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# EXISTENCE OF SOLUTIONS FOR MIXED VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this article, we give some results concerning the continuity of the nonlinear Volterra and Fredholm integral operators on the space $L^1[0,\infty)$. Then by using the concept of measure of weak noncompactness, we prove an existence result for a functional integral equation which includes several classes of nonlinear integral equations. Our results extend some previous works.

## 1. Introduction

Integral Equations occur in mechanics and many related fields of engineering and mathematical physics [6, 7, 8, 11, 12, 13, 14, 17, 22, 24, 25, 26, 27]. They also form one of useful mathematical tools in many branches of pure analysis such as functional analysis [21, 26]. Recently many papers have been devoted to the existence of solutions of nonlinear functional integral equations [1, 2, 4, 5, 8, 11]. Our main purpose is to prove an existence theorem for a class of functional integral equations which contains many integral or functional integral equations. For example, we can mention the nonlinear Volterra integral equations, mixed Volterra-Fredholm integral equations and Fredholm integral equations on the unbounded interval $[0,\infty)$.

The concept of measure of weak noncompactness was developed by De Blasie [16]. Banaś and Knap [6] introduced a measure of weak noncompactness in the space of real Lebesgue integrable functions on an interval which is convenient for our purpose. In the proof of main result we will use a measure of weak noncompactness given by Banaś and Knap to find a special subset of $L^1[0,\infty)$ and also by applying the Schauder fixed point theorem on this set, the existence result which generalizes several previous works [3, 7, 8, 9, 11, 13, 17, 27] will be proven.

Organization of this article: Section 2 gives some definitions and preliminary results about continuous operator on $L^1(\mathbb{R}_+)$, Section 3 describes the concept of measure of weak noncompactness and weakly compact sets in $L^1(\mathbb{R}_+)$ and finally in Section 4 we give our main result and some examples.

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## 2. Notations and auxiliary results

In this paper, $\mathbb{R}_+$ indicates the interval $[0,\infty)$ and for the Lebesgue measurable subset $D$ of $\mathbb{R}$, $m(D)$ implies the Lebesgue measure of $D$. Also, let $L^1(D)$ be the space of all Lebesgue integrable functions $y$ on $D$ equipped with the standard norm $\|y\|_{L^1(D)} = \int_D |y(x)|dx$.

**Lemma 2.1** ([20]). *Let $\Omega$ be a Lebesgue measurable subset of $\mathbb{R}$ and $1 \le p \le \infty$. If $\{f_n\}$ is convergent to $f \in L^p(\Omega)$ in the $L^p$-norm, then there is a subsequence $\{f_{n_k}\}$ which converges to $f$ a.e., and there is $g \in L^p(\Omega)$, $g \ge 0$, such that*

$$|f_{n_k}(x)| \le g(x), \quad a.e.\ x \in \Omega.$$

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \times \mathbb{R} \to \mathbb{R}$ is said to have the Carathéodory property if

- (M) for all $x \in \mathbb{R}$ the function $t \mapsto f(t,x)$ is Lebesgue measurable on $I$;
- (C) for almost all $t \in I$ the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R}$.

One of the most important nonlinear mappings is the so-called Nemytski operator which is also called the substitution (or superposition) operator [6, 8, 20, 27]. By substituting the function $x : I \to \mathbb{R}$ into the function $f$ the Nemytski operator $F : x \to f(.,x(.))$ has been obtained which acts on a space containing functions $x$. Krasnosel'skii [22] and Appell and Zabreiko [3] have proven the following assertion when $I$ is a bounded and an unbounded domain respectively.

