Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 136, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

BOUNDARY BEHAVIOR OF LARGE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN BORDERLINE CASES

ZHIJUN ZHANG

Abstract. In this article, we analyze the boundary behavior of solutions to the boundary blow-up elliptic problem

$$\Delta u = b(x)f(u), \quad u \geq 0,\ x \in \Omega,\ u|_{\partial\Omega} = \infty,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $f(u)$ grows slower than any u^p $(p > 1)$ at infinity, and $b \in C^\alpha(\bar{\Omega})$ which is non-negative in Ω and positive near $\partial\Omega$, may be vanishing on the boundary.

1. Introduction

In this article, we consider the boundary behavior of solutions to the boundary blow-up elliptic problem

$$\Delta u = b(x)f(u),\ u \geq 0, \quad x \in \Omega, \quad u|_{\partial\Omega} = \infty, \tag{1.1}$$

where the last condition means that $u(x) \to \infty$ as $d(x) = \text{dist}(x, \partial\Omega) \to 0$, Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, f satisfies

- (F1) $f \in C[0,\infty) \cap C^1(0,\infty)$, $f(0) = 0$ and $f(s)$ is increasing on $(0,\infty)$;
- (F2) the Keller-Osserman ([11], [15]) condition

$$\Theta(r) := \int_r^\infty \frac{ds}{\sqrt{2F(s)}} < \infty, \quad \forall r > 0, \quad F(s) = \int_0^s f(\tau)d\tau;$$

the function b satisfies

- (B1) $b \in C^\alpha(\bar{\Omega})$, is non-negative in Ω and positive near $\partial\Omega$.

The model problem (1.1) arises from many branches of mathematics and has generated a good deal of research, see, for instance, [1]-[3], [5]-[9], [11]-[13], [15]-[18] and the references therein.

When $b \equiv 1$ in Ω and f satisfies (F1), it is well-known that (1.1) has one solution $u \in C^2(\Omega)$ if and only if (F2) holds. Moreover, the blow-up rate of $u(x)$ near $\partial\Omega$

2000 *Mathematics Subject Classification.* 35J55, 35J60, 35J65.
Key words and phrases. Semilinear elliptic equations; boundary blow-up; boundary behavior; borderline cases.

Submitted June 30, 2012. Published August 19, 2012.
Supported by grant 10671169 from NNSF of China and 2009ZRB01795 from NNSF of Shandong Province.

can be described by (see, e.g., [3] and [8, Theorem 6.8])

$$\frac{\Theta(u(x))}{d(x)} \to 1 \quad \text{as } d(x) \to 0. \tag{1.2}$$

Moreover, if one assumes that

$$\liminf_{r\to\infty} \frac{\Theta(\lambda r)}{\Theta(r)} > 1, \quad \forall \lambda \in (0,1), \tag{1.3}$$

then it holds (see [3])

$$\frac{u(x)}{\phi(d(x))} \to 1 \quad \text{as } d(x) \to 0, \tag{1.4}$$

where ϕ is the inverse of Θ; i.e., ϕ satisfies

$$\int_{\phi(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t, \quad \forall t > 0. \tag{1.5}$$

However, there are less results for the boundary behavior of the solution to problem (1.1) under the condition that

$$\lim_{r\to\infty} \frac{\Theta(\lambda r)}{\Theta(r)} = 1, \quad \forall \lambda \in (0,1). \tag{1.6}$$

When f satisfies

- (A) f is locally Lipschitz continuous and non-negative on $[0,\infty)$, and $f(s)/s$ is increasing on $(0,\ \infty)$;
- (B) $f(s) = C_1^2 s(\ln s)^{2\alpha} + C_2 s(\ln s)^{2\alpha-1}(1+o(1))$ as $s \to \infty$ with $C_1 > 0$, $\alpha > 1$ and $C_2 \in \mathbb{R}$,

Cîrstea and Du [5] first showed that problem (1.1) has a unique solution u satisfying

$$\lim_{d(x)\to\infty} \frac{u(x)}{\exp\big((C_1(\alpha-1)K(d(x)))^{-1/(\alpha-1)}\big)} = \exp(\xi_0), \tag{1.7}$$

where

$$\xi_0 = \frac{1}{2} - \frac{C_2}{2\alpha C_1^2}. \tag{1.8}$$

Then they extended the above result to weight b which can be vanishing on the boundary.

It is worthwhile to point out that (1.7) depends not only on $C_1^2 s(\ln s)^{2\alpha}$ but also on the lower term $C_2 s(\ln s)^{2\alpha-1}$ in (B). This is completely different from the case $f(s) = s^p[C_1 + o(1)]$ as $s \to \infty$ for some $p > 1$, since problem (1.1) has a unique positive solution u which satisfies

$$\lim_{d(x)\to 0} u(x)(d(x))^{2/(p-1)} = \Big(\frac{2(p+1)}{C_1(p-1)^2}\Big)^{1/(p-1)}$$

in such a situation and $b \equiv 1$ in Ω (see [3]).

