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POSITIVE SOLUTIONS FOR A SYSTEM OF SECOND-ORDER BOUNDARY-VALUE PROBLEMS INVOLVING FIRST-ORDER DERIVATIVES

KUN WANG, ZHILIN YANG

Abstract. In this article we study the existence and multiplicity of positive solutions for the system of second-order boundary value problems involving first order derivatives

$$\begin{gathered}-u'' = f(t,u,u',v,v'),\\ -v'' = g(t,u,u',v,v'),\\ u(0)=u'(1)=0, \quad v(0)=v'(1)=0.\end{gathered}$$

Here $f,g \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)(\mathbb{R}_+ := [0,\infty))$. We use fixed point index theory to establish our main results based on a priori estimates achieved by utilizing Jensen's integral inequality for concave functions and $\mathbb{R}_+^2$-monotone matrices.

1. Introduction

In this article we study the existence and multiplicity of positive solutions for the system of second-order boundary value problems involving first order derivatives

$$\begin{gathered}-u'' = f(t,u,u',v,v'),\\ -v'' = g(t,u,u',v,v'),\\ u(0)=u'(1)=0, \quad v(0)=v'(1)=0,\end{gathered} \tag{1.1}$$

where $f \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)$ and $g \in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)$. By a positive solution of (1.1), we mean a pair of functions $(u,v)\in C^2[0,1]\times C^2[0,1]$ that solve (1.1) and satisfy $u(t)\ge 0$, $v(t)\ge 0$ for all $t\in[0,1]$, with at least one of them positive on $(0,1]$.

Boundary-value problems for systems of nonlinear second-order ordinary differential equations arise from physics, biology, chemistry, and other applied sciences, and, as a result, play an important role in both theory and application. Recently, there are many articles in this direction. We refer the reader to [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 20] and the references cited therein. It should remarked that in the works cited above, only a few of them involve first-order derivatives in their nonlinearities.

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In [13], the authors study the existence and multiplicity of positive solutions for the system

$$
\begin{aligned}
&(-1)^m w^{(2m)} \\
&= f(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}), \\
&(-1)^n z^{(2n)} \\
&= g(t, w, w', -w''', \dots, (-1)^{m-1} w^{(2m-1)}, z, z', -z''', \dots, (-1)^{n-1} z^{(2n-1)}), \\
&\qquad w^{(2i)}(0) = w^{(2i+1)}(1) = 0 \quad (i = 0, 1, \dots, m-1), \\
&\qquad z^{(2j)}(0) = z^{(2j+1)}(1) = 0 \quad (j = 0, 1, \dots, n-1).
\end{aligned}
$$

The hypotheses imposed on the nonlinearities f and g are formulated in terms of two linear functions $h_1(x)$ and $h_2(y)$. The main results in [13] are established by using fixed point index theory based on a priori estimates of positive solutions achieved by utilizing new integral inequalities and nonnegative matrices.

In [17], motivated by [18], Yang and Kong studied the system of second-order boundary value problems involving first-order derivatives

$$
-u_i'' = f_i(t, u_1, u_1', \dots, u_n, u_n'), u_i(0) = u_i'(1) = 0, i = 1, \dots, n. \tag{1.2}
$$

To obtain the a priori estimates of positive solutions, the authors develop some integral identities and inequalities so that the main conditions imposed on f_i's in [17] can be formulated in terms of simple linear functions of the form $g_i(x_1, \dots, x_{2n}) := \sum_{i=1}^n a_i(x_{2i-1} + 2x_{2i})$. More precisely, for example, (H2) in [17] states that there exist a nonnegative matrix $A = (a_{ij})_{n\times n}$ and a constant $c > 0$ such that the matrix $A - I$ is an $\mathbb{R}^n_+$-monotone matrix and

$$
f_i(t, x) \geq \sum_{j=1}^n a_{ij}(x_{2j-1} + 2x_{2j}) - c
$$

for all $(t, x) \in [0, 1] \times \mathbb{R}^{2n}_+$, $i = 1, \dots, n$.

Motivated by [13, 17, 18], in this paper, we study the existence and multiplicity of positive solutions for (1.1). We use fixed point index theory to establish our main results based on a priori estimates of positive solutions for some associated problems, generalizing the corresponding ones for the single boundary value problem

$$
-u'' = f(t, u, u'), \quad u(0) = u'(1) = 0
$$

in [18]. Our generalizations are not routine, as our conditions imposed on the nonlinearities f and g, unlike these in [18], involve both linear functions on $\mathbb{R}^4_+$ and concave functions on $\mathbb{R}_+$; these functions describe how the nonlinearities f, g grow and enable us to treat the three cases of them: one with both superlinear, one with both sublinear and the last with one superlinear and the other sublinear. Also, it is of interest to note that, for nonnegative constants p, q, ξ, η and nonnegative concave functions φ, ψ, we have to prove the ratio

$$
\frac{\int_0^1 (pu(t) + 2qu'(t))\varphi(t)dt}{\int_0^1 (\xi u(t) + 2\eta u'(t))\psi(t)dt}
$$

is bounded away from both 0 and ∞ (see Lemma 2.2 below for more details). This is a great difference between this article and [13, 17].

We use fixed point index theory to establish our main results based on a priori estimates of positive solutions achieved by utilizing Jensen's integral equality for concave functions and $\mathbb{R}_+^2$-monotone matrices. More precisely, Jensen's inequality is mainly applied to derive the boundedness of weighted integrals of positive solutions for some problems associated to (1.1), whereas $\mathbb{R}_+^2$-monotone matrices are employed to solve systems of inequalities resulting from some weighted integrals and thereby achieve the boundedness of associated weighted integrals.

This article is organized as follows. Section 2 contains some preliminary results, including two new integral inequalities and a new integral identity. Our main results, namely Theorems 3.4–3.6, are stated and proved in Section 3. Finally, in Section 4, we presented four examples of nonlinearities to illustrate our main results.

2. Preliminaries

Let $E := C^1([0,1],\mathbb{R})$ and

$$P := \{u \in E : u(t) \geq 0, u'(t) \geq 0, \forall t \in [0,1]\}, \|u\| := \max\{\|u\|_0, \|u'\|_0\},$$

where $\|u\|_0 := \max\{|u(t)| : t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a real Banach space and P is a cone in E. For $(u,v) \in E^2$, let

$$\|(u,v)\| := \max\{\|u\|, \|v\|\}.$$

Then E^2 is also a real Banach space under the above norm and P^2 is a cone in E^2.

Let $k(t,s) := \min\{t,s\}$ and

$$(Tu)(t) := \int_0^1 k(t,s)u(s)ds.$$

Then $T : E \to E$ is a completely continuous, positive, linear operator, with the spectral radius $r(T) = \frac{4}{\pi^2}$ and

$$(T\varphi)(s) = \int_0^1 k(t,s)\varphi(t)dt = r(T)\varphi(s) \tag{2.1}$$

where $\varphi(t) := \sin\frac{\pi}{2}t$.