**Theorem 2.2.** *The superposition operator $F$ generated by function $f(t,x)$ maps the space $L^1(I)$ continuously into itself if and only if $|f(t,x)| \le g(t) + c|x|$ for all $t$ in an interval $I$, and $x \in \mathbb{R}$, where $g$ is a function from the space $L^1(I)$ and $c$ is a nonnegative constant.*

**Remark 2.3.** The Carathéodory property can be generalized to functions $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ where $\Omega$ is a measurable subset of $\mathbb{R}^n$. Theorem 2.2 holds similarly if and only if there exist $c \in \mathbb{R}$ and $g \in L^1(\Omega)$ such that

$$|f(x,y)| \le g(x) + c\sum_{i=1}^{m} |y_i|, \tag{2.1}$$

for almost all $x \in \Omega$ and all $y = (y_1, \dots, y_m) \in \mathbb{R}^m$.

Now we are going to review a theorem from [6] about the continuity of the linear Volterra integral operator on the space $L^1 = L^1(\mathbb{R}_+)$. Let $\Delta = \{(t,s) : 0 \le s \le t\}$ and $k : \triangle \to \mathbb{R}$ be a measurable function with respect to both variables. Consider

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds, \quad t \in \mathbb{R}_+,\ x \in L^1(\mathbb{R}_+).$$

We notice that $K$ is a linear Volterra integral operator generated by $k$.

**Theorem 2.4.** *Let $k$ be measurable on $\Delta$ and such that*

$$\operatorname{ess\,sup}_{s\ge0} \int_s^\infty |k(t,s)|dt < \infty. \tag{2.2}$$

*Then the Volterra integral operator $K$ generated by $k$ maps (continuously) the space $L^1(\mathbb{R}_+)$ into itself and the norm $\|K\|$ of this operator is majorized by the number $\operatorname{ess\,sup}_{s\ge0} \int_s^\infty |k(t,s)|dt$.*

Assume that $A$ is a measurable subset of $\mathbb{R}_+$, we denote by $\|K\|_A$ the norm of linear Volterra operator $K : L^1(A) \to L^1(A)$.

Now we give a result concerning the continuity of the nonlinear Volterra operator on $L^1(\mathbb{R}_+)$. In what follows we suppose that $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies:

(a) $t \to u(t, s, x)$ is measurable for all $s \in \mathbb{R}_+$ and $x \in \mathbb{R}$;
(b) $(s, x) \to u(t, s, x)$ is continuous for almost all $t \in \mathbb{R}_+$.

**Theorem 2.5.** *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function such that*

$$|u(t, s, x)| \le k_1(t, s) + k_2(t, s)|x|, \quad t, s \in \mathbb{R}_+ \; x \in \mathbb{R}, \tag{2.3}$$

*where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (i=1,2) are measurable functions. Moreover, the integral operator $K_2$ generated by $k_2$ is a continuous map from $L^1(\mathbb{R}_+)$ into itself and $\int_0^t k_1(t, s)ds \in L^1(\mathbb{R}_+)$. Then the operator*

$$(Ux)(t) = \int_0^t u(t, s, x(s))ds,$$

*maps $L^1(\mathbb{R}_+)$ continuously into itself*

*Proof.* Let $\{x_n\}$ be an arbitrary sequence in $L^1 = L^1(\mathbb{R}_+)$ which converges to $x \in L^1$ in the $L^1$-norm. By using Lemma 2.1 there is a subsequence $\{x_{n_k}\}$ which converges to $x$ a.e., and there is $g \in L^1$, $g \ge 0$, such that

$$|x_{n_k}(s)| \le g(s), \quad \text{a.e. on } \mathbb{R}_+. \tag{2.4}$$

Since $x_{n_k} \to x$ almost everywhere in $\mathbb{R}_+$, it readily follows from (b) that

$$u(t, s, x_{n_k}(s)) \to u(t, s, x(s)), \quad \text{for almost all } s, t \in \mathbb{R}_+. \tag{2.5}$$

From inequalities (2.3) and (2.4), we infer that

$$|u(t, s, x_{n_k}(s))| \le k_1(t, s) + k_2(t, s)g(s), \text{ for almost all } s, t \in \mathbb{R}_+. \tag{2.6}$$

