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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A STEKLOV PROBLEM INVOLVING THE P(X)-LAPLACE OPERATOR

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Abstract. In this article we study the nonlinear Steklov boundary-value problem

$$\Delta_{p(x)}u = |u|^{p(x)-2}u \quad \text{in } \Omega,$$
$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) \quad \text{on } \partial\Omega.$$

Using the variational method, under appropriate assumptions on f, we obtain results on existence and multiplicity of solutions.

1. Introduction

Motivated by the developments in elastic mechanics, electrorheological fluids and image restoration [4, 20, 22, 26, 27], the interest in variational problems and differential equations with variable exponent has grown in recent decades; see for example [5, 13, 14, 19]. We refer the reader to [3, 6, 7, 10, 11, 12, 18, 23, 24, 25] for developments in $p(x)$-Laplacian equations.

The purpose of this article is to study the existence and multiplicity of solutions for the Steklov problem involving the $p(x)$-Laplacian,

$$\begin{gathered}\Delta_{p(x)}u = |u|^{p(x)-2}u \quad \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $\lambda > 0$ is a real number, p is a continuous function on $\overline{\Omega}$ with $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a generalization of the classical p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant. Many authors have studied the inhomogeneous Steklov problems involving the p-Laplacian [17]. The authors have studied this class of inhomogeneous Steklov problems in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$, respectively.

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We make the following assumptions on the function f:

(H0) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the carathéodory condition and there exists a constant $C \geq 0$ such that:
$$|f(x,s)| \leq C(1+|s|^{\beta(x)-1}) \quad \text{for all } (x,s) \in \partial\Omega \times \mathbb{R},$$
where $\beta(x) \in C(\partial\Omega)$, $\beta(x) > 1$ and $\beta(x) < p^{\partial}(x)$ for all $x \in \partial\Omega$.

(H1) There exist $R > 0$, $\mu > p^+$ such that for all $|s| \geq R$ and $x \in \partial\Omega$,
$$0 < \mu F(x,s) \leq f(x,s)s.$$

(H2) $f(x,s) = o(|s|^{p^+-1})$ as $s \to 0$ and uniformly for $x \in \partial\Omega$.

(H3) $f(x,-s) = -f(x,s)$, $x \in \partial\Omega$, $s \in \mathbb{R}$.

The main results of this paper are as follows.

Theorem 1.1. *If* (H0), (H1), (H2) *hold and* $\beta^- > p^+$*, then for any* $\lambda \in (0,+\infty)$, (1.1) *has at least a nontrivial weak solution.*

Theorem 1.2. *If* (H0), (H1), (H3) *hold and* $\beta^- > p^+$*, then for any* $\lambda \in (0,+\infty)$, (1.1) *has infinite many pairs of weak solutions.*

For the next theorem we assume that f satisfies the following conditions:

(F1) $|f(x,s)| \leq a(x) + b|s|^{\alpha(x)-1}$, for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $a(x)$ is in $L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)$, $b \geq 0$ is a constant, $\alpha(x) \in C(\partial\Omega)$, $1 < \alpha^- := \inf_{x\in\overline{\Omega}} \alpha(x) \leq \alpha^+ := \sup_{x\in\overline{\Omega}} \alpha(x) < p^-$ and $p(x) > N$.

(F2) $f(x,t) < 0$, when $|t| \in (0,1)$, $f(x,t) \geq m > 0$, when $t \in (t_0,\infty)$, $t_0 > 1$.

Theorem 1.3. *If* (F1), (F2) *hold, then there exist an open interval* $\Lambda \subset (0,\infty)$ *and a positive real number* ρ *such that each* $\lambda \in \Lambda$, (1.1) *has at least three solutions whose norms are less than* ρ.

The special features of the of the problems considered in this paper are that they involve the variable exponent. To prove theorems (1.1)-(1.3) we use the theory of variable exponent Sobolev spaces, established first by Kováčik and Rákosník [16], and some research results obtained recently for the $p(x)$-Laplacian equations. For the proof of theorem (1.1), we will use the Mountain Pass Theorem. For the proof of theorem (1.2), we will use the Fountain theorem. For the proof of theorem (1.3), we will use Ricceri three-critical-points theorem.