In our setting, problem (1.1) is equivalent to the system of nonlinear integral equations

$$\begin{aligned} u(t) &= \int_0^1 k(t,s)f(s,u(s),u'(s),v(s),v'(s))ds, \\ v(t) &= \int_0^1 k(t,s)g(s,u(s),u'(s),v(s),v'(s))ds. \end{aligned} \tag{2.2}$$

Define the operators $A_i(i=1,2) : P^2 \to P$ and $A : P^2 \to P^2$ by

$$\begin{gathered} A_1(u,v)(t) := \int_0^1 k(t,s)f(s,u(s),u'(s),v(s),v'(s))ds, \\ A_2(u,v)(t) := \int_0^1 k(t,s)g(s,u(s),u'(s),v(s),v'(s))ds, \\ A(u,v)(t) := (A_1(u,v), A_2(u,v)). \end{gathered}$$

Now $f \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)$ and $g \in C([0,1] \times \mathbb{R}_+^4, \mathbb{R}_+)$ imply that A_i and A are completely continuous operators. Clearly, the existence of positive solutions for (1.1) is equivalent to that of positive fixed points of $A : P^2 \to P^2$.

To establish the priori estimates of positive solutions for some problems associated with (1.1), we need two transcendental equations(see [18]).

For any $\xi > \eta > 0$, let $\mu(\xi,\eta) \in (1/\xi, 1/\eta)$ denote the minimal positive solution of the transcendental equation

$$\eta\mu \sin\sqrt{\xi\mu - \eta^2\mu^2} - \sqrt{\xi\mu - \eta^2\mu^2}\cos\sqrt{\xi\mu - \eta^2\mu^2} = 0. \tag{2.3}$$

Also, for any $\eta > \xi > 0$, let $\nu(\xi,\eta) \in (1/\eta, 1/\xi)$ denote the unique solution of the transcendental equation

$$\eta\nu \sinh\sqrt{\eta^2\nu^2 - \xi\nu} - \sqrt{\eta^2\nu^2 - \xi\nu}\cosh\sqrt{\eta^2\nu^2 - \xi\nu} = 0. \tag{2.4}$$

in $(1/\eta, \infty)$. Let

$$\varphi_{\xi,\eta}(t) := \begin{cases} \frac{\pi}{2}\sin\frac{\pi t}{2}, & \xi > 0, \eta = 0, \\ te^t, & \xi = \eta > 0, \\ \frac{\xi\mu(\xi,\eta)}{\sqrt{\xi\mu(\xi,\eta) - \eta^2\mu^2(\xi,\eta)}}\exp(\eta\mu(\xi,\eta)t) & \\ \times \sin(\sqrt{\xi\mu(\xi,\eta) - \eta^2\mu^2(\xi,\eta)}t), & \xi > \eta > 0, \\ \frac{\xi\nu(\xi,\eta)}{\sqrt{\eta^2\nu^2(\xi,\eta) - \xi\nu(\xi,\eta)}}\exp(\eta\nu(\xi,\eta)t) & \\ \times \sinh(\sqrt{\eta^2\nu^2(\xi,\eta) - \xi\nu(\xi,\eta)}t), & \eta > \xi > 0, \end{cases} \tag{2.5}$$

$$\lambda(\xi,\eta) := \begin{cases} \frac{\pi^2}{4\xi}, & \xi > 0, \eta = 0 \\ \frac{1}{\xi}, & \xi = \eta > 0, \\ \mu(\xi,\eta), & \xi > \eta > 0, \\ \nu(\xi,\eta), & \eta > \xi > 0. \end{cases} \tag{2.6}$$

for all $\xi > 0, \eta \geq 0$. Direct calculation shows

$$\int_0^1 \varphi_{\xi,\eta}(t)dt = 1. \tag{2.7}$$

Lemma 2.1. *Suppose $\psi \in C([0,1], \mathbb{R}_+)$ is not identically vanishing on $[0,1]$, and $v \in C([0,1], \mathbb{R}_+)$ is a concave function. Let $\varrho(\psi) := \int_0^1 t\psi(t)dt > 0$. Then we have*

$$\int_0^1 \psi(t)v(t)dt \geq v(1)\varrho(\psi). \tag{2.8}$$

Proof. By the concavity of v and the nonnegativity of ψ, we have

$$\int_0^1 \psi(t)v(t)dt = \int_0^1 \psi(t)v(t\cdot 1 + (1-t)\cdot 0)dt \geq v(1)\int_0^1 t\psi(t)dt = v(1)\varrho(\psi).$$

This completes the proof. □

Denote

$$P_0 := \{u \in P : u \text{ is concave on } [0,1], u(0) = u'(1) = 0\}.$$

Lemma 2.2. *Let $\xi_i > 0, \eta_i \geq 0, \varphi_{(\xi_i,\eta_i)} (i = 1,2,3)$ be defined by (2.5). Define*

$$\beta(\xi_1,\eta_1,\xi_2,\eta_2,\xi_3,\eta_3) := \sup_{u \in P_0\setminus\{0\}} \frac{\int_0^1 (\xi_1 u(t) + 2\eta_1 u'(t))\varphi_{\xi_2,\eta_2}(t)dt}{\int_0^1 (\xi_3 u(t) + 2\eta_3 u'(t))\varphi_{\xi_3,\eta_3}(t)dt}.$$

Then $0 < \beta(\xi_1,\eta_1,\xi_2,\eta_2,\xi_3,\eta_3) < \infty$.

Proof. If $u \in P_0$, then by Lemma 2.1, we have

$$\begin{aligned} &\int_0^1 (\xi_3 u(t) + 2\eta_3 u'(t))\varphi_{\xi_3,\eta_3}(t)dt \\ &\geq \int_0^1 \xi_3 u(t)\varphi_{\xi_3,\eta_3}(t)dt = \xi_3 \int_0^1 u(t)\varphi_{\xi_3,\eta_3}(t)dt \\ &\geq u(1)\xi_3 \varrho(\varphi_{\xi_3,\eta_3}) = \xi_3\|u\|_0 \varrho(\varphi_{\xi_3,\eta_3}) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\int_0^1 (\xi_1 u(t) + 2\eta_1 u'(t))\varphi_{\xi_2,\eta_2}(t)dt \\ &= \xi_1 \int_0^1 u(t)\varphi_{\xi_2,\eta_2}(t)dt + 2\eta_1 \int_0^1 u'(t)\varphi_{\xi_2,\eta_2}(t)dt \\ &\leq \xi_1\|\varphi_{\xi_2,\eta_2}\|_0 \int_0^1 u(t)dt + 2\eta_1\|\varphi_{\xi_2,\eta_2}\|_0 \int_0^1 u'(t)dt \\ &\leq \xi_1\|\varphi_{\xi_2,\eta_2}\|_0\|u\|_0 + 2\eta_1\|\varphi_{\xi_2,\eta_2}\|_0 u(1) \\ &= (\xi_1 + 2\eta_1)\|\varphi_{\xi_2,\eta_2}\|_0\|u\|_0. \end{aligned} \tag{2.10}$$

Combining (2.9) and (2.10), we obtain

$$\beta(\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3) \leq \frac{(\xi_1 + 2\eta_1)\|\varphi_{\xi_2,\eta_2}\|_0}{\xi_3 \varrho(\varphi_{\xi_3,\eta_3})} < \infty.$$

This completes the proof. □

Lemma 2.3. *If $u \in C^2[0,1]$, $u(0) = u'(1) = 0$, $\xi > 0$, $\eta \geq 0$, then*

$$\int_0^1 -u''(t)\varphi_{\xi,\eta}(t)dt = \lambda(\xi,\eta)\int_0^1 (\xi u(t) + 2\eta u'(t))\varphi_{\xi,\eta}(t)dt, \tag{2.11}$$

where $\varphi_{\xi,\eta}$ and $\lambda(\xi,\eta)$ are defined by (2.5) and (2.6) respectively.