As a consequence of the Lebesgue's Dominated Convergence Theorem, (2.5) and (2.6) yield

$$\int_0^t u(t, s, x_{n_k}(s))ds \to \int_0^t u(t, s, x(s))ds,$$

for almost all $t \in \mathbb{R}_+$. Inequality (2.6) implies that

$$|(Ux_{n_k})(t)| \le \int_0^t |u(t, s, x_{n_k}(s))|ds \le \int_0^t k_1(t, s)ds + \int_0^t k_2(t, s)g(s)ds, \tag{2.7}$$

for almost all $t \in \mathbb{R}_+$. Regarding the assumptions on $k_1$ and $k_2$, we obtain

$$\int_0^\infty \int_0^t k_1(t, s)\, ds\, dt + \int_0^\infty \int_0^t k_2(t, s)g(s)\, ds\, dt < \infty. \tag{2.8}$$

Then inequalities (2.7)-(2.8) and the Lebesgue's Dominated Convergence Theorem imply

$$\|Ux_{n_k} - Ux\|_{L^1} \to 0.$$

Since any sequence $\{x_n\}$ converging to $x$ in $L^1$ has a subsequence $\{x_{n_k}\}$ such that $Ux_{n_k} \to Ux$ in $L^1$, we can conclude that $U : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ is a continuous operator. □

Similar to the above theorem, we can prove the following theorem for the nonlinear Fredholm integral operators.

**Theorem 2.6.** *Let $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a function satisfying* (a)–(b) *such that*

$$|v(t,s,x)| \le k_1(t,s) + k_2(t,s)|x|, \quad t,s \in \mathbb{R}_+\ x \in \mathbb{R}, \tag{2.9}$$

*where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1,2)$ are measurable functions. Moreover, the integral operator $(K_2x)(t) = \int_0^\infty k_2(t,s)|x(s)|ds$ maps $L^1(\mathbb{R}_+)$ continuously into itself and $k_1(t,s) \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$. Then the operator*

$$(Vx)(t) = \int_0^\infty v(t,s,x(s))ds,$$

*maps $L^1(\mathbb{R}_+)$ continuously into itself.*

## 3. Measure of weak noncompactness

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with zero element $\theta$. We write $B(x,r)$ to denote the closed ball centered at $x$ with radius $r$ and $\operatorname{conv} X$ to denote the closed convex hull of $X$. Further let:

$\mathbf{m}_E$ be the family of all nonempty bounded subsets of $E$, $\mathbf{n}_E^w$ : the subfamily of $\mathbf{m}_E$ consisting of all relatively weakly compact sets, and $\overline{X}^w$ : the weak closure of a set $X$.

In this paper, we use the following definition of the measure of weak noncompactness [9].

**Definition 3.1.** *A mapping $\mu : \boldsymbol{m}_E \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:*

1. *The family* $\ker\mu = \{X \in \boldsymbol{m}_E : \mu(X) = 0\}$ *is nonempty and* $\ker\mu \subset \boldsymbol{n}_E^w$,
2. $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,
3. $\mu(\operatorname{conv} X) = \mu(X)$,
4. $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ *for* $\lambda \in [0,1]$,
5. *If $X_n \in \boldsymbol{m}_E$, $X_n = \overline{X}_n^w$ for $n = 1,2,\dots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty.*

In the sequel, we will use a measure of weak noncompactness represented by a convenient formula in the space $L^1(\mathbb{R}_+)$ [10]. For $X \in \mathbf{m}_{L^1(\mathbb{R}_+)}$ define:

$$c(X) = \lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\int_D |x(t)|dt : D \subset \mathbb{R}_+,\ m(D) \le \varepsilon]\}\},$$
$$d(X) = \lim_{T\to\infty}\{\sup[\int_T^\infty |x(t)|dt : x \in X]\},$$
$$\mu(X) = c(X) + d(X).$$

In [10], it is shown that $\mu$ is a measure of weak noncompactness on $L^1(\mathbb{R}_+)$. By using the following theorem [19], we can infer that $\ker\mu = \mathbf{n}^w_{L^1(\mathbb{R}_+)}$.