On the other hand, when $b \equiv 1$ in Ω, f satisfies (F1), (F2) and the conditions that

- (F03) there exists $\alpha > 1$ such that
$$\frac{2F(s)f'(s)}{f^2(s)} = 1 - (\alpha + o(1))(\ln s)^{-1} \quad \text{as } s \to \infty;$$
- (F04) there exist $\theta_0 \in (0,1)$ and $S_0 > 1$ such that
$$\theta f(s) \ge f(\theta s), \quad \forall \theta \in (\theta_0, 1),\ \forall s > S_0;$$

(F05) there exist $C_0 > 0$ and $S_1 \geq S_0$ such that

$$\frac{s^2|f''(\theta s)|}{f''(s)} \leq C_0(\ln s)^{-1}, \quad \forall s > S_1, \ \forall \theta \in (1/2, 2),$$

Anedda and Porru [2] showed that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that the solution of problem (1.1) satisfying

$$\begin{aligned} &1 + \frac{(\alpha-1)(N-1)}{2(2\alpha-1)}K(x)d(x) - \varepsilon d(x) - C_\varepsilon d^2(x) \\ &< \frac{u(x)}{\phi(d(x))} \\ &< 1 + \frac{(\alpha-1)(N-1)}{2(2\alpha-1)}K(x)d(x) + \varepsilon d(x) + C_\varepsilon d^2(x), \end{aligned}$$

where $K(x)$ is the mean curvature of the surface $\{x \in \Omega : d(x) = \text{constant}\}$.

We also note that an example which satisfies the above requirements is the following

$$f(s) = 0, \quad s \in [0,1], \quad f(s) = s(\ln s)^{2\alpha}, \quad s > 1, \quad \alpha > 1.$$

Inspired by the above works, in this article, we analyze the boundary behavior of solutions to problem (1.1) for more general f which satisfies the condition (1.6). In particular, we consider functions f which satisfy (F1), (F2) and the following conditions that

(F3) there exist two functions $f_1 \in C^1[S_0, \infty)$ for some large $S_0 > 0$ and f_2 such that

$$f(s) := f_1(s) + f_2(s), \quad s \geq S_0;$$

(F4)

$$\frac{f_1'(s)s}{f_1(s)} := 1 + g(s), \quad s \geq S_0, \tag{1.9}$$

with $g \in C^1[S_0, \infty)$ satisfying

$$g(s) > 0, \quad s \geq S_0, \quad \lim_{s\to\infty} g(s) = 0, \tag{1.10}$$

$$\lim_{s\to\infty} \frac{sg'(s)}{g(s)} = 0, \quad \lim_{s\to\infty} \frac{sg'(s)}{g^2(s)} = C_g \in \mathbb{R}, \quad \lim_{s\to\infty} \frac{\sqrt{\frac{s}{f_1(s)}}}{g(s)} = 0; \tag{1.11}$$

(F5) either there exists a constant $E_1 \neq 0$ such that

$$\lim_{s\to\infty} \frac{f_2(s)}{g(s)f_1(s)} = E_1 \tag{1.12}$$

or

$$\lim_{s\to\infty} \frac{f_2(s)}{g(s)f_1(s)} = 0 \tag{1.13}$$

and there exists a constant $\mu \leq 1$ such that

$$\lim_{s\to\infty} \frac{f_2(\xi s)}{f_2(s)} = \xi^\mu, \quad \forall \xi > 0. \tag{1.14}$$

Our main result is stated using the assumption

(B2) There exist $k \in \Lambda$ and a positive constant b_0 such that

$$\lim_{d(x)\to 0} \frac{b(x)}{(k(d(x)))^2} = b_0^2,$$

where Λ denotes the set of all positive non-decreasing functions in $C^1(0,\delta_0)$ ($\delta_0 > 0$) which satisfy

$$\lim_{t\to 0^+} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) := D_k \in [0,\infty), \quad K(t) = \int_0^t k(s)ds, \tag{1.15}$$

Theorem 1.1. *Let f satisfy* (F1)–(F5). *If b satisfies* (B1)–(B2), *then for any solution u of problem* (1.1),

$$\lim_{d(x)\to 0} \frac{u(x)}{\psi(b_0 K(d(x)))} = \exp(\xi_0), \tag{1.16}$$

where

$$\begin{gathered} \xi_0 = \frac{1}{2} - E_2 - (1-D_k)\big(\frac{1}{2} + C_g\big), \\ E_2 = \begin{cases} E_1 & \text{if (1.12) holds;} \\ 0, & \text{if (1.13) and (1.14) hold,} \end{cases} \end{gathered} \tag{1.17}$$

and ψ is the unique solution of the problem

$$\int_{\psi(t)}^{\infty} \frac{ds}{\sqrt{s f_1(s)}} = t, \quad \forall t > 0. \tag{1.18}$$

Remark 1.2. (F3), (1.10), and (1.12) or (1.13) imply

$$\lim_{s\to\infty} \frac{f_2(s)}{f(s)} = 0, \quad \lim_{s\to\infty} \frac{f_1(s)}{f(s)} = 1.$$

Remark 1.3. Some basic examples which satisfy all our requirements are the following:

(1) $f_1(s) = C_1^2 s(\ln s)^{2\alpha}$ in (F3), where $\alpha > 1$,

$$g(s) = 2\alpha(\ln s)^{-1}; \quad \lim_{s\to\infty} \frac{\sqrt{\frac{s}{f_1(s)}}}{g(s)} = \frac{1}{2\alpha C_1} \lim_{s\to\infty} (\ln s)^{-(\alpha-1)} = 0;$$

$$\frac{sg'(s)}{g^2(s)} \equiv C_g = -\frac{1}{2\alpha}; \quad \lim_{s\to\infty} \frac{f_2(s)}{g(s)f_1(s)} = \frac{1}{2\alpha C_1^2} \lim_{s\to\infty} \frac{f_2(s)}{s(\ln s)^{2\alpha-1}} = E_2;$$

$$\psi(t) = \exp\big(C_1(\alpha-1)t\big)^{-1/(\alpha-1)}.$$

In particular, when $f_2(s) = C_2 s^{\mu}(\ln s)^{\beta}$ with $\beta \le 2\alpha - 1$, $E_1 = 0$ for $\mu < 1$ or $\mu = 1$ and $\beta < 2\alpha - 1$, and $E_1 = \frac{C_2}{2\alpha C_1^2}$ for $\mu = 1$ and $\beta = 2\alpha - 1$.

(2) $f_1(s) = C_1^2 s e^{(\ln s)^q}$ in (F3), where $q \in (0,1)$,

$$g(s) = q(\ln s)^{-(1-q)}; \quad \lim_{s\to\infty} \frac{\sqrt{\frac{s}{f_1(s)}}}{g(s)} = \frac{1}{qC_1} \lim_{s\to\infty} \frac{\exp(-\frac{1}{2}(\ln s)^q)}{(\ln s)^{-(1-q)}} = 0;$$

$$\lim_{s\to\infty} \frac{sg'(s)}{g^2(s)} = -\frac{1-q}{q} \lim_{s\to\infty} (\ln s)^{-q} = C_g = 0;$$

$$\lim_{s\to\infty} \frac{f_2(s)}{g(s)f_1(s)} = \frac{1}{qC_1^2} \lim_{s\to\infty} \frac{f_2(s)}{s(\ln s)^{-(1-q)} \exp((\ln s)^q)} = E_2;$$

$$\int_{\ln(\psi(t))}^{\infty} \exp(-s^q/2)ds = C_1 t.$$

(3) $f_1(s) = C_1^2 s(\ln s)^2(\ln(\ln s))^{2\alpha}$ in (F3), where $\alpha > 1$,

$$g(s) = 2(\ln s)^{-1}\big(1+\alpha(\ln(\ln s))^{-1}\big);$$

$$\lim_{s\to\infty}\frac{\sqrt{\frac{s}{f_1(s)}}}{g(s)} = \frac{1}{2C_1}\lim_{s\to\infty}\frac{(\ln(\ln s))^{-\alpha}}{1+\alpha(\ln(\ln s))^{-1}} = 0;$$

$$\lim_{s\to\infty}\frac{sg'(s)}{g^2(s)} = -\lim_{s\to\infty}\frac{1+\alpha(\ln(\ln s))^{-1}+\alpha(\ln(\ln s))^{-2}}{2(1+\alpha(\ln(\ln s))^{-1})^2} = C_g = -\frac{1}{2};$$

$$\lim_{s\to\infty}\frac{f_2(s)}{g(s)f_1(s)} = \frac{1}{2C_1^2}\lim_{s\to\infty}\frac{f_2(s)}{s\ln s(\ln(\ln s))^{2\alpha}(1+\alpha(\ln(\ln s))^{-1})} = E_2;$$

$$\psi(t) = \exp\big(\exp\big(C_1(\alpha-1)t\big)^{-1/(\alpha-1)}\big).$$

Remark 1.4. When f further satisfies the condition $f(s)/s$ being increasing on $(0,\infty)$, in a similar proof in [5], problem (1.1) has a unique solution.

Remark 1.5. For the existence of the minimal solution to problem (1.1), see [12].

Remark 1.6. For each $k \in \Lambda$, $D_k \in [0,1]$ and

$$\lim_{t\to 0^+}\frac{K(t)}{k(t)} = 0, \quad \lim_{t\to 0^+}\frac{K(t)k'(t)}{k^2(t)} = 1 - \lim_{t\to 0^+}\frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) = 1 - D_k. \tag{1.19}$$

2. Preliminaries

Our approach relies on Karamata regular variation theory established by Karamata in 1930 which is a basic tool in stochastic process (see, for instance, Bingham, Goldie and Teugels [4], Maric [14] and the references therein.), and has been applied to study the asymptotic behavior of solutions to differential equations and problem (1.1) (see Maric [14], Cîrstea and Rădulescu [6], Rădulescu [16], Cîrstea and Du [5], the authors [18] and the references therein.). In this section, we present some bases of Karamata regular variation theory.