Proof. We just prove (2.11) in the case $\xi > \eta > 0$; the remaining cases can be proved in the same way. Let

$$a := \eta\mu(\xi,\eta), \quad b := \sqrt{\xi\mu(\xi,\eta) - \eta^2\mu^2(\xi,\eta)}.$$

Then $\varphi_{\xi,\eta}(t) = \frac{\xi a}{\eta b} e^{at}\sin bt$, and

$$a^2 + b^2 = \xi\mu(\xi,\eta), a\sin b - b\cos b = 0. \tag{2.12}$$

Integrate by parts over $[0,1]$ and use (2.12) to obtain

$$\begin{aligned} &\int_0^1 -u''(t)\varphi_{\xi,\eta}(t)dt \\ &= \frac{\xi a}{\eta b}\int_0^1 -u''(t)e^{at}\sin bt dt \\ &= \frac{\xi a}{\eta b}\int_0^1 u'(t)e^{at}(a\sin bt + b\cos bt)dt \\ &= \mu(\xi,\eta)\int_0^1 2\eta u'(t)\varphi_{\xi,\eta}(t)dt + \frac{\xi a}{\eta b}\int_0^1 u'(t)e^{at}(b\cos bt - a\sin bt)dt \\ &= \mu(\xi,\eta)\int_0^1 (\xi u(t) + 2\eta u'(t))\varphi_{\xi,\eta}(t)dt. \end{aligned} \tag{2.13}$$

This completes the proof. □

Lemma 2.4 ([3]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \bigcap P \to P$ is a completely continuous operator. If there exists $w_0 \in P \backslash \{0\}$ such that*

$$w - Tw \neq \lambda w_0, \forall \lambda \geq 0, \omega \in \partial\Omega \cap P,$$

then $i(T, \Omega \bigcap P, P) = 0$, where i indicates the fixed point index on P.

Lemma 2.5 ([3]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $T : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If*

$$w - \lambda Tw \neq 0, \forall \lambda \in [0, 1], \quad w \in \partial\Omega \cap P,$$

then $i(T, \Omega \cap P, P) = 1$.

Lemma 2.6 ([19, Lemma 2.4]). *If p is concave on $[d, \infty)$, with $\lim_{y\to\infty} p(y)/y \geq 0$, then p is increasing on $[d, \infty)$ and*

$$p(y + z - d) \leq p(y) + p(z) - p(d) \tag{2.14}$$

for all $y, z \in [d, \infty)$.

3. Existence of positive solutions for (1.1)

Definition A real matrix B is said to be nonnegative if all elements of B are nonnegative.

Definition (see [1, p.112]) A real square matrix $M = (m_{ij})_{2\times 2}$ is called $\mathbb{R}^2_+$-monotone, if for any column vector $x \in \mathbb{R}^2$, $Mx \in \mathbb{R}^2_+ \Rightarrow x \in \mathbb{R}^2_+$.

For simplicity, we denote by $x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+$ and $I_\rho := [0, \rho]$ for $\rho > 0$. Now we list our hypotheses on f and g.

(H1) $f, g \in C([0, 1] \times \mathbb{R}^4_+, \mathbb{R}_+)$.

(H2) There exist $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $q \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that: (1) p is concave; (2) There are two constants $c > 0$ and $\mu_1 > 1$ such that

$$f(t, x) \geq p(x_3) - c, g(t, x) \geq q(x_1) - c, \quad \forall (t, x) \in [0, 1] \times \mathbb{R}^4_+,$$

and

$$p(q(t)) \geq \frac{\pi^4 \mu_1}{16} t - c, \quad \forall t \in \mathbb{R}_+.$$

(H3) For every $N > 0$, there exist two functions $\Phi_N, \Psi_N \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$f(t, x) \leq \Phi_N(x_2 + x_4), \quad g(t, x) \leq \Psi_N(x_2 + x_4)$$

for all $x \in I_N \times \mathbb{R}_+ \times I_N \times \mathbb{R}_+$, $t \in [0, 1]$, and

$$\int_0^\infty \frac{\tau d\tau}{\Phi_N(\tau) + \Psi_N(\tau) + \delta} = \infty$$

for all $\delta > 0$.

(H4) There are constants $a_i > 0$, $b_i \geq 0$, $c_i > 0$, $d_i \geq 0$ $(i = 1, 2)$ and $r > 0$ such that

$$\begin{pmatrix} f(t, x) \\ g(t, x) \end{pmatrix} \leq \begin{pmatrix} a_1 & 2b_1 & c_1 & 2d_1 \\ a_2 & 2b_2 & c_2 & 2d_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

for all $(t,x) \in [0,1] \times I_r^4$ and the matrix

$$B_1 := \begin{pmatrix} \lambda(a_1,b_1)-1, & -\beta(c_1,d_1,a_1,b_1,c_2,d_2) \\ -\beta(a_2,b_2,c_2,d_2,a_1,b_1), & \lambda(c_2,d_2)-1 \end{pmatrix}$$

is an $\mathbb{R}_+^2$-monotone matrix, where the entries $\beta(c_1,d_1,a_1,b_1,c_2,d_2)$ and $\beta(a_2,b_2,c_2,d_2,a_1,b_1)$ are defined as in Lemma 2.2.

(H5) There exist $\widetilde{p} \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $\widetilde{q} \in C(\mathbb{R}_+,\mathbb{R}_+)$ such that: (1) $\widetilde{p}$ is concave, $\widetilde{p}(0)=\widetilde{q}(0)=0$; (2) There are two constants $r_2>0$ and $\mu_2>1$ such that

$$f(t,x) \geq \widetilde{p}(x_3), g(t,x) \geq \widetilde{q}(x_1), \forall (t,x) \in [0,1] \times I_{r_2}^4,$$
$$\widetilde{p}(\widetilde{q}(t)) \geq \frac{\pi^4\mu_2}{16}t, \forall t \in [0,r_2]$$

(H6) There are nonnegative constants $a_i>0$, $b_i \geq 0$, $c_i>0$, $d_i \geq 0$ $(i=3,4)$ and $c>0$ such that

$$\begin{pmatrix} f(t,x) \\ g(t,x) \end{pmatrix} \leq \begin{pmatrix} a_3 & 2b_3 & c_3 & 2d_3 \\ a_4 & 2b_4 & c_4 & 2d_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} c \\ c \end{pmatrix}$$

for all $(t,x) \in [0,1] \times \mathbb{R}_+^4$ and the matrix

$$B_2 := \begin{pmatrix} \lambda(a_3,b_3)-1, & -\beta(c_3,d_3,a_3,b_3,c_4,d_4) \\ -\beta(a_4,b_4,c_4,d_4,a_3,b_3), & \lambda(c_4,d_4)-1 \end{pmatrix}$$

is an $\mathbb{R}_+^2$-monotone matrix, where the entries $\beta(c_3,d_3,a_3,b_3,c_4,d_4)$ and $\beta(a_4,b_4,c_4,d_4,a_3,b_3)$ are defined as in Lemma 2.2.

