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BOUNDS AND COMPACTNESS FOR SOLUTIONS OF SECOND-ORDER ELLIPTIC EQUATIONS

CARLOS C. ARANDA

Dedicated to my mother Gregoria Ynes Zalazar

Abstract. In this article, we establish some connections between Sobolev spaces and nonlinear singular elliptic problems, to obtain bounds and compactness results for solutions of second-order elliptic equations.

1. Introduction and results

The main purpose of this paper is to develop some connections between Sobolev spaces and nonlinear singular elliptic problems to obtain bounds and compactness results for solutions of second-order elliptic equations, where the structure of the imbedding is nonlinear. The theory of singular nonlinear elliptic problems is fairly well developed. (See for example [2, 6, 7, 9, 10, 11, 12] for a survey and bibliography.) In [15] it is stated that

> During the past half century, linear second order elliptic equations on bounded regions have been studied, if not exhaustively, at least with reasonable completeness and the fundamental questions concerning them have received rather simple solutions. In the works of Giraud and Schauder in the thirties, it was shown that the basic boundary value problems are solvable for such equations under the assumption of sufficient smoothness of the coefficients and of the boundary of the region. Then, there were interpreted from the standpoint of functional analysis. This approach was initiated by the article [8] of Friedrichs in 1934 on semibounded extensions of symmetric elliptic operators. This article and further studies of Friedrichs, Mikhlin, Vishik, and others during the late forties showed that the solution of the classical boundary-value problems for elliptic equations (we are only speaking of second order equations) was equivalent to solving equations of the form $x + Ax = f$, for a completely continuous operator A in certain Hilbert spaces constructed from the quadratic form of the principal symmetric part of a differential operator.

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For a review of the state of the art on this topic, see [3, 4, 5, 14, 16, 17].

Theorem 1.1 ([13, Theorem 7.26])**.** *Let Ω be a $C^{0,1}$ domain in $\mathbb{R}^N$. Then*

- (i) *if $kp < N$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^{p^*}(\Omega)$, $p^* = \frac{Np}{N-kp}$, and compactly imbedded in $L^q(\Omega)$ for any $q < p^*$.*
- (ii) *if $0 \le m < k - \frac{N}{p} < m+1$, the space $W^{k,p}$ is continuously imbedded in $C^{m,\alpha}(\overline{\Omega})$, $\alpha = k - \frac{N}{p} - m$, and compactly imbedded in $C^{m,\beta}(\overline{\Omega})$ for any $\beta < \alpha$*

Theorem 1.2 ([13, Theorem 6.6])**.** *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^N$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solutions of the equation*

$$\mathcal{L}u \equiv \sum_{i,j=1}^{N} a_{ij} u_{x_i,x_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu = f \tag{1.1}$$

where $f \in C^{\alpha}(\overline{\Omega})$ and the coefficients of $\mathcal{L}$ satisfy, for positive constants λ, Λ,

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^N,$$
$$|a_{i,j}|_{0,\alpha;\Omega},\ |b_i|_{0,\alpha;\Omega},\ |c|_{0,\alpha;\Omega} \le \Lambda\,.$$

Let $\varphi \in C^{2,\alpha}(\overline{\Omega})$, and suppose $u = \varphi$ on $\partial\Omega$. Then

$$|u|_{2,\alpha;\Omega} \le C\{|u|_{0,\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}\} \tag{1.2}$$

where $C = C(n, \alpha, \lambda, \Lambda)$.

Our main concern is related to a quotation from [15]:

> We pose the following question: To what classes $L^s(\Omega)$ must the functions a_i, b_i, c, f_i and f belong in order that all generalized solutions $u(x)$ of the equations
>
> $$Lu \equiv \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[a_{ij}(x)u_{x_j} + a_i(x)u] + \sum_{i=1}^{n} b_i(x)u_{x_i} + a(x)u = f - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \tag{1.3}$$
>
> in $W^{1,2}(\Omega)$ be bounded functions? To ascertain the necessary conditions, we again take the function $u = \log|\log r|$ and regard it in the sphere $r \le R < 1$ as a solution of any one of the following equations
>
> $$\Delta u = F(r), \quad \Delta u - \frac{\partial}{\partial x_i}\Big(\frac{x_i}{r^2 \log r}\Big) = 0,$$
> $$\Delta u - \frac{F(r)}{\log|\log r|}u = 0, \quad \Delta u - \frac{\partial}{\partial x_i}\Big(\frac{x_i u}{r^2 \log r \log|\log r|}\Big) = 0$$
>
> where $F(r)$ has the same meaning as above. It is easy to see that in these functions $f, a \in L^{\frac{N}{2}}(K_R)$ and $f_i, a_i \in L^N(K_R)$. Therefore these last conditions does not ensures boundedness of the generalized solutions. Therefore the requirements $\|a_i, b_i, f_i\|_{L^q(\Omega)} \le \mu < \infty$ $\|a, f\|_{L^{\frac{q}{2}}(\Omega)} \le \mu < \infty$; $q > N$ are necessary.