This article is organized as follows. First, we will introduce some basic preliminary results and lemmas in Section 2. In Section 3, we will give the proofs of our main results.

2. Preliminaries

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For more details, see [8, 9]. Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where
$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} p(x) > 1\}.$$
Denote by $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by
$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ is a measurable and } \int_\Omega |u|^{p(x)} dx < +\infty\},$$

with the norm
$$|u|_{p(x)} = \inf\{\tau > 0; \int_\Omega |\frac{u}{\tau}|^{p(x)} dx \le 1\}.$$
Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by
$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$
with the norm
$$\|u\| = \inf\{\tau > 0; \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)} + |\frac{u}{\tau}|^{p(x)}) dx \le 1\},$$
$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$
We refer the reader to [8, 9] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1 ([9]). *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|)$ are separable, reflexive and uniformly convex Banach spaces.*

Lemma 2.2 ([9]). *Hölder inequality holds, namely*
$$\int_\Omega |uv| dx \le 2|u|_{p(x)}|v|_{q(x)} \quad \forall u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega),$$
where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Lemma 2.3 ([9]). *Let $I(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)}) dx$, for $u \in W^{1,p(x)}(\Omega)$ we have*

- $\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$.
- $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$.
- $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$.

Lemma 2.4 ([8]). *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*
$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N; \\ +\infty, & \text{if } p(x) \ge N. \end{cases}$$

Lemma 2.5 ([9]). *If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function and*
$$|f(x,s)| \le a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad \forall (x,s) \in \overline{\Omega} \times \mathbb{R},$$
where $p_1(x), p_2(x) \in C(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $p_2(x) > 1$, $p_2(x) > 1$, $a(x) \ge 0$ and $b \ge 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $N_f(u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Let $a : \partial\Omega \to \mathbb{R}$ be a measurable. Define the weighted variable exponent Lebesgue space by
$$L^{p(x)}_{a(x)}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma < +\infty\},$$
with the norm
$$|u|_{(p(x),a(x))} = \inf\{\tau > 0; \int_{\partial\Omega} |a(x)|\,|\frac{u}{\tau}|^{p(x)} d\sigma \le 1\},$$
where $d\sigma$ is the measure on the boundary. Then $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space. In particular, when $a \in L^\infty(\partial\Omega)$, $L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Lemma 2.6 ([5]). *Let* $\rho(u)=\int_{\partial\Omega}|a(x)||u|^{p(x)}d\sigma$ *for* $u\in L^{p(x)}_{a(x)}(\partial\Omega)$ *we have*

- $|u|_{(p(x),a(x))}\geq 1\Rightarrow |u|^{p^-}_{(p(x),a(x))}\leq\rho(u)\leq |u|^{p^+}_{(p(x),a(x))}$.
- $|u|_{(p(x),a(x))}\leq 1\Rightarrow |u|^{p^+}_{(p(x),a(x))}\leq\rho(u)\leq |u|^{p^-}_{(p(x),a(x))}$.

For $A\subset\overline{\Omega}$, denote by $p^-(A)=\inf_{x\in A}p(x)$, $p^+(A)=\sup_{x\in A}p(x)$. Define

$$p^\partial(x)=(p(x))^\partial:=\begin{cases}\frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x)<N,\\ \infty, & \text{if } p(x)\geq N.\end{cases}$$

$$p^\partial_{r(x)}(x):=\frac{r(x)-1}{r(x)}p^\partial(x),$$

where $x\in\partial\Omega$, $r\in C(\partial\Omega,\mathbb{R})$ and $r(x)>1$.

Lemma 2.7 ([5]). *Assume that the boundary of* Ω *possesses the cone property and* $p\in C(\overline{\Omega})$ *with* $p^->1$. *Suppose that* $a\in L^{r(x)}(\partial\Omega)$, $r\in C(\partial\Omega)$ *with* $r(x)>\frac{p^\partial(x)}{p^\partial(x)-1}$ *for all* $x\in\partial\Omega$. *If* $q\in C(\partial\Omega)$ *and* $1\leq q(x)<p^\partial_{r(x)}(x)$, $\forall x\in\partial\Omega$. *Then there is a compact embedding* $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$. *In particular, there is a compact embedding* $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\partial\Omega)$, *where* $1\leq q(x)<p^\partial(x)$, $\forall x\in\partial\Omega$.