(H7) $f(t,x)$ and $g(t,x)$ are increasing in $x \in \mathbb{R}_+^4$, and there is a constant $\omega>0$ such that

$$\int_0^1 f(s,\omega,\omega,\omega,\omega)ds < \omega, \quad \int_0^1 g(s,\omega,\omega,\omega,\omega)ds < \omega.$$

Remark 3.1 ([1, p.113]). A real square matrix M is $\mathbb{R}_+^2$-monotone if and only if M is nonsingular and M^{-1} is nonnegative.

Remark 3.2. Let $l_{ij}(i,j=1,2)$ be four nonnegative constants. Then it is easy to see that the matrix $D := \begin{pmatrix} l_{11}-1 & -l_{12} \\ -l_{21} & l_{22}-1 \end{pmatrix}$ is an $\mathbb{R}_+^2$-monotone matrix if and only if $l_{11}>1$, $l_{22}>1, \det D = (l_{11}-1)(l_{22}-1)-l_{12}l_{21}>0$.

Remark 3.3. $f(t,x)$ is said to be increasing in x if

$$f(t,x) \leq f(t,y)$$

holds for every pair $x,y \in \mathbb{R}_+^4$ with $x \leq y$ for all $t \in [0,1]$, where the partial ordering $\leq$ in $\mathbb{R}_+^4$ is understood componentwise.

We adopt the convention in the sequel that $\widehat{c}_1,\widehat{c}_2,\dots$ stand for different positive constants and $\Omega_\rho := \{v \in E : \|v\| < \rho\}$ for $\rho>0$.

Theorem 3.4. *If* (H1)–(H4) *hold, then* (1.1) *has at least one positive solution.*

Proof. By (H2), we obtain

$$A_1(u,v)(t) \ge \int_0^1 k(t,s)p(v(s))ds - \widehat{c}_1, \quad A_2(u,v)(t) \ge \int_0^1 k(t,s)q(u(s))ds - \widehat{c}_1 \tag{3.1}$$

for all $(u,v) \in P^2$, $t \in [0,1]$. We claim that the set

$$\mathcal{M}_1 := \{(u,v) \in P^2 : (u,v) = A(u,v) + \lambda(\sigma,\sigma), \lambda \ge 0\}$$

is bounded, where $\sigma(t) := te^{-t}$. Indeed, if $(u_0,v_0) \in \mathcal{M}_1$, then there exist a constant $\lambda_0 \ge 0$ such that $(u_0,v_0) = A(u_0,v_0) + \lambda_0(\sigma,\sigma)$, which can be written in the form

$$\begin{aligned} u_0(t) &= \int_0^1 k(t,s)f(s,u_0(s),u_0'(s),v_0(s),v_0'(s))ds + \lambda_0\sigma(t), \\ v_0(t) &= \int_0^1 k(t,s)g(s,u_0(s),u_0'(s),v_0(s),v_0'(s))ds + \lambda_0\sigma(t). \end{aligned}$$

By (H2) and (3.1), we have

$$u_0(t) \ge \int_0^1 k(t,s)p(v_0(s))ds - \widehat{c}_1, v_0(t) \ge \int_0^1 k(t,s)q(u_0(s))ds - \widehat{c}_1 \tag{3.2}$$

for all $t \in [0,1]$. The nonnegativity and concavity of p imply $\lim_{y\to\infty} p(y)/y \ge 0$. We also note $\max_{(t,s)\in[0,1]\times[0,1]} k(t,s) = 1$. Now Lemma 2.6 and Jensen's inequality imply

$$p(v_0(t)) \ge p(v_0(t) + \widehat{c}_1) - p(\widehat{c}_1) \ge \int_0^1 k(t,s)p(q(u_0(s)))ds - p(\widehat{c}_1). \tag{3.3}$$

This, together with (3.2) and (H2), implies

$$\begin{aligned} u_0(t) &\ge \int_0^1 k(t,s)\Big[\int_0^1 k(s,\tau)p(q(u_0(\tau)))d\tau - p(\widehat{c}_1)\Big]ds - \widehat{c}_1 \\ &\ge \int_0^1\int_0^1 k(t,s)k(s,\tau)p(q(u_0(\tau)))\,ds\,d\tau - \widehat{c}_2 \\ &\ge \int_0^1\int_0^1 k(t,s)k(s,\tau)\Big[\frac{\pi^4\mu_1}{16}u_0(\tau) - c\Big]ds\,d\tau - \widehat{c}_2 \\ &\ge \frac{\pi^4\mu_1}{16}\int_0^1\int_0^1 k(t,s)k(s,\tau)u_0(\tau)\,ds\,d\tau - \widehat{c}_3. \end{aligned} \tag{3.4}$$

Multiply both sides of the last inequality by $\varphi(t) := \sin(\pi t/2)$ and integrate over $[0,1]$ and use (2.1) twice to obtain

$$\begin{aligned} \int_0^1 \varphi(t)u_0(t)dt &\ge \frac{\pi^4\mu_1}{16}\int_0^1\int_0^1\int_0^1 \varphi(t)k(t,s)k(s,\tau)u_0(\tau)dt\,ds\,d\tau - \frac{2\widehat{c}_3}{\pi} \\ &= \mu_1\int_0^1 \varphi(t)u_0(t)dt - \frac{2\widehat{c}_3}{\pi}, \end{aligned} \tag{3.5}$$

so that

$$\int_0^1 \varphi(t)u_0(t)dt \le \frac{2\widehat{c}_3}{\pi(\mu_1 - 1)}. \tag{3.6}$$

By Lemma 2.1, we have

$$\|u_0\|_0 = u_0(1) \le \frac{2\widehat{c}_3}{\pi\varrho(\varphi)(\mu_1 - 1)} = \frac{\widehat{c}_3\pi}{2(\mu_1 - 1)}. \tag{3.7}$$