We introduce now the equation

$$-\mathcal{L}u = g(u), \text{ in } \Omega, \quad u = 0 \text{ on } \Omega, \tag{1.4}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, $c \leq 0$ in Ω and $g : (0, \infty) \to (0, \infty)$ is non-increasing locally Hölder continuous function singular at the origin. It is well known that problem 1.4 in the case $\mathcal{L} = \Delta$ and $g(u) = u^{-\gamma}$ has a unique classical bounded solution u for all $\gamma > 0$. This solution u belongs to the Sobolev space $H_0^1(\Omega)$ if and only if $0 < \gamma < 3$. Moreover $\gamma > 1$ implies $C_0\varphi_1^{-\frac{2\gamma s}{1+\gamma}} \leq u^{-s\gamma} \leq C_1\varphi_1^{-\frac{2\gamma s}{1+\gamma}}$ where φ_1 is the principal eigenfunction of the laplacian operator ($-\Delta\varphi_1 = \lambda_1\varphi$, in Ω, $\varphi_1 = 0$ on $\partial\Omega$) [2, 11]. Therefore $u^{-\gamma}$ not belong to any $L^s(\Omega)$, $s \geq 1$ for $\gamma > 1$ because $\int_\Omega \varphi_1^{-r} dx < \infty$ for $r \geq 0$ if and only if $0 \leq r < 1$. Nevertheless, we have an unexpected nonlinear compact imbedding:

Theorem 1.3 (Aranda-Godoy [2])**.** *Let P be the positive cone in $L^\infty(\Omega)$. Let $S_\epsilon : P \to P$ be the solution operator for the problem*

$$-\Delta u = g(u) + w \text{ in } \Omega, \quad u = \epsilon \text{ on } \partial\Omega, \tag{1.5}$$

gives $S_\epsilon(w) = u$ where $\epsilon \geq 0$. Then $S_\epsilon : P \to P$ is a continuous, non decreasing and compact map with $S_{\epsilon_0}(w) \leq S_{\epsilon_1}(w)$ for $\epsilon_0 < \epsilon_1$.

The derivations of our results are very elementary using a Schauder approach. We set

$$\mathcal{C}_{\rm loc}^{\alpha,g,+}(\Omega) = \{f \in C_{\rm loc}^\alpha(\Omega) | 0 \leq f \leq g(u) \text{ where } u \text{ solves 1.4}\}$$

Our main result follows.

Theorem 1.4. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$. Then the equation*

$$-\mathcal{L}v = f \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \tag{1.6}$$

where $f \in \mathcal{C}_{\rm loc}^{\alpha,g,+}(\Omega)$ has a unique solution $v \in C_{\rm loc}^\alpha(\Omega) \cap C^0(\overline{\Omega}) \cap C^2(\Omega)$ with $0 \leq v \leq u$ in Ω and u solves equation 1.4.

Our imbedding theorem is as follows.

Theorem 1.5. *Let $\mathcal{P}$ be the cone of positive functions in $C^0(\overline{\Omega})$. Let $\mathcal{S} : \mathcal{C}_{\rm loc}^{\alpha,g,+}(\Omega) \to \mathcal{P}$ the solution operator of problem 1.6 gives $\mathcal{S}(f) = v$. Then $\mathcal{S}$ is continuous and compact. Moreover $g_m \leq g_{m+j}$ implies $\mathcal{C}_{\rm loc}^{\alpha,g_m,+}(\Omega) \subset \mathcal{C}_{\rm loc}^{\alpha,g_{m+j},+}(\Omega)$.*

Finally our last result is the infinite tower property.

Theorem 1.6. *Let us consider the equation*

$$-\Delta u_m = g_m(u_m) \text{ in } B_R(0), \quad u_{\epsilon,m} = \epsilon \text{ on } \partial B_R(0), \tag{1.7}$$

where $g_m : (0, \infty) \to (0, \infty)$ is non increasing locally Hölder continuous function singular at the origin with the properties $g_m(s) = g(s)$ for all $s \geq 1$ and $\lim_{m\to\infty} g_m(s) = \infty$ for all $s \in (0, 1)$, $m = 1, \dots, \infty$. Then there exists $\delta > 0$ and u_∞ such that $\lim_{m\to\infty} u_m = u_\infty$ where $-\Delta u_\infty = \lim_{m\to\infty} g_m(u_m) = \infty$ on the annulus $A(R - \delta, R)$. Therefore the tower

$$\mathcal{C}_{\rm loc}^{\alpha,g_1,+}(\Omega) \subset \dots \subset \mathcal{C}_{\rm loc}^{\alpha,g_m,+}(\Omega) \subset \dots \subset \mathcal{C}_{\rm loc}^{\alpha,g_{m+j},+}(\Omega) \subset \cdots$$

actually goes to infinite on the annulus $A(R - \delta, R)$.