Lemma 2.8 ([2, 15, 21]). *Let* X *be a separable and reflexive real Banach space,* $\phi:X\to\mathbb{R}$ *is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*; $\Psi:X\to\mathbb{R}$ *is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that:*

(i) $\lim_{\|u\|_X\to\infty}(\phi(u)+\lambda\psi(u))=\infty$ *for all* $\lambda>0$,
(ii) *there exist* $r\in\mathbb{R}$ *and* $u_0,u_1\in X$ *such that* $\phi(u_0)<r<\phi(u_1)$,
(iii)
$$\inf_{u\in\phi^{-1}(-\infty,r]}\psi(u)>\frac{(\phi(u_1)-r)\psi(u_0)+(r-\phi(u_0))\psi(u_1)}{\phi(u_1)-\phi(u_0)}.$$

Then there exist an open interval $\Lambda\subset(0,\infty)$ *and a positive constant* $\rho>0$ *such that for any* $\lambda\in\Lambda$ *the equation* $\phi'(u)+\lambda\psi'(u)=0$ *has at least three solutions in* X *whose norms are less than* ρ.

Theorem 2.9. *If* $f:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ *is a carathéodory function and*

(F1) $|f(x,s)|\leq a(x)+b|s|^{\alpha(x)-1}$, *for all* $(x,s)\in\partial\Omega\times\mathbb{R}$,

where $a(x)\in L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)$ *and* $b\geq 0$ *is a constant,* $\alpha(x)\in C_+(\partial\Omega)$ *such that for all* $x\in\partial\Omega$,

$$\alpha(x)<\begin{cases}\frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x)<N;\\ +\infty, & \text{if } p(x)\geq N.\end{cases}\tag{2.1}$$

Set $X=W^{1,p(x)}(\Omega)$, $F(x,u)=\int_0^u f(x,t)dt$, $\psi(u)=-\int_{\partial\Omega}F(x,u(x))d\sigma$, *then* $\psi(u)\in C^1(X,\mathbb{R})$ *and* $D\psi(u,\varphi)=<\psi'(u),\varphi>=-\int_{\partial\Omega}f(x,u(x))\varphi d\sigma$, *moreover, the operator* $\psi':X\to X^*$ *is compact.*

Proof. By the Mean-value theorem, we have

$$
\begin{aligned}
D\psi(u,\varphi) &= \lim_{t\to 0}\frac{\psi(u+t\varphi)-\psi(u)}{t} \\
&= -\lim_{t\to 0}\int_{\partial\Omega}\frac{F(x,u(x)+t\varphi(x))-F(x,u(x)}{t}d\sigma \\
&= -\lim_{t\to 0}\int_{\partial\Omega} f(x,u(x)+t\theta\varphi(x))\varphi(x)d\sigma,
\end{aligned}
\tag{2.2}
$$

where $0\le\theta=\theta(u(x),t\varphi(x))\le 1$. If $u,\varphi\in X$, then by condition (2.1) and the embedding theorem (lemma2.7), we have $u,\varphi\in L^{\alpha(x)}(\partial\Omega)$. Then there is some constant C such that

$$
\|w\|_{L^{\alpha(x)}(\partial\Omega)}\le C\|w\|_X \quad \forall w\in X. \tag{2.3}
$$

By (F1) and Young's inequality, we have

$$
\begin{aligned}
&|f(x,u(x)+t\theta\varphi(x))\varphi(x)| \\
&\le [a(x)+b|u(x)+t\theta\varphi(x)|^{\alpha(x)-1}]|\varphi(x)| \\
&\le \frac{\alpha(x)-1}{\alpha(x)}[a(x)+b|u(x)+t\theta\varphi(x)|^{\alpha(x)-1}]^{\frac{\alpha(x)}{\alpha(x)-1}}+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)}.
\end{aligned}
\tag{2.4}
$$