**Theorem 3.2.** *A bounded set $X$ is relatively weakly compact in $L^1(\mathbb{R}_+)$ if and only if the following two conditions are satisfied:*

1. *for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(D) \le \delta$ then $\int_D |x(t)|dt \le \varepsilon$ for all $x \in X$,*
2. *for any $\varepsilon > 0$ there exists $T > 0$ such that $\int_T^\infty |x(t)|dt \le \varepsilon$ for any $x \in X$.*

## 4. Main results

In this section, we study the existence of solutions to the functional integral equation

$$x(t) = f\Big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\Big), \quad t \geq 0. \tag{4.1}$$

This equation is a general form of many integral equations, such as the mixed Volterra-Fredholm integral equation

$$x(t) = g(t) + \int_0^t k(t,s)u(s,x(s))ds + a(t)\int_0^\infty v(s,x(s))ds, \quad t \geq 0. \tag{4.2}$$

Equations like (4.2) have been considered by many authors; see for example [12, 15, 18, 23, 25] and references cited therein. Moreover, (4.1) contains the nonlinear Volterra and Fredholm integral equations on $\mathbb{R}_+$ such as:

$$\begin{gathered} x(t) = g(t) + \int_0^t u(t,s,x(s))ds, \quad t \geq 0, \\ x(t) = f(t) + a(t)\int_0^\infty v(s,x(s))ds, \quad t \geq 0. \end{gathered}$$

We consider equation (4.1) under the following assumptions:

(i) The function $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions and there exist constant $B \in \mathbb{R}_+$ and function $a_1 \in L^1(\mathbb{R}_+)$ such that
$$|f(t,x,y)| \leq a_1(t) + B(|x| + |y|), \quad t \in \mathbb{R}_+,\ x,y \in \mathbb{R}. \tag{4.3}$$

(ii) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies (a)–(b) and $|u(t,s,x)| \leq k_1(t,s) + k_2(t,s)|x|$ for $(t,s,x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, where $k_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ (i=1,2) satisfies Carathéodory conditions. Moreover, the integral operator $K_2$ generated by $k_2$ i.e.
$$(K_2x)(t) = \int_0^t k_2(t,s)x(s)ds, \tag{4.4}$$
is a continuous map from $L^1(\mathbb{R}_+)$ into itself and $\int_0^t k_1(t,s)ds \in L^1(\mathbb{R}_+)$.

(iii) $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $|v(s,x)| \leq n(s) + b|x|$ for $(s,x) \in \mathbb{R}_+ \times \mathbb{R}$ where $n \in L^1(\mathbb{R}_+)$ and $b$ is a positive constant.

(iv) $a_2 : \mathbb{R}_+ \to \mathbb{R}$ is a function in $L^1(\mathbb{R}_+)$.

(v) $B(b\|a_2\| + \|K_2\|) < 1$, where $\|K_2\|$ denotes the norm of operator $K_2$.

To prove the main result of this paper, we need the next lemma.

Let $X$ be a nonempty, closed, convex, bounded and weakly compact subset of $L^1 = L^1(\mathbb{R}_+)$ and $I = [0,a]$ where $a > 0$. Moreover, we define the operator $F$ on $L^1 = L^1(\mathbb{R}_+)$ as follows:

$$(Fx)(t) = f\Big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\Big). \tag{4.5}$$

**Lemma 4.1.** *Suppose that assumptions (i)–(iv) hold and the operator $F$ takes $X$ into itself. Then for any $\varepsilon > 0$ there exists $D_\varepsilon \subset I$ with $m(I \setminus D_\varepsilon) \leq \varepsilon$ such that $F(\operatorname{conv} FX)$ on $D_\varepsilon$ is a relatively compact subset of $C(D_\varepsilon)$.*

