*

$$\lim_{x \to +\infty} \frac{\phi(x)}{\phi_p(x)} = 1, \tag{3.4}$$

$$c(p) < \liminf_{x \to +\infty} \Big(\min_{t \in [0,1]} \frac{f(t,x)}{\phi_p(x)} \Big) = l_\infty \leq \limsup_{x \to +\infty} \Big(\max_{t \in [0,1]} \frac{f(t,x)}{\phi_p(x)} \Big) = l^\infty < \infty,$$

Then Problem (1.1) *admits a positive solution.*

Proof. Let $\epsilon > 0$ be such that $(f^0 + \epsilon)\Gamma < 1$. There exists $r_0 > 0$ such that

$$f(s,u) \leq \phi((f^0 + \epsilon)u) \quad \text{for all } (s,u) \in [0,1] \times [0, r_0].$$

Let $u \in K \cap \partial B(0,r)$ with $0 < r \leq r_0$. We have

$$\begin{aligned} \|Tu\| &= Tu(1) \\ &\leq \frac{\alpha}{1-\alpha} \int_0^\eta \psi\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi((f^0+\epsilon)u(s))ds\Big)dt \\ &\quad + \int_0^1 \psi\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi((f^0+\epsilon)u(s))ds\Big)dt \\ &\leq \Gamma(f^0 + \epsilon)\|u\| < \|u\|. \end{aligned}$$

So, by Lemma 1.1, $i(T, B(0,r) \cap K, K) = 1$ for all $r \in (0, r_0]$.

Now let us prove that there exists $r_\infty > R_\infty(p)$ such that $i(T, B(0,r)\cap K, K) = 0$. Let for $\theta \in [0,1]$, $\phi_\theta = \theta\phi + (1-\theta)\phi_p$, $\psi_\theta = \phi_\theta^{-1}$ and consider the equation

$$u = T_\theta u, \tag{3.5}$$

where $T_\theta : K \to K$ is given for $u \in K$ by

$$\begin{aligned} T_\theta u(x) = {}& \frac{\alpha}{1-\alpha}\int_0^\eta \psi_\theta\Big(\frac{1}{a(t)}\int_t^1 b(s)f(s,u(s))ds\Big)dt \\ &+ \int_0^x \psi_\theta\Big(\frac{1}{a(t)}\int_t^1 b(s)f(s,u(s))ds\Big)dt. \end{aligned}$$

It is clear that u is a positive solution of

$$\begin{gathered} -(a\phi_\theta(u'))'(x) = b(x)f(x,u(x)), \quad x \in (0,1), \\ u(0) = \alpha u(\eta), \quad u'(1) = 0, \end{gathered}$$

if and only if u is a nontrivial fixed point of T_θ, that T_θ is completely continuous, that $T_1 = T$ and $T_0 = T_p$.

To use the homotopy property of the fixed point index, let us prove that there exists $r_\infty > R_\infty(p)$ such that (3.5) has no solution in $\partial B(0, r_\infty) \cap K$. Assume to the contrary. Then there exists sequences $(\theta_n) \subset [0,1]$, $(r_n) \subset (R_\infty(p), +\infty)$ and $(u_n) \subset K$ with $\lim r_n = +\infty$, $u_n \in \partial B(0, r_n) \cap K$ such that

$$\frac{u_n}{\|u_n\|} = \frac{T_{\theta_n} u_n}{\|u_n\|}. \tag{3.6}$$

It is easy to see that hypothesis (3.4) implies $\lim_{x\to+\infty}$ $phi_\theta(x)/\phi_p(x) = 1$. Then $\lim_{x\to+\infty}\psi_\theta(x)/\psi_p(x) = 1$. Set $\psi_\theta = \psi_p + \delta_\theta$ and $T_\theta = T_p + \widetilde{T}_\theta$, where $\widetilde{T}_\theta : K \to E$ is given for $u \in K$ by

$$\begin{aligned} \widetilde{T}_\theta u(x) = {}& \frac{\alpha}{1-\alpha}\int_0^\eta \delta_\theta\Big(\frac{1}{a(t)}\int_t^1 b(s)f(s,u(s))ds\Big)dt \\ &+ \int_0^x \delta_\theta\Big(\frac{1}{a(t)}\int_t^1 b(s)f(s,u(s))ds\Big)dt. \end{aligned}$$

Then (3.6) becomes

$$\frac{u_n}{\|u_n\|} = N_p\Big(\frac{Fu_n}{\phi_p(\|u_n\|)}\Big) + \frac{\widetilde{T}_{\theta_n} u_n}{\|u_n\|}. \tag{3.7}$$

At this stage, we claim that $\lim_{n\to\infty}\widetilde{T}_{\theta_n}u_n/\|u_n\| = 0$. Indeed, because of $l_\infty \le l^\infty < \infty$, there exists $c_1 > 0$ such that

$$\frac{Fu_n}{\phi_p(\|u_n\|)} \le c_1.$$

Also, see that $\lim_{x\to+\infty}(|\delta_\theta(x)|/\psi_p(x)) = 0$ means that for arbitrary $\epsilon > 0$ there exists $c_\epsilon > 0$ such that for all $x > 0$

$$|\delta_\theta(x)| \le \epsilon\psi_p(x) + c_\epsilon.$$

Thus, we have from the definition of T_θ that for all $x \in [0,1]$

$$\Big|\frac{T_\theta u_n(x)}{\|u_n\|}\Big| \le \frac{\epsilon}{1-\alpha}\int_0^1 \psi_p\Big(\frac{1}{a(t)}\int_t^1 b(s)\frac{f(s,u_n(s))}{\phi_p(\|u_n\|)}ds\Big)dt + \frac{c_\epsilon}{\|u_n\|}$$

which implies that

$$\limsup_{n\to\infty} \frac{\|T_\theta u_n\|}{\|u_n\|} \leq \epsilon \frac{c_1}{1-\alpha} \int_0^1 \psi_p\Big(\frac{1}{a(t)} \int_t^1 b(s)ds\Big)dt$$

and since ϵ is arbitrary $\lim_{n\to\infty}(\widetilde{T}_{\theta_n} u_n/\|u_n\|) = 0$.