Using the inequality

$$
(a+b)^p\le 2^{p-1}(|a|^p+|b|^p),\quad p\ge 1,
$$

we have

$$
\begin{aligned}
&\frac{\alpha(x)-1}{\alpha(x)}[a(x)+b|u(x)+t\theta\varphi(x)|^{\alpha(x)-1}]^{\frac{\alpha(x)}{\alpha(x)-1}}+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)} \\
&\le \frac{(\alpha(x)-1)}{\alpha(x)}2^{\frac{1}{\alpha(x)-1}}[(a(x))^{\frac{\alpha(x)}{\alpha(x)-1}}+b^{\frac{\alpha(x)}{\alpha(x)-1}}|u(x)+t\theta\varphi(x)|^{\alpha(x)}]+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)} \\
&\le \frac{(\alpha(x)-1)}{\alpha(x)}2^{\frac{1}{\alpha(x)-1}}[(a(x))^{\frac{\alpha(x)}{\alpha(x)-1}}+2^{\alpha(x)-1}b^{\frac{\alpha(x)}{\alpha(x)-1}}[|u(x)|^{\alpha(x)}+|\varphi(x)|^{\alpha(x)}]] \\
&\quad +\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)},
\end{aligned}
$$

for $|t|\le 1$. Note that the right hand side of the above inequality is independent of t and integrable on $\partial\Omega$, then by the Lebesgue dominated convergence theorem, we have

$$
D\psi(u,\varphi)=-\int_{\partial\Omega} f(x,u(x))\varphi(x)d\sigma. \tag{2.5}
$$

Obviously the operator $D\psi(u,\varphi)$ is a linear operator for a given u. We know that the Nemytskii operator $N_f : u(x)\mapsto f(x,u(x))$ is a continuous bounded operator from $L^{\alpha(x)}(\partial\Omega)$ into $L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)$. Then by (2.3) and (2.5) we have

$$
D\psi(u,\varphi)=-\int_{\partial\Omega} f(x,u(x))\varphi(x)d\sigma\le 2C\|f(x,u)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}\|\varphi(x)\|_X.
$$

So $D\psi(u,\varphi)$ is a linear bounded functional, therefore the Gâteaux derivative of the linear bounded functional $\psi(u)$ exists and

$$
D\psi(u,\varphi)=<D\psi(u),\varphi>=-\int_{\partial\Omega} f(x,u(x))\varphi(x)d\sigma \quad \forall u,\varphi\in X. \tag{2.6}
$$

We will prove that $\psi' : X \to X^*$ is completely continuous. For $u, v, \varphi \in X$, from (2.5) and (2.6), we obtain

$$|\langle D\psi(u) - D\psi(v), \varphi\rangle| \leq 2C\|f(x,u) - f(x,v)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}\|\varphi\|_X.$$

Then

$$\|D\psi(u) - D\psi(v)\|_{X^*} \leq 2C\|f(x,u) - f(x,v)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}.$$

The above inequality shows that the operator $T : L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega) \to X^*$ defined by $T(f(x,u)) = D\psi(u)$ is continuous. Then the composite operator $D\psi = ToN_f oI$: $u \to D\psi(u)$ from X into X^* is continuous. Therefore, ψ is Frèchet differentiable and its Frèchet derivative $\psi'(u) = D\psi(u)$. This shows that $\psi(u) \in C^1(X, \mathbb{R})$, $D\psi(u,\varphi) =< \psi'(u), \varphi >= -\int_{\partial\Omega} f(x, u(x))\varphi(x)d\sigma$ and $\psi' : X \to X^*$ is compact. $\square$

We say that $u \in X$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v\,dx + \int_\Omega |u|^{p(x)-2}uv\,dx = \lambda\int_{\partial\Omega} f(x,u)vd\sigma \quad \text{for all } v \in X.$$

Let

$$\phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx,$$

$$\psi(u) = -\int_{\partial\Omega} F(x,u)d\sigma,$$

$$J(u) = \phi(u) + \lambda\psi(u),$$

where $F(x,t) = \int_0^t f(x,s)ds$. Then we have

$$(\phi'(u), v) = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv)dx,$$

$$(\psi'(u), v) = -\int_{\partial\Omega} f(x,u)vd\sigma.$$

3. Proof of main results

For the proof of theorem 1.1, we will use the Mountain Pass Theorem. We start with the following lemmas.