Definition 2.1. A positive measurable function f defined on $[a,\infty)$, for some $a > 0$, is called *regularly varying at infinity* with index ρ, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t\to\infty}\frac{f(\xi t)}{f(t)} = \xi^\rho. \tag{2.1}$$

In particular, when $\rho = 0$, f is called *slowly varying at infinity*. Clearly, if $f \in RV_\rho$, then $L(t) := f(t)/t^\rho$ is slowly varying at infinity.

Some basic examples of slowly varying functions at infinity are

(i) every measurable function on $[a,\infty)$ which has a positive limit at infinity;
(ii) $(\ln t)^q$ and $\big(\ln(\ln t)\big)^q$, $q \in \mathbb{R}$;
(iii) $e^{(\ln t)^q}$, $0 < q < 1$.

We also say that a positive measurable function g defined on $(0,a)$ for some $a > 0$, is *regularly varying at zero* with index ρ (and denoted by $g \in RVZ_\rho$) if $t \to g(1/t)$ belongs to $RV_{-\rho}$.

Proposition 2.2 (Uniform convergence theorem). *If $f \in RV_\rho$, then (2.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. Moreover, if $\rho < 0$, then uniform convergence holds on intervals of the form (a_1, ∞) with $a_1 > 0$; if $\rho > 0$, then uniform convergence holds on intervals $(0, a_1]$ provided f is bounded on $(0, a_1]$ for all $a_1 > 0$.*

Proposition 2.3 (Representation theorem). *A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(t) = \varphi(t) \exp\Big(\int_{a_1}^{t} \frac{y(\tau)}{\tau} d\tau\Big), \quad t \geq a_1, \tag{2.2}$$

for some $a_1 \geq a$, where the functions φ and y are measurable and for $t \to \infty$, $y(t) \to 0$ and $\varphi(t) \to c_0$, with $c_0 > 0$.

We say that

$$\hat{L}(t) = c_0 \exp\Big(\int_{a_1}^{t} \frac{y(\tau)}{\tau} d\tau\Big), \quad t \geq a_1, \tag{2.3}$$

is *normalized* slowly varying at infinity and

$$f(t) = t^\rho \hat{L}(t), \quad t \geq a_1, \tag{2.4}$$

is *normalized* regularly varying at infinity with index ρ (and written $f \in NRV_\rho$).

A function $f \in RV_\rho$ belongs to NRV_ρ if and only if

$$f \in C^1[a_1, \infty), \text{ for some } a_1 > 0 \text{ and } \lim_{t \to \infty} \frac{tf'(t)}{f(t)} = \rho. \tag{2.5}$$

Then, we see that $f_1 \in NRV_1$, $f_2 \in RV_\mu$, $f \in RV_1$ and g is normalized slowly varying at infinity in (F3)-(F5).

Similarly, g is called *normalized* regularly varying at zero with index ρ, and denoted by $g \in NRVZ_\rho$, if $t \to g(1/t)$ belongs to $NRV_{-\rho}$.

Proposition 2.4. *If functions L, L_1 are slowly varying at infinity, then*

- (i) *L^ρ (for every $\rho \in \mathbb{R}$), $L \circ L_1$ (if $L_1(t) \to \infty$ as $t \to \infty$) , are also slowly varying at infinity.*
- (ii) *For every $\rho > 0$ and $t \to \infty$,*
$$t^\rho L(t) \to \infty, \quad t^{-\rho} L(t) \to 0.$$
- (iii) *For $\rho \in \mathbb{R}$ and $t \to \infty$, $\ln(L(t))/\ln t \to 0$ and $\ln(t^\rho L(t))/\ln t \to \rho$.*

Our results in the section are summarized as follows.

Lemma 2.5 ([18, Lemma 2.1]). *Let $k \in \Lambda$.*

- (i) *When $D_k \in (0, 1)$, k is* normalized *regularly varying at zero with index $(1 - D_k)/D_k$;*
- (ii) *when $D_k = 1$, k is normalised slowly varying at zero;*
- (iii) *when $D_k = 0$, k grows faster than any t^p $(p > 1)$ near zero.*

Denote

$$\Theta(r) = \int_r^\infty \frac{ds}{\sqrt{2F(s)}}, \quad \Theta_1(r) = \int_r^\infty \frac{ds}{\sqrt{sf_1(s)}}, \quad r > 0. \tag{2.6}$$

Then

$$\Theta'(r) = -\frac{1}{\sqrt{2F(r)}}, \quad \Theta_1'(r) = -\frac{1}{\sqrt{rf_1(r)}}, \quad r > 0. \tag{2.7}$$

Lemma 2.6. *Under the hypotheses in Theorem 1.1:*

(i)
$$\lim_{r\to\infty}\frac{\Theta(\lambda r)}{\Theta(r)}=\lim_{r\to\infty}\frac{\Theta_1(\lambda r)}{\Theta_1(r)}=1,\quad \forall\lambda\in(0,1);$$

(ii)
$$\lim_{r\to\infty}\frac{(r/f_1(r))^{1/2}}{\Theta_1(r)g(r)}=\frac{1}{2}+C_g;$$

(iii)
$$\lim_{r\to\infty}\frac{\frac{f_1(\xi r)}{\xi f_1(r)}-1}{g(r)}=\ln\xi$$
uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$;

(iv)
$$\lim_{r\to\infty}\frac{f_2(\xi r)}{\xi g(r)f_1(r)}=E_2$$
uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$.

