Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 129, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

DECAY OF SOLUTIONS FOR A PLATE EQUATION WITH p-LAPLACIAN AND MEMORY TERM

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Abstract. In this note we show that the assumption on the memory term g in Andrade [1] can be modified to be $g'(t) \le -\xi(t)g(t)$, where $\xi(t)$ satisfies

$$\xi'(t) \le 0, \quad \int_0^{+\infty} \xi(t)\mathrm{d}t = \infty.$$

Then we show that rate of decay for the solution is similar to that of the memory term.

1. Introduction

Consider a bounded domain Ω in $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial\Omega$, and study the solutions to the problem

$$u_{tt} + \Delta^2 u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)\mathrm{d}s - \Delta u_t + f(u) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = \Delta u = 0 \quad \text{on } \Gamma \times \mathbb{R}^+, \tag{1.2}$$

$$u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \quad \text{in } \Omega, \tag{1.3}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.

This problem without the memory term models elastoplastic flows. We refer to [1] for a motivation and references concerning the study of problem (1.1)-(1.3). We will us the following assumptions:

(A1) The memory kernel g has typical properties

$$g(0) > 0, \quad l = 1 - \mu_1 \int_0^\infty g(s)\mathrm{d}s > 0, \tag{1.4}$$

where $\mu_1 > 0$ is the embedding constant for $\|\nabla u\|_2^2 \le \mu_1 \|\Delta u\|_2^2$. There exists a constant $k_1 > 0$ such that

$$g'(t) \le -k_1 g(t), \quad \forall\ t \ge 0. \tag{1.5}$$

(A2) The forcing term f satisfies

$$f(0) = 0, \quad |f(u) - f(v)| \le k_2(1 + |u|^\rho + |v|^\rho)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{1.6}$$

$$0 \le \widehat{f}(u) \le f(u)u, \quad \forall\ u \in \mathbb{R}, \tag{1.7}$$

2000 *Mathematics Subject Classification.* 35L75, 35B40.

Key words and phrases. Rate of decay; plate equation; p-Laplacian; memory term.

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Submitted April 20, 2012. Published August 15, 2012.

where k_2 is a positive constant, $\widehat{f}(z)=\int_0^z f(s)\mathrm{d}s$, and

$$0<\rho\leq\frac{4}{N-4}\text{ if } N\geq 5 \quad\text{and}\quad \rho>0 \text{ if } 1\leq N\leq 4.$$

(A3) The constant p satisfies

$$2\leq p\leq\frac{2N-2}{N-2}\text{ if } N\geq 3 \quad\text{and}\quad p\geq 2 \text{ if } N=1,2. \tag{1.8}$$

Theorem 1.1 ([1, Theorem 2.1])**.** *Assume that* (A1)–(A3) *hold.*

(i) *If the initial data* $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$*, then problem* (1.1)-(1.3) *has a unique weak solution*

$$u\in C(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))\cap C^1(\mathbb{R}^+;L^2(\Omega)).$$

(ii) *If the initial data* $(u_0,u_1)\in H_\Gamma^3(\Omega)\times H_0^1(\Omega)$*, where*

$$H_\Gamma^3(\Omega)=\{u\in H^3(\Omega)|u=\Delta u=0 on\ \Gamma\},$$

then problem (1.1)-(1.3) *has a unique strong solution satisfying*

$$u\in L^\infty(\mathbb{R}^+;H_\Gamma^3(\Omega)),\quad u_t\in L^\infty(\mathbb{R}^+;H_0^1(\Omega)),\quad u_{tt}\in L^2(0,T;H^{-1}(\Omega)).$$

(iii) *In both cases, the energy* $E(t)$ *of problem* (1.1)-(1.3) *satisfies the decay rate*

$$E(t)\leq CE(0)e^{-\gamma t},\quad t\geq 0,$$

for some $C,\gamma>0$*, where*

$$E(t)=\frac{1}{2}\|u_t(t)\|_2^2+\frac{1}{2}\|\Delta u(t)\|_2^2+\frac{1}{p}\|\nabla u(t)\|_p^p+\int_\Omega\widehat{f}(u(t))\mathrm{d}x. \tag{1.9}$$

In this note, we shall extend the above exponential rate of decay to the general case, which is similar to that of g. We use the following assumption which is weaker than (1.5).

(A4) There exists a positive differentiable function $\xi(t)$ such that

$$g'(t)\leq-\xi(t)g(t),\quad \forall t\geq 0,$$

and $\xi(t)$ satisfies

$$\xi'(t)\leq 0,\ \forall\ t>0,\ \int_0^{+\infty}\xi(t)\mathrm{d}t=\infty.$$

Then, we can prove the following main result.

Theorem 1.2. *Assume that* (A2)–(A4) *and* (1.4) *hold. If the initial data* $(u_0,u_1)\in(H^2(\Omega)\cap H_0^1(\Omega))\times L^2(\Omega)$ *or* $(u_0,u_1)\in H_\Gamma^3(\Omega)\times H_0^1(\Omega)$*, then the energy* $E(t)$ *of problem* (1.1)-(1.3) *satisfies the inequality*

$$E(t)\leq KE(0)e^{-k\int_0^t\xi(s)\mathrm{d}s},\quad t\geq 0, \tag{1.10}$$

for some $K,k>0$.

Remark 1.3. We note that a similar decay rate was given in [5, Theorem 3.5]. However, unlike [5, (G2)] and [6, (A1)], we do not use the condition of $|\frac{\xi'(t)}{\xi(t)}|\leq k$ here.