Lemma 3.1. *If* (H0), (H1) *hold, then for any* $\lambda \in (0, +\infty)$ *the functional* J *satisfies the Palais Smale condition (PS).*

Proof. Suppose that $(u_n) \subset X$ is a (PS) sequence; i.e.,

$$\sup|J(u_n)| \leq M, J'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Let us show that (u_n) is bounded in X. Using hypothesis (H1), since $J(u_n)$ is bounded, we have for n large enough:

$$\begin{aligned} M + 1 &\geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n\rangle + \frac{1}{\mu}\langle J'(u_n), u_n\rangle \\ &= \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx - \lambda\int_{\partial\Omega} F(x,u_n)d\sigma \\ &\quad - \frac{1}{\mu}\Big[\int_\Omega (|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx - \lambda\int_{\partial\Omega} f(x,u_n)u_n d\sigma\Big] + \frac{1}{\mu}\langle J'(u_n), u_n\rangle \end{aligned}$$

$$\begin{aligned}&\geq (\frac{1}{p^+}-\frac{1}{\mu})\|u_n\|^{p^-}-\frac{1}{\mu}\|J'(u_n)\|_{X^*}\|u_n\|-C\\&\geq (\frac{1}{p^+}-\frac{1}{\mu})\|u_n\|^{p^-}-\frac{C_1}{\mu}\|u_n\|-C,\end{aligned}$$

where C and C_1 are two constants. From the inequality above, we know that u_n is bounded in X since $\mu > p^+$. The proof is complete. □

Lemma 3.2. *There exist $r_1, C' > 0$ such that $J(u) \geq C'$ for all $u \in X$ such that $\|u\| = r_1$.*

Proof. Conditions (H0) and (H2) assure that

$$|F(x,s)| \leq \varepsilon |s|^{p^+} + C(\varepsilon)|s|^{\beta(x)} \quad \text{for all } (x,s) \in \partial\Omega \times \mathbb{R}.$$

For $\|u\|$ small enough, we have

$$\begin{aligned}J(u) &\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\int_{\partial\Omega} F(x,u)d\sigma\\&\geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\int_{\partial\Omega} (\varepsilon|u|^{p^+} + C(\varepsilon)|u|^{\beta(x)})d\sigma.\end{aligned} \tag{3.1}$$

Since $p^+ < \beta^- \leq \beta(x) < p^\partial(x)$, for all $x \in \partial\Omega$, we have

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{p^+}(\partial\Omega),$$

with a continuous and compact embedding, which implies the existence of $C_3 > 0$ such that

$$|u|_{L^{p^+}(\partial\Omega)} \leq C_3\|u\|, \quad \forall u \in X. \tag{3.2}$$

From (3.1) and (3.2), we have for $\|u\|$ small enough

$$J(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \lambda\varepsilon C_3\|u\|^{p^+} - \lambda C(\varepsilon)C_3\|u\|^{\beta^-}.$$

Choose $\varepsilon > 0$ small enough that $0 < \lambda\varepsilon C_3 < \frac{1}{2p^+}$, we obtain

$$\begin{aligned}J(u) &\geq \frac{1}{2p^+}\|u\|^{p^+} - C(\lambda,\varepsilon)C_3\|u\|^{\beta^-}\\&\geq \|u\|^{p^+}(\frac{1}{2p^+} - C(\lambda,\varepsilon)C_3\|u\|^{\beta^- - p^+}).\end{aligned}$$

Since $p^+ < \beta^-$, the function $t \mapsto (\frac{1}{2p^+} - C(\lambda,\varepsilon)C_3 t^{\beta^- - p^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $r_1 > 0$ and $C' > 0$ such that

$$J(u) \geq C' \quad \forall u \in X : \|u\| = r_1.$$

The proof is complete. □

Proof of theorem 1.1. To apply the Mountain Pass Theorem, we must prove that $J(tu) \to -\infty$ as $t \to +\infty$, for a certain $u \in X$. From condition (H1), we obtain

$$F(x,s) \geq c|s|^\mu \quad \text{for all } (x,s) \in \partial\Omega \times \mathbb{R}.$$