Multiply the first inequality of (3.2) by $\varphi(t)$, integrate over $[0,1]$ and use (2.1) to obtain
$$\int_0^1 u_0(t)\varphi(t)dt \geq \frac{4}{\pi^2}\int_0^1 p(v_0(t))\varphi(t)dt - \frac{2}{\pi}\widehat{c}_1.$$
This, along with (3.6), implies
$$\int_0^1 p(v_0(t))\varphi(t)dt \leq \frac{\pi^2}{4}\Big(\frac{2\widehat{c}_1}{\pi} + \int_0^1 u_0(t)\varphi(t)dt\Big) \leq \frac{\widehat{c}_1\pi}{2} + \frac{\widehat{c}_3\pi}{2(\mu_1-1)}. \tag{3.8}$$
By Lemma 2.1, we have
$$\begin{aligned}
\|v_0\|_0 = v_0(1) &\leq \frac{1}{\varrho(\varphi)}\int_0^1 v_0(t)\varphi(t)dt \\
&= \frac{\|v_0\|_0}{\varrho(\varphi)p(\|v_0\|_0)}\int_0^1 \varphi(t)\frac{v_0(t)}{\|v_0\|_0}p(\|v_0\|_0)dt \\
&\leq \frac{\|v_0\|_0}{\varrho(\varphi)p(\|v_0\|_0)}\int_0^1 \varphi(t)p(v_0(t))dt,
\end{aligned} \tag{3.9}$$
so that
$$p(\|v_0\|_0) \leq \frac{1}{\varrho(\varphi)}\int_0^1 \varphi(t)p(v_0(t))dt \leq \frac{\widehat{c}_1\pi^3}{8} + \frac{\widehat{c}_3\pi^4}{16(\mu_1-1)}.$$
(H2) implies that p is strictly increasing and $\lim_{x\to\infty} p(x) = \infty$ (see Lemma 2.6). Consequently, there exists $\widehat{c}_4 > 0$ such that
$$\|v_0\|_0 \leq \widehat{c}_4.$$
Let $N := \max\{\frac{\widehat{c}_3\pi}{2(\mu_1-1)}, \widehat{c}_4\}$. Then
$$\|u\|_0 \leq N, \quad \|v\|_0 \leq N, \quad \forall (u,v) \in \mathcal{M}_1. \tag{3.10}$$
This establishes the a priori bound of $\mathcal{M}_1$ for $\|(u,v)\|_0$. Now it remains to derive the a priori bound of $\mathcal{M}_1$ for $\|(u',v')\|_0$. To this end, we let
$$\Lambda := \{\mu \geq 0 : \text{there exists } (u,v) \in P^2 \text{ such that } (u,v) = A(u,v) + \mu(\sigma,\sigma)\}.$$
Now (3.10) imply that $\mu_0 := \sup\Lambda < \infty$. By (H3), there are two functions $\Phi_N, \Psi_N \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that
$$\begin{gathered}
f(t,u(t),u'(t),v(t),v'(t)) \leq \Phi_N(u'(t)+v'(t)), \\
g(t,u(t),u'(t),v(t),v'(t)) \leq \Psi_N(u'(t)+v'(t))
\end{gathered}$$
for all $(u,v) \in \mathcal{M}_1$, $t \in [0,1]$. Hence, for all $(u,v) \in \mathcal{M}_1$ and for some $\mu \geq 0$, we have
$$\begin{aligned}
-u''(t) &= f(t,u(t),u'(t),v(t),v'(t)) + \mu(2-t)e^{-t} \\
&\leq \Phi_N(u'(t)+v'(t)) + \mu(2-t)e^{-t} \\
&\leq \Phi_N(u'(t)+v'(t)) + 2\mu_0,
\end{aligned}$$
$$\begin{aligned}
-v''(t) &= g(t,u(t),u'(t),v(t),v'(t)) + \mu(2-t)e^{-t} \\
&\leq \Psi_N(u'(t)+v'(t)) + \mu(2-t)e^{-t} \\
&\leq \Psi_N(u'(t)+v'(t)) + 2\mu_0,
\end{aligned}$$
so that
$$-(u''(t)+v''(t))(u'(t)+v'(t)) \leq (u'(t)+v'(t))(\Phi_N(u'(t)+v'(t))$$

$$+ \Psi_N(u'(t)+v'(t)) + 4\mu_0).$$

This implies

$$\int_0^{u'(0)+v'(0)} \frac{\tau d\tau}{\Phi_N(\tau)+\Psi_N(\tau)+4\mu_0} \le \int_0^1 (u'(t)+v'(t))dt = u(1)+v(1) \le 2N$$

for all $(u,v) \in \mathcal{M}_1$. By (H3) again, there exists a constant $N_1 > 0$ such that

$$\|u'+v'\|_0 = u'(0)+v'(0) \le N_1, \quad \forall (u,v) \in \mathcal{M}_1.$$

This establishes the a priori bound of $\mathcal{M}_1$ for $\|(u',v')\|_0$ and, in turn, implies that $\mathcal{M}_1$ is bounded(notice that we have achieved the a priori bound of $\mathcal{M}_1$ for $\|(u,v)\|_0$ in (3.10)). Taking $R > \max\{\sup\{\|(u,v)\| : (u,v) \in \mathcal{M}_1\}, r\}$, we have

$$(u,v) \ne A(u,v) + \lambda(\sigma,\sigma), \quad \forall (u,v) \in \partial\Omega_R \cap P^2, \lambda \ge 0.$$

Now Lemma 2.4 yields

$$i(A, \Omega_R \cap P^2, P^2) = 0. \tag{3.11}$$

Let

$$\mathcal{M}_2 := \{(u,v) \in \overline{\Omega}_r \cap P^2 : (u,v) = \lambda A(u,v), \lambda \in [0,1]\}.$$

Now we want to prove that $\mathcal{M}_2 = \{0\}$. Indeed, if $(u,v) \in \mathcal{M}_2$, then $(u,v) \in P_0^2$ and $(u,v) = \lambda A(u,v)$ for some $\lambda \in [0,1]$, written componentwise as

$$\begin{aligned} u(t) &= \lambda \int_0^1 k(t,s) f(s,u(s),u'(s),v(s),v'(s))ds, \\ v(t) &= \lambda \int_0^1 k(t,s) g(s,u(s),u'(s),v(s),v'(s))ds, \end{aligned}$$

which are equivalent to

$$-u''(t) = \lambda f(t,u(t),u'(t),v(t),v'(t)), -v''(t) = \lambda g(t,u(t),u'(t),v(t),v'(t)).$$

By (H4), we have

$$\begin{aligned} -u''(t) &\le a_1 u(t) + 2b_1 u'(t) + c_1 v(t) + 2d_1 v'(t), \\ -v''(t) &\le a_2 u(t) + 2b_2 u'(t) + c_2 v(t) + 2d_2 v'(t). \end{aligned}$$

Multiply the last two inequalities by $\varphi_{a_1,b_1}(t)$ and $\varphi_{c_2,d_2}(t)$ respectively and integrate over $[0,1]$ and use Lemmas 2.2 and 2.3 to obtain

$$\begin{aligned} &\lambda(a_1,b_1) \int_0^1 (a_1 u(t) + 2b_1 u'(t))\varphi_{a_1,b_1}(t)dt \\ &\le \int_0^1 (a_1 u(t) + 2b_1 u'(t))\varphi_{a_1,b_1}(t)dt + \int_0^1 (c_1 v(t) + 2d_1 v'(t))\varphi_{a_1,b_1}(t)dt \\ &\le \int_0^1 (a_1 u(t) + 2b_1 u'(t))\varphi_{a_1,b_1}(t)dt \\ &\quad + \beta(c_1,d_1,a_1,b_1,c_2,d_2) \int_0^1 (c_2 v(t) + 2d_2 v'(t))\varphi_{c_2,d_2}(t)dt, \end{aligned}$$

$$\begin{aligned} &\lambda(c_2,d_2) \int_0^1 (c_2 v(t) + 2d_2 v'(t))\varphi_{c_2,d_2}(t)dt \\ &\le \int_0^1 (a_2 u(t) + 2b_2 u'(t))\varphi_{c_2,d_2}(t)dt + \int_0^1 (c_2 v(t) + 2d_2 v'(t))\varphi_{c_2,d_2}(t)dt \end{aligned}$$