Proof. (i) By $f,f_1\in RV_1$ and the l'Hospital's rule, we have
$$\lim_{r\to\infty}\frac{F(\lambda r)}{F(r)}=\lambda\lim_{r\to\infty}\frac{f(\lambda r)}{f(r)}=\lambda^2,$$
$$\lim_{r\to\infty}\frac{\Theta(\lambda r)}{\Theta(r)}=\lambda\lim_{r\to\infty}\frac{\Theta'(\lambda r)}{\Theta'(r)}=\lambda\lim_{r\to\infty}\Big(\frac{F(\lambda r)}{F(r)}\Big)^{-1/2}=1;$$
$$\lim_{r\to\infty}\frac{\Theta_1(\lambda r)}{\Theta_1(r)}=\lambda\lim_{r\to\infty}\frac{\Theta_1'(\lambda r)}{\Theta_1'(r)}=\lambda\lim_{r\to\infty}\Big(\frac{\lambda f_1(\lambda r)}{f_1(r)}\Big)^{-1/2}=1.$$

(ii) By (1.11) and the l'Hospital's rule, we obtain
$$\begin{aligned}
&\lim_{r\to\infty}\frac{\big(\frac{r}{f_1(r)}\big)^{1/2}}{\Theta_1(r)g(r)}\\
&=\lim_{r\to\infty}\frac{(g(r))^{-1}\big(\frac{r}{f_1(r)}\big)^{1/2}}{\Theta_1(r)}\\
&=\lim_{r\to\infty}\frac{-(g(r))^{-2}g'(r)\big(\frac{r}{f_1(r)}\big)^{1/2}+\frac{1}{2}(g(r))^{-1}\big(\frac{r}{f_1(r)}\big)^{-1/2}\frac{f_1(r)-rf_1'(r)}{f_1^2(r)}}{-(rf_1(r))^{-1/2}}\\
&=\lim_{r\to\infty}\Big(\frac{1}{2g(r)}\frac{rf_1'(r)-f_1(r)}{f_1(r)}+\frac{rg'(r)}{g^2(r)}\Big)=\frac{1}{2}+C_g.
\end{aligned}$$

(iii) When $\xi=1$, the result is obvious. Let $\xi\neq1$. By $f_1\in RV_1$, one can see that
$$\frac{f_1(\xi r)}{\xi f_1(r)}-1=\exp\Big(\int_r^{\xi r}\frac{g(\tau)}{\tau}d\tau\Big)-1.$$
It follows by $g\in NRV_0$ and Proposition 2.3 that
$$\lim_{r\to\infty}\frac{g(r\nu)}{\nu}=0,\quad \lim_{r\to\infty}\frac{g(r\nu)}{g(r)}=1$$
uniformly with respect to $\nu\in[c_1,c_2]$. So
$$\lim_{r\to\infty}\int_r^{\xi r}\frac{g(\tau)}{\tau}d\tau=\lim_{r\to\infty}\int_1^{\xi}\frac{g(r\nu)}{\nu}d\nu=0,$$

$$\lim_{r\to\infty}\int_1^{\xi}\frac{g(r\nu)}{g(r)\nu}d\nu=\int_1^{\xi}\nu^{-1}d\nu=\ln\xi.$$

Since $e^s-1\cong s$ as $s\to 0$, this leads to

$$\frac{f_1(\xi r)}{\xi f_1(r)}-1\cong g(r)\ln\xi \quad \text{as } r\to\infty$$

uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$ by Proposition 2.3.

(iv) Note that

$$\lim_{r\to\infty}\frac{f_2(\xi r)}{\xi g(r)f_1(r)}=\lim_{r\to\infty}\frac{f_2(\xi r)}{\xi f_2(r)}\lim_{r\to\infty}\frac{f_2(r)}{g(r)f_1(r)}.$$

When (1.13) and (1.14) hold,

$$\lim_{r\to\infty}\frac{f_2(\xi r)}{\xi g(r)f_1(r)}=0.$$

When (1.12) holds. Let

$$\frac{f_2(s)}{g(s)f_1(s)}-E_1=h(s) \quad \text{with } \lim_{s\to\infty}h(s)=0.$$

It follows by $g\in NRV_0$ and $f_1\in NRV_1$ that

$$\lim_{r\to\infty}\frac{f_2(\xi r)}{f_2(r)}=\lim_{r\to\infty}\frac{f_1(\xi r)}{f_1(r)}\frac{g(\xi r)}{g(r)}\frac{E_1+h(\xi s)}{E_1+h(s)}=\xi;$$

thus

$$\lim_{r\to\infty}\frac{f_2(\xi r)}{\xi g(r)f_1(r)}=E_1.$$

□

Lemma 2.7. *Under the hypotheses of Theorem 1.1, let ψ be the solution to the problem*

$$\int_{\psi(t)}^{\infty}\frac{ds}{\sqrt{sf_1(s)}}=t, \quad \forall t>0.$$

Then

(i) $-\psi'(t)=\sqrt{\psi(t)f_1(\psi(t))}$, $\psi(t)>0$, $t>0$, $\psi(0):=\lim_{t\to 0^+}\psi(t)=\infty$, $\psi''(t)=\frac{1}{2}\big(f_1(\psi(t))+\psi(t)f_1'(\psi(t))\big)$, $t>0$;