Let $u \in X$ and $t > 1$, we have

$$\begin{aligned}J(tu) &= \int_\Omega \frac{t^{p(x)}}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - \lambda\int_{\partial\Omega} F(x,tu)d\sigma\\&\leq t^{p^+}\int_\Omega \frac{1}{p(x)}[|\nabla u|^{p(x)} + |u|^{p(x)}]dx - ct^\mu\lambda\int_{\partial\Omega}|u|^\mu d\sigma.\end{aligned}$$

The fact $\mu > p^+$, implies for any $\lambda \in (0, +\infty)$ $J(tu) \to -\infty$ as $t \to +\infty$.

It follows that there exists $e \in X$ such that $\|e\| > r_1$ and $J(e) < 0$. According to the Mountain Pass Theorem, J admits a critical value $\tau \geq C'$ which is characterized by

$$\tau = \inf_{h\in\Gamma} \sup_{t\in[0,1]} J(h(t))$$

where

$$\Gamma = \{h \in C([0,1], X) : h(0) = 0 \text{ and } h(1) = e\}.$$

This completes the proof. $\square$

Since X is a separable and reflexive Banach space [3, 8], there exist $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases}$$

$$X = \overline{\text{span}}\{e_n : n = 1, 2, \dots\}, \quad X^* = \overline{\text{span}}^{w^*}\{f_n : n = 1, 2, \dots\}.$$

For $k = 1, 2, \dots$ denote by

$$X_n = \text{span}\{e_n\}, \quad Y_n = \oplus_{j=1}^{n} X_j, \quad Z_n = \overline{\oplus_{j=n}^{\infty} X_j}.$$

Lemma 3.3 ([5, 11]). *For $\beta(x) \in C_+(\partial\Omega)$, $\beta(x) < p^\partial(x)$ and $x \in \partial\Omega$, let $\beta_k = \sup\{|u|_{L^{\beta(x)}(\partial\Omega)} : \|u\| = 1,\ u \in Z_k\}$. Then $\lim_{k\to\infty} \beta_k = 0$.*

Proof of theorem 1.2. We use the Fountain theorem [1]. Obviously, J is an even functional and satisfies the (PS) condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that:

- (A1) $b_k := \inf\{J(u)/u \in Z_k, \|u\| = r_k\} \to +\infty$ as $k \to +\infty$,
- (A2) $a_k := \max\{J(u)/u \in Y_k, \|u\| = \rho_k\} \leq 0$ as $k \to +\infty$.

(A1): For $u \in Z_k$ such that $\|u\| = r_k > 1$, by condition (H0), we have

$$\begin{aligned} J(u) &= \int_\Omega \frac{1}{p(x)} \Big[|\nabla u|^{p(x)} + |u|^{p(x)}\Big] dx - \lambda \int_{\partial\Omega} F(x,u) d\sigma \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda \int_{\partial\Omega} C(1 + |u|^{\beta(x)}) d\sigma \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda C \max\{|u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}\} - C_1. \end{aligned}$$

It follows that

$$\begin{aligned} J(u) &\geq \begin{cases} \frac{1}{p^+}\|u\|^{p^-} - (C_2(\lambda) + C_1) & \text{if } |u|_{L^{\beta(x)}(\partial\Omega)} \leq 1 \\ \frac{1}{p^+}\|u\|^{p^-} - C_2(\lambda)(\beta_k\|u\|)^{\beta^+} - C_1 & \text{if } |u|_{L^{\beta(x)}(\partial\Omega)} > 1 \end{cases} \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - C_2(\lambda)(\beta_k\|u\|)^{\beta^+} - C_3. \end{aligned}$$