*Proof.* Consider an arbitrary but fixed $\varepsilon > 0$. Then using Lusin theorem and generalized version of Scorza-Dragoni theorem [14] we can find the closed set $D_\varepsilon \subset I$ with $m(I \setminus D_\varepsilon) \le \varepsilon$, such that the functions $a_i\big|_{D_\varepsilon}$, $k\big|_{D_\varepsilon \times \mathbb{R}_+}$, $u\big|_{D_\varepsilon \times \mathbb{R}_+ \times \mathbb{R}}$ and $f\big|_{D_\varepsilon \times \mathbb{R}_+ \times \mathbb{R}}$ are continuous. Let us take an arbitrary $x \in X$. Then for $t \in D_\epsilon$ we have

$$\begin{aligned} \Big|\int_0^t u(t,s,x(s))ds\Big| &\le \int_0^t k_1(t,s) + \int_0^t k_2(t,s)|x(s)|ds \\ &\le \bar{k}_1 a + \bar{k}_2\|x\| \le \bar{k}_1 a + \bar{k}_2\|X\| =: U_\varepsilon, \end{aligned} \tag{4.6}$$

and

$$\Big|\int_0^\infty a_2(t)v(s,x(s))ds\Big| \le \bar{a}_2(\|n\| + b\|x\|) \le \bar{a}_2(\|n\| + b\|X\|) =: V_\varepsilon, \tag{4.7}$$

where $\|X\| = \sup\{\|x\| : x \in X\}$, $\bar{a}_i = \sup\{|a_i(t)| : t \in D_\varepsilon\}$ and $\bar{k}_i = \sup\{|k_i(t,s)| : (t,s) \in D_\varepsilon \times I\}$ for $i = 1, 2$. Now let $y \in FX$. Then there exists $x \in X$ such that $y = Fx$. Using the inequalities (4.6) and (4.7) for $t \in D_\varepsilon$ we obtain

$$\begin{aligned} |y(t)| = |(Fx)(t)| &\le a_1(t) + \Big|\int_0^t u(t,s,x(s))ds\Big| + \Big|\int_0^\infty a_2(t)v(s,x(s))ds\Big| \\ &\le \bar{a}_1 + U_\varepsilon + V_\varepsilon =: Y_\varepsilon. \end{aligned} \tag{4.8}$$

We can easily deduce that the inequality (4.8) is true, for any $y \in Y = \operatorname{conv} FX$. Now assume that $\{y_n\}$ is a sequence in $Y$ and let $t_1, t_2 \in D_\varepsilon$. Without loss of generality we can assume that $t_1 \le t_2$. Relatively weakly compactness of the set $\{y_n\}$ implies that for $\varepsilon_0 = t_2 - t_1$ there exists $0 < \delta_0 \le \varepsilon_0$ such that for any measurable subset $D$ of $[0, t_1]$ with $m(D) \le \delta_0$, we have:

$$\int_D |y_n(t)|dt \le \varepsilon_0 \quad \text{for } n = 1, 2, \dots. \tag{4.9}$$

We see that the estimate (4.8) does not depend on the choice of $\varepsilon$. Thus for $\varepsilon = \delta_0$ there exists a closed set $D_{\delta_0} \subset [0, t_1]$ with $m([0, t_1] \setminus D_{\delta_0}) \le \delta_0$ such that

$$|y_n(t)| \le Y_{\delta_0} \quad \text{for } t \in D_{\delta_0},\ n = 1, 2, \dots. \tag{4.10}$$