(ii)
$$\lim_{t\to 0}\big(g(\psi(t))\big)^{-1}\Big(\frac{1}{2}\big(1+\frac{\psi(t)f_1'(\psi(t))}{f_1(\psi(t))}\big)-\frac{f_1(\xi\psi(t))}{\xi f_1(\psi(t))}\Big)=\frac{1}{2}-\ln\xi;$$

(iii)
$$\lim_{t\to 0}\frac{\sqrt{\psi(t)f_1(\psi(t))}}{tg(\psi(t))f_1(\psi(t))}=\frac{1}{2}+C_g;$$

(iv)
$$\lim_{t\to 0}\frac{f_2(\xi\psi(t))}{\xi g(\psi(t))f_1(\psi(t))}=E_2$$
uniformly for $\xi\in[c_1,c_2]$ with $0<c_1<c_2$.

Proof. By the definition of ψ and a direct calculation, we can show (i). Statements (ii)–(iv) follow by Lemma 2.6, letting $u=\psi(t)$. □

3. Proof of Theorem 1.1

First, by the same proof of [7, Lemma 2.4], we have the following result.

Lemma 3.1 (Comparison principle [7, Lemma 2.1]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and* (F1), (B1) *be satisfied. Assume that $u_1, u_2 \in C^2(\Omega)$ satisfy $\Delta u_1 \geq b(x)f(u_1)$ and $\Delta u_2 \leq b(x)f(u_2)$ in Ω. If $\liminf_{x\to\partial\Omega}(u_2 - u_1)(x) \geq 0$, then $u_2 \geq u_1$ in Ω.*

Let $v_0 \in C^{2+\alpha}(\Omega) \cap C^1(\bar{\Omega})$ be the unique solution of the problem

$$-\Delta v = 1, \quad v > 0, \quad x \in \Omega, \quad v|_{\partial\Omega} = 0. \tag{3.1}$$

By the Höpf maximum principle [10, Lemma 3.4], we see that

$$\nabla v_0(x) \neq 0,\ \forall x \in \partial\Omega \text{ and } c_1 d(x) \leq v_0(x) \leq c_2 d(x),\ \forall x \in \Omega, \tag{3.2}$$

where c_1, c_2 are positive constants.

Denote $\varsigma_0 = \exp(\xi_0)$, where ξ_0 is given in (1.17),

$$\varsigma_2 = \varsigma_0 + \varepsilon, \quad \varsigma_1 = \varsigma_0 - \varepsilon, \quad \varepsilon \in (0, \min\{\varsigma_0,\ b_0^2\}/2).$$

It follows that

$$\varsigma_0/2 < \varsigma_1 < \varsigma_2 < 2\varsigma_0, \quad \lim_{\varepsilon\to 0}\varsigma_1 = \lim_{\varepsilon\to 0}\varsigma_2 = \varsigma_0.$$

Since $\ln(1+s) \cong s$ as $s \to 0^+$, we can choose ε sufficiently small such that

$$\ln(\varsigma_0) - \ln(\varsigma_2) = \ln\big(1 - \frac{\varepsilon}{\varsigma_0 + \varepsilon}\big) < -\frac{1}{4\varsigma_0}\varepsilon; \tag{3.3}$$

$$\ln(\varsigma_0) - \ln(\varsigma_1) = \ln\big(1 + \frac{\varepsilon}{\varsigma_0 - \varepsilon}\big) > \frac{1}{4\varsigma_0}\varepsilon. \tag{3.4}$$

Fix the above ε. For any $\delta > 0$, we define $\Omega_\delta = \{x \in \Omega : 0 < d(x) < \delta\}$. Since Ω is C^2-smooth, choose $\delta_1 \in (0, \delta_0)$ such that (see, 14.6. Appendix: Boundary Curvatures and the Distance Function in [10])

$$d \in C^2(\Omega_{\delta_1}), \quad |\nabla d(x)| = 1, \quad \Delta d(x) = -(N-1)H(\bar{x}) + o(1), \quad \forall x \in \Omega_{\delta_1}. \tag{3.5}$$

Proof of Theorem 1.1. By Lemma 2.7, (1.19), (3.3), (3.5) and $K \in C[0, \delta_0)$ with $K(0) = 0$, we see that there are $\delta_{1\varepsilon}$, $\delta_{2\varepsilon} \in \big(0, \min\{1, \delta_1/2\}\big)$ (which are corresponding to ε) sufficiently small such that

(i) $(b_0^2 - \varepsilon)k^2(d(x) - \sigma) \leq (b_0^2 - \varepsilon)k^2(d(x)) < b(x)$, $x \in D_\sigma^- = \Omega_{2\delta_{1\varepsilon}}/\bar{\Omega}_\sigma$; $b(x) < (b_0^2 + \varepsilon)k^2(d(x)) \leq (b_0^2 + \varepsilon)k^2(d(x) + \sigma)$, $x \in D_\sigma^+ = \Omega_{2\delta_{1\varepsilon} - \sigma}$, where $\sigma \in (0, \delta_{1\varepsilon})$;