For $r_k = (C_2(\lambda)\beta^+\beta_k^{\beta^+})^{\frac{1}{p^- - \beta^+}}$, we have

$$J(u) \geq r_k^{p^-}\Big(\frac{1}{p^+} - \frac{1}{\beta^+}\Big) - C_3.$$

Since $\beta_k \to 0$ and $p^+ < \beta^+$, we have $r_k \to +\infty$ as $k \to +\infty$. Consequently,

$$J(u) \to +\infty \quad \text{as } \|u\| \to +\infty, u \in Z_k.$$

So $(A1)$ holds.
(A2): Condition (H1) implies

$$F(x,s) \geq C_1|s|^{\mu} - C_2, \quad \forall (x,s) \in \partial\Omega \times \mathbb{R}.$$

Let $u \in Y_k$ be such that $\|u\| = \rho_k > r_k > 1$. Then

$$\begin{aligned} J(u) &\leq \frac{1}{p^-}\|u\|^{p^+} - \lambda \int_{\partial\Omega} C_1|s|^{\mu} - C_2 d\sigma \\ &\leq \frac{1}{p^-}\|u\|^{p^+} - \lambda C_1 \int_{\partial\Omega} |u|^{\mu} d\sigma + C_3. \end{aligned}$$

Note that the space Y_k has finite dimension, then all norms are equivalents and we obtain

$$J(u) \leq \frac{1}{p^-}\|u\|^{p^+} - \lambda C_2\|u\|^{\mu} + C_3.$$

Finally,

$$J(u) \to -\infty \quad \text{as } \|u\| \to +\infty, u \in Y_k$$

because $\mu > p^+$. The assertion (A2) is then satisfied and the proof of theorem 1.2 is complete. □

Proof of theorem 1.3. For proving our result we use lemma 2.8. It is well known that ϕ is a continuous convex functional, then it is weakly lower semicontinuous and its inverse derivative is continuous, from theorem 2.9 the precondition of lemma 2.8 is satisfied. In following we must verify that the conditions (i), (ii) and (iii) in lemma 2.8 are fulfilled.

For $u \in X$ such that $\|u\|_X \geq 1$, we have

$$\begin{aligned} \psi(u) &= -\int_{\partial\Omega} F(x,u) d\sigma = -\int_{\partial\Omega} [\int_0^{u(x)} f(x,t)dt] d\sigma \\ &\leq \int_{\partial\Omega} [a(x)|u(x)| + \frac{b}{\alpha(x)}|u|^{\alpha(x)}] d\sigma \\ &\leq 2\|a\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)} \|u\|_{L^{\alpha(x)}(\partial\Omega)} + \frac{b}{\alpha^-}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \\ &\leq 2C\|a\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)} \|u\|_X + \frac{b}{\alpha^-}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma. \end{aligned}$$

By the embedding theorem, we have $u \in L^{\alpha(x)}(\partial\Omega)$; therefore,

$$\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma \leq \max\{\|u\|^{\alpha^+}_{L^{\alpha(x)}(\partial\Omega)}, \|u\|^{\alpha^-}_{L^{\alpha(x)}(\partial\Omega)}\} \leq C'\|u\|_X^{\alpha^+}.$$

Then

$$|\psi(u)| \leq 2C\|a\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)} \|u\|_X + \frac{b}{\alpha^-}C'\|u\|_X^{\alpha^+}.$$

On the other hand,

$$\phi(u) = \int_{\Omega} \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)}) dx \geq \frac{1}{p^+}\|u\|_X^{p^-}.$$

Which implies that for any $\lambda > 0$,

$$\phi(u) + \lambda\psi(u) \geq \frac{1}{p^+}\|u\|_X^{p^-} - 2\lambda C\|a\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)} \|u\|_X - \frac{\lambda b C'}{\alpha^-}\|u\|_X^{\alpha^+}.$$

For $p^- > \alpha^+$ we have

$$\lim_{\|u\|_X \to \infty} (\phi(u) + \lambda\psi(u)) = \infty,$$

then (i) of lemma 2.8 is verified.

lt remains to show (ii) and (iii) of this lemma (Ricceri). By (F2), it is clear that F(x,t) is increasing for $t \in (t_0, \infty)$ and decreasing for $t \in (0,1)$ uniformly for $x \in \partial\Omega$, and $F(x,0) = 0$ is obvious, $F(x,t) \to +\infty$ when $t \to +\infty$ because ($F(x,t) \geq mt$ uniformly on x). Then, there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau) \quad \forall u \in X, t > \delta, \tau \in (0,1).$$

Let a, b be two real numbers such that $0 < a < \min\{1, c_1\}$ where c_1 is a constant which satisfies

$$\|u\|_{C(\overline{\Omega})} \leq c_1 \|u\|_X,$$
$$\|u\|_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |u(x)|.$$

This inequality is well defined due to compactly embedding from $W^{1,p(x)}(\Omega)$ to $C(\overline{\Omega})$ (because $N < p^-$).