$$\le \beta(a_2,b_2,c_2,d_2,a_1,b_1)\int_0^1 (a_1u(t)+2b_1u'(t))\varphi_{a_1,b_1}(t)dt + \int_0^1 (c_2v(t)+2d_2v'(t))\varphi_{c_2,d_2}(t)dt,$$

which can be written in the form

$$\begin{aligned}&\begin{pmatrix}\lambda(a_1,b_1)-1 & -\beta(c_1,d_1,a_1,b_1,c_2,d_2)\\ -\beta(a_2,b_2,c_2,d_2,a_1,b_1) & \lambda(c_2,d_2)-1\end{pmatrix}\\ &\cdot\begin{pmatrix}\int_0^1(a_1u(t)+2b_1u'(t))\varphi_{a_1,b_1}(t)dt\\ \int_0^1(c_2v(t)+2d_2v'(t))\varphi_{c_2,d_2}(t)dt\end{pmatrix}\\ &= B_1\begin{pmatrix}\int_0^1(a_1u(t)+2b_1u'(t))\varphi_{a_1,b_1}(t)dt\\ \int_0^1(c_2v(t)+2d_2v'(t))\varphi_{c_2,d_2}(t)dt\end{pmatrix}\le\begin{pmatrix}0\\0\end{pmatrix}.\end{aligned}$$

(H4) again implies

$$\begin{pmatrix}\int_0^1(a_1u(t)+2b_1u'(t))\varphi_{a_1,b_1}(t)dt\\ \int_0^1(c_2v(t)+2d_2v'(t))\varphi_{c_2,d_2}(t)dt\end{pmatrix}\le B_1^{-1}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Consequently,

$$\int_0^1 (a_1u(t)+2b_1u'(t))\varphi_{a_1,b_1}(t)dt=\int_0^1 (c_2v(t)+2d_2v'(t))\varphi_{c_2,d_2}(t)dt=0$$

and $u=v=0$, whence $\mathcal{M}_2=\{0\}$, as required. As a result of this, we have

$$(u,v)\neq\lambda A(u,v),\quad \forall (u,v)\in\partial\Omega_r\cap P^2,\lambda\in[0,1].$$

Now Lemma 2.5 yields

$$i(A,\Omega_r\cap P^2,P^2)=1.$$

This together with (3.11) implies

$$i(A,(\Omega_R\backslash\overline{\Omega}_r)\cap P^2,P^2)=0-1=-1.$$

Therefore, A has at least one fixed point (u,v) on $(\Omega_R\backslash\overline{\Omega}_r)\cap P^2$ and thus (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.5. *If* (H1), (H5), (H6) *hold, then* (1.1) *has at least one positive solution.*

Proof. By (H5), for all $(u,v)\in\overline{\Omega}_{r_2}\cap P^2$, we have

$$A_1(u,v)(t)\ge\int_0^1 k(t,s)\widetilde{p}(v(s))ds,\quad A_2(u,v)(t)\ge\int_0^1 k(t,s)\widetilde{q}(u(s))ds.$$

Let

$$\mathcal{M}_3:=\{(u,v)\in\overline{\Omega}_{r_2}\cap P^2:(u,v)=A(u,v)+\lambda(\sigma,\sigma),\lambda\ge 0\}.$$

Now we want to prove that $\mathcal{M}_3\subset\{0\}$, where $\sigma(t):=te^{-t}$. If $(\widetilde{u},\widetilde{v})\in\mathcal{M}_3$, there exists $\widetilde{\lambda}\ge 0$ such that $(\widetilde{u},\widetilde{v})=A(\widetilde{u},\widetilde{v})+\widetilde{\lambda}(\sigma,\sigma)$, which implies

$$\widetilde{u}(t)\ge\int_0^1 k(t,s)\widetilde{p}(\widetilde{v}(s))ds,\widetilde{v}(t)\ge\int_0^1 k(t,s)\widetilde{q}(\widetilde{u}(s))ds \tag{3.12}$$

Note $\max_{(t,s)\in[0,1]\times[0,1]}k(t,s)=1$. By (H5) and Jensen's inequality, we have

$$\widetilde{u}(t)\ge\int_0^1 k(t,s)\widetilde{p}(\widetilde{v}(s))ds$$

$$
\begin{aligned}
&\geq \int_0^1 k(t,s)\widetilde{p}(\int_0^1 k(s,\tau)\widetilde{q}(\widetilde{u}(\tau))d\tau)ds \\
&\geq \int_0^1 \int_0^1 k(t,s)k(s,\tau)\widetilde{p}(\widetilde{q}(\widetilde{u}(\tau)))d\tau ds \\
&\geq \int_0^1 \int_0^1 k(t,s)k(s,\tau)\widetilde{p}(\widetilde{q}(\widetilde{u}(\tau)))\,ds\,d\tau \\
&\geq \frac{\pi^4\mu_2}{16}\int_0^1 \int_0^1 k(t,s)k(s,\tau)\widetilde{u}(\tau)\,ds\,d\tau\,.
\end{aligned}
$$

Multiply both sides of the above inequality by $\varphi(t) := \sin\frac{\pi}{2}t$ and integrate over $[0,1]$ and use (2.1) to obtain

$$\int_0^1 \widetilde{u}(t)\varphi(t)dt \geq \mu_2\int_0^1 \widetilde{u}(t)\varphi(t)dt,$$

so that $\int_0^1 \widetilde{u}(t)\varphi(t)dt = 0$, and whence $\widetilde{u}(t) \equiv 0$. This, together with (3.12), yields $\widetilde{p}(\widetilde{v}(t)) \equiv 0$, and, in particular,

$$\widetilde{p}(\|\widetilde{v}\|_0) = \widetilde{p}(\widetilde{v}(1)) = 0.$$

Note that (H5) implies that $\widetilde{p}$ is strictly increasing on $[0,\varepsilon]$ for sufficiently small $\varepsilon > 0$ (see Lemma 2.6) and thus $\widetilde{v}(1) = 0$. Hence $\widetilde{u} = \widetilde{v} = 0$, and $\mathcal{M}_3 \subset \{0\}$, as required. As a result of this, we have

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \forall (u,v) \in \partial\Omega_{r_2} \cap P^2, \lambda \geq 0.$$

Now Lemma 2.4 yields

$$i(A,\Omega_{r_2} \cap P^2, P^2) = 0. \tag{3.13}$$

Let

$$\mathcal{M}_4 := \{(u,v) \in P^2 : (u,v) = \lambda A(u,v), \lambda \in [0,1]\}.$$

We now assert that $\mathcal{M}_4$ is bounded. Indeed, if $(u,v) \in \mathcal{M}_4$, then $(u,v) \in P_0^2$ and $(u,v) = \lambda A(u,v)$ for some $\lambda \in [0,1]$, which can be written componentwise as

$$
\begin{aligned}
u(t) &= \lambda\int_0^1 k(t,s)f(s,u(s),u'(s),v(s),v'(s))ds, \\
v(t) &= \lambda\int_0^1 k(t,s)g(s,u(s),u'(s),v(s),v'(s))ds.
\end{aligned}
$$