2. Auxiliary results

Let us consider the problem

$$-\mathcal{L}u_m = g_m(u_m) \text{ in } \Omega, \quad u_m = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where $g_{m+j} \geq g_m$, are non-increasing locally Hölder continuous functions on $(0,\infty)$ and singular at zero.

Lemma 2.1. *Let u_m be a solution of* (2.1)*. Then $u_{m+j} \geq u_m$.*

Proof. Suppose that there exists $x_0 \in \Omega$ such that $u_m(x_0) > u_{m+j}(x_0)$. We define $\Omega_\nu = \{x \in \Omega | \nu + u_m(x) > u_{m+j}(x)\}$. Then $\Omega_\nu \neq \emptyset$ for all $\nu \geq 0$. Moreover $g_m(u_m(x)+\nu) \leq g_{m+j}(u_m(x)+\nu) < g_{m+j}(u_{m+j}(x))$ for all $x \in \Omega_\nu$. Setting

$$\Omega_\tau = \{x \in \Omega | u_m(x) > \tau + u_{m+j}(x)\},$$

we deduce that $\Omega_\tau \neq \emptyset$ and $\Omega_\tau \subset \Omega_\nu$ for τ small enough. Moreover, $g_m(u_m(x)) \leq g_{m+j}(u_m(x)) \leq g_{m+j}(u_{m+j}(x))$ for all $x \in \Omega_\tau$. Therefore

$$-\mathcal{L}u_m \leq -\mathcal{L}(u_{m+j}+\tau) \text{ in } \Omega_\tau, \quad u_m = u_{m+j}+\tau \text{ on } \partial\Omega_\tau.$$

and we obtain $u_m \leq u_{m+j} + \tau$ in Ω_τ [13, Theorem 3.3], a contradiction. □

Lemma 2.2. *Let u_m be a solution of* (2.1)*. Then $g_{m+j}(u_{m+j}(x)) \geq g_m(u_m(x))$.*

Proof. Suppose that there exists $x_0 \in \Omega$ such that $g_m(u_m(x_0)) > g_{m+j}(u_{m+j}(x_0))$. Then there exists $\hat{\Omega} \subset \Omega$ such that

$$-\mathcal{L}u_m \geq -\mathcal{L}u_{m+j} \text{ in } \hat{\Omega}, \quad u_m = u_{m+j} \text{ on } \partial\hat{\Omega}.$$

We infer that $u_m \geq u_{m+j}$ in $\hat{\Omega}$ [13, Theorem 3.3]. Therefore, $g_m(u_m(x)) \leq g_m(u_{m+j}(x)) \leq g_{m+j}(u_{m+j}(x))$ for all $x \in \hat{\Omega}$. A contradiction. □

Proof of Theorem 1.4. For any $f \in \mathcal{C}^{\alpha,g,+}_{\rm loc}(\Omega)$, we have $f_k = \min(k,f) \in C^\alpha(\overline{\Omega})$. Therefore there exist a unique solution $v_k \in C^{2,\alpha}(\overline{\Omega})$ of the problem

$$-\mathcal{L}v_k = f_k \text{ in } \Omega, \quad v_k = 0 \text{ on } \partial\Omega \tag{2.2}$$

Using [13, Corollary 6.3], we obtain

$$d|Dv_k|_{0;\Omega'} + d^2|D^2v_k|_{0;\Omega'} + d^{2+\alpha}[D^2v_k]_{\alpha;\Omega'} \leq C(|v_k|_{0;\Omega''} + |f_k|_{0,\alpha;\Omega''})$$

where $\Omega' \subset \Omega'' \subset \Omega$, $d = \operatorname{dist}(\Omega', \partial\Omega'')$ and C is independent of k. Moreover $v_k \leq u$, it follows that $v_k \to v$ in $C^2_{\rm loc}(\Omega) \cap C^0(\overline{\Omega})$ were v solves equation 1.6. □

Proof of Theorem 1.5. This theorem is a direct consequence of the proof of Theorem 1.4 and Lemmas 2.1 and 2.2 . □

Proof of Theorem 1.6. This theorem is a direct consequence of [1]. □

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Carlos Cesar Aranda
Blue Angel Navire Research laboratory, Rue Eddy 113 Gatineau QC, Canada
E-mail address: carloscesar.aranda@gmail.com