We choose $b > \delta$ satisfying $b^{p^-}|\Omega| > 1$. When $t \in [0, a]$ we have

$$F(x,t) \leq F(x,0) = 0.$$

Then

$$\int_{\partial\Omega} \sup_{0<t<a} F(x,t)d\sigma \leq \int_{\partial\Omega} F(x,0)d\sigma = 0.$$

Furthermore, since $b > \delta$ we have

$$\int_{\partial\Omega} F(x,b)d\sigma > 0.$$

Moreover,

$$\frac{1}{c_1^{p^+}} . \frac{a^{p^+}}{b^{p^-}} \int_{\partial\Omega} F(x,b)d\sigma > 0.$$

Which implies

$$\int_{\partial\Omega} \sup_{0<t<a} F(x,t)d\sigma \leq 0 < \frac{1}{c_1^{p^+}} \frac{a^{p^+}}{b^{p^-}} \int_{\partial\Omega} F(x,b)d\sigma.$$

Let $u_0, u_1 \in X$, $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \overline{\Omega}$. We define $r = \frac{1}{p^+}(\frac{a}{c_1})^{p^+}$. Clearly $r \in (0,1)$, $\phi(u_0) = \psi(u_0) = 0$,

$$\phi(u_1) = \int_\Omega \frac{1}{p(x)} b^{p(x)} dx \geq \frac{1}{p^+} b^{p^-} |\Omega| > \frac{1}{p^+} 1 > \frac{1}{p^+} (\frac{a}{c_1})^{p^+} = r,$$

and

$$\psi(u_1) = -\int_{\partial\Omega} F(x, u_1(x))d\sigma = \int_{\partial\Omega} F(x,b)d\sigma < 0.$$

So we have $\phi(u_0) < r < \phi(u_1)$. Then (ii) of lemma 2.8 is verified.

On the other hand, we have

$$\begin{aligned} -\frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)} &= -r \frac{\psi(u_1)}{\phi(u_1)} \\ &= r \frac{\int_{\partial\Omega} F(x,b)d\sigma}{\int_\Omega \frac{1}{p(x)} b^{p(x)} dx} > 0. \end{aligned}$$

Let $u \in X$ be such that $\phi(u) \le r < 1$. Set $I(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})dx$. Since $\frac{1}{p^+} I(u) \le \phi(u) \le r$, for $u \in W^{1,p(x)}(\Omega)$, we obtain

$$I(u) \le p^+ . r = (\frac{a}{c_1})^{p^+} < 1.$$

It follows that $\|u\|_X < 1$ by lemma 2.3. We have

$$\frac{1}{p^+}\|u\|^{p^+} \le \frac{1}{p^+} I(u) \le \phi(u) \le r.$$

Then

$$|u(x)| \le c_1 \|u\|_X \le c_1 (p^+ . r)^{\frac{1}{p^+}} = a \quad \forall u \in X, x \in \overline{\Omega}, \phi(u) \le r.$$

The above inequality shows that

$$-\inf_{u \in \phi^{-1}(-\infty, r]} \psi(u) = \sup_{u \in \phi^{-1}(-\infty, r]} -\psi(u) \le \int_{\partial\Omega} \sup_{0<t<a} F(x,t) d\sigma \le 0.$$

Then

$$\inf_{u \in \phi^{-1}(-\infty, r]} \psi(u) > \frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)}.$$

Which means that condition (iii) in lemma 2.8 is obtained. Since the assumptions of lemma 2.8 are verified, there exist an open interval $\Lambda \subset (0, \infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ. □

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