Differentiate the last equations twice to obtain

$$-u''(t) = \lambda f(t,u(t),u'(t),v(t),v'(t)), \quad -v''(t) = \lambda g(t,u(t),u'(t),v(t),v'(t)),$$

for $t \in [0,1]$. By (H6), we have

$$-u''(t) \leq a_3u(t) + 2b_3u'(t) + c_3v(t) + 2d_3v'(t) + \widetilde{c}, \tag{3.14}$$

$$-v''(t) \leq a_4u(t) + 2b_4u'(t) + c_4v(t) + 2d_4v'(t) + \widetilde{c} \tag{3.15}$$

Multiply the last two inequalities by $\varphi_{a_3,b_3}(t)$ and $\varphi_{c_4,d_4}(t)$ respectively and integrate over $[0,1]$, and use Lemmas 2.2 and 2.3 to obtain

$$\begin{aligned}
&\lambda(a_3,b_3)\int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\
&\le \int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt + \int_0^1 (c_3v(t)+2d_3v'(t))\varphi_{a_3,b_3}(t)dt+\widetilde{c}\\
&\le \int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\
&\quad + \beta(c_3,d_3,a_3,b_3,c_4,d_4)\int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt+\widetilde{c},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda(c_4,d_4)\int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt\\
&\le \int_0^1 (a_4u(t)+2b_4u'(t))\varphi_{c_4,d_4}(t)dt + \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt+\widetilde{c}\\
&\le \beta(a_4,b_4,c_4,d_4,a_3,b_3)\int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\
&\quad + \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt+\widetilde{c},
\end{aligned}$$

which can be written in the form

$$\begin{aligned}
&\begin{pmatrix} \lambda(a_3,b_3)-1 & -\beta(c_3,d_3,a_3,b_3,c_4,d_4)\\ -\beta(a_4,b_4,c_4,d_4,a_3,b_3) & \lambda(c_4,d_4)-1\end{pmatrix}\\
&\cdot\begin{pmatrix} \int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\ \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt\end{pmatrix}\\
&= B_2\begin{pmatrix} \int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\ \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt\end{pmatrix} \le \begin{pmatrix}\widetilde{c}\\ \widetilde{c}\end{pmatrix}.
\end{aligned}$$

(H6) again implies

$$\begin{pmatrix} \int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt\\ \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt\end{pmatrix} \le B_2^{-1}\begin{pmatrix}\widetilde{c}\\ \widetilde{c}\end{pmatrix} := \begin{pmatrix}\widetilde{c}_1\\ \widetilde{c}_2\end{pmatrix}.$$

Let $\widetilde{c}_3 := \max\{\widetilde{c}_1,\widetilde{c}_2\} > 0$. Then we have

$$\int_0^1 (a_3u(t)+2b_3u'(t))\varphi_{a_3,b_3}(t)dt \le \widetilde{c}_3,\ \int_0^1 (c_4v(t)+2d_4v'(t))\varphi_{c_4,d_4}(t)dt \le \widetilde{c}_3,$$

for all $(u,v)\in\mathcal{M}_4$. By Lemma 2.1, we obtain

$$\|u\|_0 = u(1) \le \frac{\widetilde{c}_3}{a_3\varrho(\varphi_{a_3,b_3})},\quad \|v\|_0 = v(1) \le \frac{\widetilde{c}_3}{c_4\varrho(\varphi_{c_4,d_4})},$$

for all $(u,v)\in\mathcal{M}_4$. Let $\widetilde{N} = \max\{\frac{\widetilde{c}_3}{a_3\varrho(\varphi_{a_3,b_3})}, \frac{\widetilde{c}_3}{c_4\varrho(\varphi_{c_4,d_4})}\} > 0$. Then by (3.14) and (3.15), we have

$$\begin{aligned}
-u''(t) &\le (a_3+c_3)\widetilde{N} + 2b_3u'(t) + 2d_3v'(t) + \widetilde{c},\\
-v''(t) &\le (a_4+c_4)\widetilde{N} + 2b_4u'(t) + 2d_4v'(t) + \widetilde{c},
\end{aligned}$$

for all $(u,v) \in \mathcal{M}_4$. Adding the above inequalities yields

$$-u''(t) - v''(t) \le (a_3 + a_4 + c_3 + c_4)\widetilde{N} + 2(b_3 + b_4)u'(t) + 2(d_3 + d_4)v'(t) + 2\widetilde{c}.$$

Let

$$\widetilde{N}_2 := (a_3 + a_4 + c_3 + c_4)\widetilde{N} + 2\widetilde{c}, \widetilde{L} := 2(b_3 + b_4 + d_3 + d_4) + 1.$$

Noticing $u'(1) = v'(1) = 0$, we obtain

$$u'(t) + v'(t) \le \frac{\widetilde{N}_2}{\widetilde{L}}(e^{\widetilde{L} - \widetilde{L}t} - 1),$$

so that

$$\|u' + v'\|_0 = u'(0) + v'(0) \le \frac{\widetilde{N}_2}{\widetilde{L}}(e^{\widetilde{L}} - 1).$$

This proves the boundedness of $\mathcal{M}_4$, as asserted. Taking $R > \max\{\sup\{\|(u,v)\| : (u,v) \in \mathcal{M}_4\}, r_2\}$, we have

$$(u,v) \ne \lambda A(u,v), \quad \forall (u,v) \in \partial\Omega_R \cap (P \times P),\ \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, \Omega_R \cap P^2, P^2) = 1. \tag{3.16}$$

Combining (3.13) and (3.16) gives

$$i(A, (\Omega_R \backslash \overline{\Omega}_{r_2}) \cap P^2, P^2) = 1.$$

Hence A has at least one fixed point on $(\Omega_R \backslash \overline{\Omega}_{r_2}) \cap P^2$. Thus (1.1) has at least one positive solution. This completes the proof. □

Theorem 3.6. *If* (H1)–(H3), (H5), (H7) *hold, then* (1.1) *has at least two positive solutions.*

Proof. By (H7), we have

$$f(t,x) \le f(t,\omega,\omega,\omega,\omega), \quad g(t,x) \le g(t,\omega,\omega,\omega,\omega),$$

for all $t \in [0,1]$ and all $x \in I_\omega^4$. Consequently, we have for all $(u,v) \in \partial\Omega_\omega \cap P^2$,

$$\begin{aligned}
\|A_1(u,v)\|_0 = A_1(u,v)(1) &= \int_0^1 sf(s,u(s),u'(s),v(s),v'(s))ds \\
&\le \int_0^1 f(s,u(s),u'(s),v(s),v'(s))ds \\
&\le \int_0^1 f(s,\omega,\omega,\omega,\omega)ds \\
&< \omega = \|(u,v)\|,
\end{aligned}$$

$$\begin{aligned}
\|A_2(u,v)\|_0 = A_2(u,v)(1) &= \int_0^1 sg(s,u(s),u'(s),v(s),v'(s))ds \\
&\le \int_0^1 g(s,u(s),u'(s),v(s),v'(s))ds \\
&\le \int_0^1 g(s,\omega,\omega,\omega,\omega)ds \\
&< \omega = \|(u,v)\|,
\end{aligned}$$

$$\begin{aligned}\|(A_1(u,v))'\|_0 = (A_1(u,v))'(0) &= \int_0^1 f(s,u(s),u'(s),v(s),v'(s))ds \\ &\le \int_0^1 f(s,\omega,\omega,\omega,\omega)ds \\ &< \omega = \|(u,v)\|,\end{aligned}$$

and

$$\begin{aligned}\|(A_2(u,v))'\|_0 = (A_2(u,v))'(0) &= \int_0^1 g(s,u(s),u'(s),v(s),v'(s))ds \\ &\le \int_0^1 g(s,\omega,\omega,\omega,\omega)ds \\ &< \omega = \|(u,v)\|.\end{aligned}$$