Set $v_n = u_n/\|u_n\|$ and $z_n = \widetilde{T}_{\theta_n} u_n/\|u_n\|$. From the compacteness of N_p and the boundness of $Fu_n/\phi_p(\|u_n\|)$ it follows that there exists subsequences (θ_{n_k}) and (v_{n_k}) converging respectively to $\overline{\theta} \in [0,1]$ and $v \in \partial B(0,1) \cap K_p$ (see that $v_{n_k} - z_{n_k} = N_p(Fu_n/\phi_p(\|u_n\|)) \in K_p$). Furthermore, it follows from $l_\infty > c(p)$ that, for $\epsilon > 0$ with $(l_\infty - \epsilon) > c(p)$, there exists a constant $c_0 > 0$ such that for all $s \in [0,1]$ and $u \geq 0$,

$$f(s,u) \geq (l_\infty - \epsilon)\phi_p(u) - c_0. \tag{3.8}$$

Inserting (3.8) into (3.7), we obtain

$$v_{n_k} - z_{n_k} = N_p\Big(\frac{Fu_n}{\phi_p(\|u_n\|)}\Big) \geq N_p\Big((l_\infty - \epsilon)\phi_p(v_{n_k}) - \frac{c_0}{\|u_{n_k}\|}\Big).$$

Letting $n \to \infty$, we get $v \geq N_p((l_\infty - \epsilon)v)$, from which follows the contradiction,

$$1 = \|v\| \geq \|N_p((l_\infty - \epsilon)v)\| \geq c(p)(l_\infty - \epsilon)\|v\| = c(p)(l_\infty - \epsilon) > 1.$$

Thus there exists $r_\infty > R_\infty(p)$ such (3.5) admits no solution in $\partial B(0,r_\infty)\cap K$ and taking into account that $c(p) > \gamma(p)$, we deduce from the homotopy property of the fixed point index and Lemma 2.3, $i(T, B(0,r_\infty)\cap K, K) = i(T_p, B(0,r_\infty)\cap K, K) = 0$. At the end by excision and solution properties of the fixed point index, we deduce that $i(T, (B(0,r_\infty) \setminus \overline{B}(0,r)) \cap K, K) = -1$, where $r > 0$ is small enough, and Problem (1.1) admits a positive solution u with $r < \|u\| < r_\infty$. □

Theorem 3.2. *Assume that in addition to* (3.1)*, the following conditions are satisfied:* $\Gamma f^\infty < 1$*, there exists* $p > 1$ *such that*

$$\lim_{x\to 0} \frac{\phi(x)}{\phi_p(x)} = 1, \tag{3.9}$$

$$c(p) < \liminf_{x\to 0} \Big(\min_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}\Big) = l_0 \leq \limsup_{x\to 0} \Big(\max_{t\in[0,1]} \frac{f(t,x)}{\phi_p(x)}\Big) = l^0 < \infty,$$

Then (1.1) *admits a positive solution.*

Proof. Let $\epsilon > 0$ be such that $(f^\infty + \epsilon)\Gamma < 1$. There exists $C_\epsilon > 0$ such that

$$f(s,u) \leq \phi((f^0 + \epsilon)u + C_\epsilon) \quad \text{for all } (s,x) \in [0,1] \times [0,+\infty).$$

We have for all $u \in K$,

$$\begin{aligned}
\|Tu\| &= Tu(1) \\
&\leq \frac{\alpha}{1-\alpha} \int_0^\eta \psi\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi((f^\infty + \epsilon)u(s) + C_\epsilon)ds\Big)dt \\
&\quad + \int_0^1 \psi\Big(\frac{1}{a(t)} \int_t^1 b(s)\phi((f^\infty + \epsilon)u(s) + C_\epsilon)ds\Big)dt \\
&\leq \Gamma\big((f^0 + \epsilon)\|u\| + C_\epsilon\big).
\end{aligned}$$

So, for all $u \in K \cap B(0,r)$ with $r > \frac{C_\epsilon \Gamma(f^0+\epsilon)}{1-\Gamma(f^0+\epsilon)}$, we have $\|Tu\| < \|u\|$, and by Lemma 1.1, $i(T, B(0,r)\cap K, K) = 1$.

Arguing as in the proof of Theorem 3.1, we prove the existence of $r_0 > 0$ small enough such that $i(T, B(0, r_0) \cap K, K) = 0$, and by excision and solution properties of the fixed point index, we deduce that $i(T, (\overline{B}(0, r_\infty) \setminus B(0, r_0)) \cap K, K) = 1$, and that (1.1) admits a positive solution u with $r_0 < \|u\| < r_\infty$. □

Remark 3.3. Theorem 3.1 (resp. Theorem 3.2) holds if $\lim_{x\to+\infty} \frac{\phi(x)}{\phi_p(x)} = l > 0$ (resp. $\lim_{x\to+\infty} \frac{\phi(x)}{\phi_p(x)} = l > 0$).

Remark 3.4. $\phi(x) = \phi_{p_1}(x) + \phi_{p_2}(x)$, where $1 < p_1 < p_2$, is a typical case where (3.1) and (3.4) or (3.9) are satisfied.

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