(ii) $b_0 K(d(x)) \leq \delta_{2\varepsilon}$, $x \in \Omega_{2\delta_{1\varepsilon}}$;

(iii) for all $(x,t) \in \Omega_{2\delta_{1\varepsilon}} \times (0, 2\delta_{2\varepsilon})$,

$$\begin{aligned}&(g(\psi(t))^{-1}\Big(\frac{1}{2}\big(1 + \frac{\psi(t)f_1'(\psi(t))}{f_1(\psi(t))}\big) - \frac{f_1(\varsigma_2\psi(t))}{\varsigma_2 f_1(\psi(t))}\Big) - \frac{f_2(\varsigma_2\psi(t))}{\varsigma_2 g(\psi(t))f_1(\psi(t))}\\ &- \frac{\sqrt{\psi(t)f_1(\psi(t))}}{tg(\psi(t))f_1(\psi(t))}\frac{K(d(x))k'(d(x))}{k^2(d(x))} \leq -\frac{1}{4\varsigma_0}\varepsilon;\end{aligned}$$

(iv)

$$\frac{\sqrt{\psi(t)f_1(\psi(t))}}{tg(\psi(t))f_1(\psi(t))}\frac{K(d(x))}{k(d(x))}|\Delta d(x)| \leq \frac{1}{8\varsigma_0}\varepsilon, \quad \forall (x,t) \in \Omega_{2\delta_{1\varepsilon}} \times (0, 2\delta_{2\varepsilon}).$$

Now we define

$$d_1(x) = d(x) - \sigma, \quad d_2(x) = d(x) + \sigma; \tag{3.6}$$

$$\bar{u}_\varepsilon = \varsigma_2 \psi(\sqrt{b_0^2 - \varepsilon} K(d_1(x))), \quad x \in D_\sigma^-; \tag{3.7}$$

$$\underline{u}_\varepsilon = \varsigma_1 \psi(\sqrt{b_0^2 + \varepsilon} K(d_2(x))), \quad x \in D_\sigma^+. \tag{3.8}$$

Then, by (i)–(iv), (3.5) and a direct calculation, we see that for $x \in D_\sigma^-$ and $r = \sqrt{b_0^2 - \varepsilon} K(d_1(x))$,

$$\begin{aligned}
&\Delta \bar{u}_\varepsilon(x) - b(x) f(\bar{u}_\varepsilon(x)) \\
&= \varsigma_2 (b_0^2 - \varepsilon) k^2(d_1(x)) \psi''(r) + \varsigma_2 \sqrt{b_0^2 - \varepsilon} \psi'(r) \big(k'(d_1(x)) + k(d_1(x)) \Delta d(x)\big) \\
&\quad - b(x)(f_1(\varsigma_2 \psi(r)) + f_2(\varsigma_2 \psi(r)) \\
&\le \varsigma_2 (b_0^2 - \varepsilon) f_1(\psi(r)) g(\psi(r)) k^2(d_1(x)) \Big[(g(\psi(r))^{-1} \Big(\frac{1}{2} \big(1 + \frac{\psi(r) f_1'(\psi(r))}{f_1(\psi(r))}\big) \\
&\quad - \frac{f_1(\varsigma_2 \psi(r))}{\varsigma_2 f_1(\psi(r))} \Big) - \frac{f_2(\varsigma_2 \psi(r))}{\varsigma_2 g(\psi(r)) f_1(\psi(r))} \\
&\quad - \frac{\sqrt{\psi(r) f_1(\psi(r))}}{r g(\psi(r)) f_1(\psi(r))} \Big(\frac{K(d_1(x)) k'(d_1(x))}{k^2(d_1(x))} + \frac{K(d_1(x))}{k(d_1(x))} \Delta d(x) \Big) \Big] \le 0;
\end{aligned}$$

i.e., $\bar{u}_\varepsilon$ is a supersolution of (1.1) in D_σ^-.

In a similar way, we can show that $\underline{u}_\varepsilon = \varsigma_1 \psi(\sqrt{b_0^2 + \varepsilon} K(d_2(x)))$ is a subsolution of (1.1) in D_σ^+. Now let u be an arbitrary solution to problem (1.1), we can choose a large M such that

$$u \le \bar{u}_\varepsilon + M v_0 \quad \text{on } \partial D_\sigma^-, \quad \underline{u}_\varepsilon \le u + M v_0 \quad \text{on } \partial D_\sigma^+, \tag{3.9}$$

where v_0 is the solution of (3.1).