The preceding inequalities imply

$$\|A(u,v)\| = \|(A_1(u,v),A_2(u,v))\| < \omega = \|(u,v)\|;$$

thus

$$(u,v) \neq \lambda A(u,v), \quad \forall (u,v) \in \partial\Omega_\omega \cap P^2,\ 0 \le \lambda \le 1.$$

Now Lemma 2.5 yields

$$i(A,\Omega_\omega \cap P^2, P^2) = 1. \tag{3.17}$$

By (H2), (H3) and (H5), we find that (3.11) and (3.13) hold. Note that we can choose $R > \omega > r_2$ in (3.11) and (3.13) (see the proofs of Theorems 3.4 and 3.5). Combining (3.11), (3.13) and (3.17), we obtain

$$\begin{gathered} i(A,(\Omega_R \backslash \overline{\Omega}_\omega) \cap P^2, P^2) = 0 - 1 = -1, \\ i(A,(\Omega_\omega \backslash \overline{\Omega}_{r_2}) \cap P^2, P^2) = 1 - 0 = 1. \end{gathered}$$

Therefore, A has at least two fixed points, with one on $(\Omega_R \backslash \overline{\Omega}_\omega) \cap P^2$ and the other on $(\Omega_\omega \backslash \overline{\Omega}_{r_2}) \cap P^2$. Hence (1.1) has at least two positive solutions. $\square$

4. Examples

In this section we present four examples to illustrate our main results.

Example 4.1. Let $(a_{ij})_{2\times 4}$ be a positive matrix with $1 < \alpha_i \le 2 (i = 1,2)$ and

$$f(t,x) := \Big(\sum_{j=1}^4 a_{1j}x_j\Big)^{\alpha_1}, \quad g(t,x) := \Big(\sum_{j=1}^4 a_{2j}x_j\Big)^{\alpha_2}, \quad x \in \mathbb{R}_+^4,\ t \in [0,1].$$

Then (H1)-(H4) holds with both f and g superlinear. By Theorem 3.4, Equaton (1.1) has at least one positive solution. It suffices to verify (H2)-(H4).

(1) Let $p(y) := y$ and $q(y) := a_{21}y^{\alpha_2}$. Then p is concave and $p(q(y))/y \to \infty (y \to \infty)$. It is easy to see that there exists $c > 0$ such that

$$f(t,x) \ge p(x_3) - c, g(t,x) \ge q(x_1) - c$$

for all $x \in \mathbb{R}_+^4$ and $t \in [0,1]$. This implies that (H2) holds true.

(2) Assumption (H3) holds with

$$\Phi_N(t) := \big((a_{12} + a_{14})t + 2N\big)^{\alpha_1}, \quad \Psi_N(t) := \big((a_{22} + a_{24})t + 2N\big)^{\alpha_2}$$

for every $N > 0$.

(3) Note that there is $r > 0$ such that

$$\begin{pmatrix} f(t,x) \\ g(t,x) \end{pmatrix} \le \frac{1}{3} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

for all $x \in I_r^4, t \in [0,1]$. Let $a := \frac{1}{3}$. Then

$$\lambda(a,a) = \frac{1}{a} = 3, \beta(a,a,a,a,a,a) = 1.$$

Obviously, the matrix

$$B_1 := \begin{pmatrix} \lambda(a,a) - 1 & -\beta(a,a,a,a,a,a) \\ -\beta(a,a,a,a,a,a) & \lambda(a,a) - 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

is $\mathbb{R}_2^+$-monotone. Therefore (H4) holds with $a_i = b_i = c_i = d_i = a (i = 1,2)$ and B_1 as defined above.

Example 4.2. Let $(b_{ij})_{2\times 4}$ be a positive matrix with $0 < \alpha_i < 1 (i = 3,4)$ and

$$f(t,x) := \Big(\sum_{j=1}^4 b_{1j}x_j\Big)^{\alpha_3}, \quad g(t,x) := \Big(\sum_{j=1}^4 b_{2j}x_j\Big)^{\alpha_4}, \quad x \in \mathbb{R}_+^4,\ t \in [0,1].$$

Now (H1), (H5) and (H6) hold with both f and g sublinear. By Theorem 3.5, (1.1) has at least one positive solution.

Example 4.3. Let $(c_{ij})_{2\times 4}$ be a positive matrix and

$$f(t,x) := \frac{\sum_{j=1}^4 c_{1j}x_j^{\frac{5}{3}}}{\sum_{j=1}^4 x_j + 1}, \quad g(t,x) := \frac{\sum_{j=1}^4 c_{2j}x_j^3}{\sum_{j=1}^4 x_j + 1}, \quad x \in \mathbb{R}_+^4,\ t \in [0,1].$$

Now (H1)–(H4) hold with f sublinear and g superlinear at ∞. By Theorem 3.4, Equation (1.1) has at least one positive solution.

Example 4.4. Let $(a_{ij})_{2\times 4}$, $(b_{ij})_{2\times 4}$ be two positive matrices, with $1 < \beta_i \le 2$, $0 < \gamma_i < 1 (i = 1,2)$ and

$$\Big(\sum_{j=1}^4 a_{1j}\Big)^{\beta_1} + \Big(\sum_{j=1}^4 b_{1j}\Big)^{\gamma_1} < 1, \quad \Big(\sum_{j=1}^4 a_{2j}\Big)^{\beta_2} + \Big(\sum_{j=1}^4 b_{2j}\Big)^{\gamma_2} < 1.$$

Let

$$\begin{aligned} f(t,x) &:= \Big(\sum_{j=1}^4 a_{1j}x_j\Big)^{\beta_1} + \Big(\sum_{j=1}^4 b_{1j}x_j\Big)^{\gamma_1}, \\ g(t,x) &:= \Big(\sum_{j=1}^4 a_{2j}x_j\Big)^{\beta_2} + \Big(\sum_{j=1}^4 b_{2j}x_j\Big)^{\gamma_2} \end{aligned}$$

for $x \in \mathbb{R}_+^4$, $t \in [0,1]$. Now (H1)–(H3), (H5) and (H7) hold with both f and g superlinear at ∞ and sublinear at 0. By Theorem 3.6, (1.1) has at least two positive solutions.

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Kun Wang

Department of Mathematics, Qingdao Technological University, Qingdao, 266033, China

E-mail address: wangkun880304@163.com

Zhilin Yang

Department of Mathematics, Qingdao Technological University, Qingdao, 266033, China

E-mail address: zhilinyang@sina.com