Remark 1.4. For $\xi(t)\equiv k_1$, (1.10) recaptures the exponential decay rate in [1, Theorem 2.1]. For $\xi(t)=a(1+t)^{-1}$, we can get polynomial decay rate, which is nt addressed in [1].

2. Proof of Theorem 1.2

Let us first prove the decay property for the strong solution u of problem (1.1)-(1.3). We modify the perturbed energy method in [1] by using the idea of [4, 5].

Assume that condition (A4) holds and define the modified energy, as in [1],

$$\begin{aligned} F(t) = &\frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{p}\|\nabla u(t)\|_p^p + \int_\Omega \widehat{f}(u(t))\mathrm{d}x \\ &- \frac{1}{2}\Big(\int_0^t g(s)\mathrm{d}s\Big)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t), \end{aligned}$$

where

$$(g\circ\nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2\mathrm{d}s.$$

Then we obtain

$$E(t) \le \frac{1}{l}F(t),$$

and $F(t)$ is decreasing because

$$\begin{aligned} F'(t) &= -\|\nabla u_t(t)\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &\le -\|\nabla u_t(t)\|_2^2 - \frac{1}{2}\xi(t)(g\circ\nabla u)(t) \le 0. \end{aligned} \tag{2.1}$$

Let

$$\Psi(t) = \int_\Omega u_t(t)u(t)\mathrm{d}x$$

and

$$F_\varepsilon(t) = F(t) + \varepsilon\Psi(t), \quad \forall\ \varepsilon > 0.$$

To obtain the decay result, we use the following lemmas which are of crucial importance in the proof.

Lemma 2.1 ([1, Lemma 4.1])**.** *There exists $C_1 > 0$ such that*

$$|F_\varepsilon(t) - F(t)| \le \varepsilon C_1 F(t), \quad \forall t \ge 0,\ \forall\ \varepsilon > 0.$$

Lemma 2.2 ([1, (27) in Lemma 4.2])**.** *There exist positive constants C_2, C_3 such that*

$$\Psi'(t) \le -F(t) + C_2\|\nabla u_t(t)\|_2^2 + C_3(g\circ\nabla u)(t). \tag{2.2}$$

Now, we conclude the proof of the decay property. Let

$$\varepsilon_0 = \min\Big\{\frac{1}{2C_1}, \frac{1}{C_2}\Big\}.$$

It follows from Lemma 2.1 that, for $\varepsilon < \varepsilon_0$,

$$\frac{1}{2}F(t) \le F_\varepsilon(t) \le \frac{3}{2}F(t), \quad t \ge 0. \tag{2.3}$$

By the definition of $F_\varepsilon(t)$, (2.1) and (2.2), we obtain

$$\begin{aligned}
\xi(t)F_\varepsilon'(t) &= \xi(t)F'(t) + \varepsilon\xi(t)\Psi'(t) \\
&\le -\xi(t)\|\nabla u_t(t)\|_2^2 - \frac{\xi^2(t)}{2}(g\circ\nabla u)(t) - \varepsilon\xi(t)F(t) \\
&\quad + \varepsilon C_2\xi(t)\|\nabla u_t(t)\|_2^2 + \varepsilon C_3\xi(t)(g\circ\nabla u)(t) \\
&\le -(1-\varepsilon C_2)\xi(t)\|\nabla u_t(t)\|_2^2 - \varepsilon\xi(t)F(t) + \varepsilon C_3\xi(t)(g\circ\nabla u)(t) \\
&\le -\varepsilon\xi(t)F(t) + \varepsilon C_3\xi(t)(g\circ\nabla u)(t) \\
&\le -\varepsilon\xi(t)F(t) - 2\varepsilon C_3F'(t).
\end{aligned} \tag{2.4}$$

We set

$$L(t) = \xi(t)F_\varepsilon(t) + 2\varepsilon C_3F(t).$$

Then, $L(t)$ is equivalent to $F(t)$. In fact, we have

$$L(t) \le \xi(0)F_\varepsilon(t) + 2\varepsilon C_3F(t) \le \Big(\frac{3}{2}\xi(0) + 2\varepsilon C_3\Big)F(t)$$

and

$$L(t) \ge \frac{1}{2}\xi(t)F(t) + 2\varepsilon C_3F(t) \ge 2\varepsilon C_3F(t).$$

Since $F(t) \ge lE(t) \ge 0$ and $\xi'(t) \le 0$, from (2.3) and (2.4) we obtain

$$\begin{aligned}
L'(t) &= \xi'(t)F_\varepsilon(t) + \xi(t)F_\varepsilon'(t) + 2\varepsilon C_3F'(t) \\
&\le \xi(t)F_\varepsilon'(t) + 2\varepsilon C_3F'(t) \\
&\le -\varepsilon\xi(t)F(t) \le -\varepsilon k\xi(t)L(t),
\end{aligned} \tag{2.5}$$

where we have used (2.4) and k is a positive constant.

A simple integration of (2.5) leads to

$$L(t) \le L(0)e^{-k\int_0^t \xi(s)\mathrm{d}s}, \quad \forall\ t \ge 0. \tag{2.6}$$

This proves the decay property for strong solutions in $H_\Gamma^3(\Omega)$.

The result can be extended to weak solutions by standard density arguments, as in Cavalcanti et al. [2, 3].

Acknowledgements. This work was partly supported by the Tianyuan Fund of Mathematics (Grant No. 11026211) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 09KJB110005).

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