Also by (F1), we that $u + M v_0$ and $\bar{u}_\varepsilon + M v_0$ are two supersolutions of equation (1.1) in Ω and in D_σ^-. Since $u < \infty$ on $d = \sigma$; $\bar{u}_\varepsilon(x) = \infty$ on $d = \sigma$; $u = \infty$ on $\partial\Omega$, it follows by (F1) and Lemma 3.1 that

$$u(x) \le M v_0(x) + \bar{u}_\varepsilon(x), \quad x \in D_\sigma^-; \quad \underline{u}_\varepsilon(x) \le u(x) + M v_0(x), \quad x \in D_\sigma^+. \tag{3.10}$$

Hence, letting $\sigma \to 0$, we have for $x \in \Omega_{2\delta_{1\varepsilon}}$,

$$1 - \frac{M v_0(x)}{\varsigma_1 \psi(\sqrt{b_0^2 + \varepsilon} K(d(x)))} \le \frac{u(x)}{\varsigma_1 \psi(\sqrt{b_0^2 + \varepsilon} K(d(x)))};$$

and

$$\frac{u(x)}{\varsigma_2 \psi(\sqrt{b_0^2 - \varepsilon} K(d(x)))} \le 1 + \frac{M v_0(x)}{\varsigma_2 \psi(\sqrt{b_0^2 - \varepsilon} K(d(x)))}.$$

Consequently, by $K(0) = 0$ and $\psi(0) = \infty$,

$$1 \le \lim_{d(x) \to 0} \inf \frac{u(x)}{\varsigma_1 \psi(\sqrt{b_0^2 + \varepsilon} K(d(x)))},$$

$$\lim_{d(x) \to 0} \sup \frac{u(x)}{\varsigma_2 \psi(\sqrt{b_0^2 - \varepsilon} K(d(x)))} \le 1.$$

Thus letting $\varepsilon \to 0$, we obtain

$$\lim_{d(x) \to 0} \frac{u(x)}{\psi(b_0 K(d(x)))} = \varsigma_0.$$

This completes the proof. □

References

[1] S. Alarcón, J. García-Melián, A. Quaas; *Keller-Osserman type conditions for some elliptic problems with gradient terms*, J. Diff. Equations 252 (2012), 886-914.

[2] C. Anedda, G. Porru; *Boundary behaviour for solutions of boundary blow-up problems in a borderline case*, J. Math. Anal. Appl. 352 (2009), 35-47.

[3] C. Bandle, M. Marcus; *Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, J. Anal. Math. 58 (1992), 9-24.

[4] N. H. Bingham, C. M. Goldie and J. L. Teugels; *Regular Variation*, Encyclopedia of Mathematics and its Applications 27, Cambridge University Press, Cambridge, 1987.

[5] F. Cîrstea, Y. Du; *Large solutions of elliptic equations with a weakly superlinear nonlinearity*, J. Anal. Math. 103 (2007), 261-277.

[6] F. Cîrstea, V. Rădulescu; *Uniqueness of the blow-up boundary solution of logistic equations with absorbtion*, C. R. Acad. Sci. Paris, Sér. I 335 (2002), 447-452.

[7] H. Dong, S. Kim, M. Safonov; *On uniqueness of boundary blow-up solutions of a class of nonlinear elliptic equations*, Comm. Partial Differential Equations 33 (2008), 177-188.

[8] Y. Du; *Order Structure and Topological Methods in Nonlinear Partial Differential Equations*, Vol. 1. Maximum Principles and Applications, Series in Partial Differential Equations and Applications, 2. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[9] S. Dumont, L. Dupaigne, O. Goubet and V. D. Radulescu; *Back to the Keller-Osserman condition for boundary blow-up solutions*, Advanced Nonlinear Studies 7 (2007), 271-298.

[10] D. Gilbarg, N. S. Trudinger; *Elliptic Partial Differential Equations of Second Order*, 3nd edition, Springer - Verlag, Berlin, 1998.

[11] J. B. Keller; *On solutions of* $\Delta u = f(u)$, *Commun. Pure Appl. Math.* 10 (1957), 503-510.

[12] A. V. Lair; *A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations*, J. Math. Anal. Appl. 240 (1999), 205-218.

[13] A. C. Lazer, P. J. McKenna; *Asymptotic behavior of solutions of boundary blowup problems*, Differential Integral Equations 7 (1994), 1001-1019.

[14] V. Maric; *Regular Variation and Differential Equations*, Lecture Notes in Math., vol. 1726, Springer-Verlag, Berlin, 2000.

[15] R. Osserman; *On the inequality* $\Delta u \geq f(u)$, *Pacific J. Math.* 7 (1957), 1641-1647.

[16] V. Rădulescu; *Singular phenomena in nonlinear elliptic problems: from boundary blow-up solutions to equations with singular nonlinearities*, in "Handbook of Differential Equations: Stationary Partial Differential Equations", vol. 4 (Michel Chipot, Editor) (2007), 483-591.

[17] L. Véron; *Semilinear elliptic equations with uniform blowup on the boundary*, J. Anal. Math. 59 (1992), 231-250.

[18] Z. Zhang, Y. Ma, L. Mi, X. Li; *Blow-up rates of large solutions for elliptic equations*, J. Diff. Equations 249 (2010), 180-199.

Zhijun Zhang

School of Mathematics and Information Science, Yantai University, Yantai, Shandong, 264005, China

E-mail address: zhangzj@ytu.edu.cn, chinazjzhang@yahoo.com.cn