Hence from (4.9) and uniform continuity of $u\big|_{D_\varepsilon \times D_{\delta_0} \times [-Y_{\delta_0}, Y_{\delta_0}]}$ and $k_i\big|_{D_\varepsilon \times [0,a]}$ ($i = 1, 2$) we infer that

$$\begin{aligned} &\int_0^{t_1} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\le \int_{D_{\delta_0}} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\quad + \int_{[0,t_1]\setminus D_{\delta_0}} |u(t_1,s,y_n(s)) - u(t_2,s,y_n(s))|ds \\ &\le O(|t_1 - t_2|) + 2\bar{k}_1 m([0,t_1] \setminus D_{\delta_0}) + 2\bar{k}_2 \int_{[0,t_1]\setminus D_{\delta_0}} |y_n(t)|dt \\ &\le O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2|. \end{aligned} \tag{4.11}$$

Here $O$ is a function which $O(\eta) \to 0$ as $\eta \to 0$. Thus from (4.11) we have:

$$
\begin{aligned}
&|\int_0^{t_1} u(t_1, s, y_n(s))ds - \int_0^{t_2} u(t_2, s, y_n(s))ds| \\
&\le \int_0^{t_1} |u(t_1, s, y_n(s)) - u(t_2, s, y_n(s))|ds + |\int_{t_1}^{t_2} u(t_2, s, y_n(s))ds| \\
&\le O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2| + \int_{t_1}^{t_2} k_1(t_2, s)ds + \int_{t_1}^{t_2} k_2(t_2, s)|y_n(s)|ds \\
&\le O(|t_1 - t_2|) + 2(\bar{k}_1 + \bar{k}_2)|t_1 - t_2| + \bar{k}_1|t_1 - t_2| + \bar{k}_2 \int_{t_1}^{t_2} |y_n(s)|ds.
\end{aligned}
\tag{4.12}
$$

Weakly compactness of the set $\{y_n\}$ implies that $\int_{t_1}^{t_2} |y_n(s)|ds$ is arbitrary small uniformly with respect to $n \in \mathbb{N}$ if $t_2 - t_1$ is small enough. Then from (4.12) and (4.6) the sequence $\{Uy_n\}$ which

$$
(Uy_n)(t) = \int_0^t u(t, s, y_n(s))ds,
$$

is equibounded and equicontinuous on the set $D_\varepsilon$. Obviously from assumption (iii) and inequality (4.7) we can easily infer that the sequence $\{Vy_n\}$ is equibounded and equicontinuous on $D_\varepsilon$ where

$$
(Vy_n)(t) = \int_0^\infty a_1(t)v(s, y_n(s))ds.
$$

Hence, uniform continuity of $f\big|_{D_\varepsilon \times [-U_\varepsilon, U_\varepsilon] \times [-V_\varepsilon, V_\varepsilon]}$ implies that the sequence $\{Fy_n\}$ is equibounded and equicontinuous on $D_\varepsilon$. Then, by Ascoli theorem the sequence $\{Fy_n\}$ has a convergent subsequence in the norm $C(D_\varepsilon)$. Therefore, $F(\operatorname{conv} FX)$ is a relatively compact subset of $C(D_\varepsilon)$. $\square$

Now we present our main result.

**Theorem 4.2.** *Under assumptions* (i)–(v)*, the functional integral equation* (4.1) *has at least one solution* $x \in L^1(\mathbb{R}_+)$.

*Proof.* At first we define the operator $F$ on $L^1 = L^1(\mathbb{R}_+)$ by

$$
(Fx)(t) = f\Big(t, \int_0^t u(t, s, x(s))ds, \int_0^\infty a_2(t)v(s, x(s))ds\Big).
$$

We prove the theorem in the following steps.

Step 1. $F : L^1(\mathbb{R}_+) \to L^1(\mathbb{R}_+)$ is a continuous operator.
Using Theorems 2.5 and 2.6, the operators

$$
(Ux)(t) = \int_0^t u(t, s, x(s))ds, \quad (Vx)(t) = \int_0^\infty v(t, s, x(s))ds,
$$

map $L^1(\mathbb{R}_+)$ continuously into itself. Also by assumptions (i)–(iv) and Remark 2.3, the Nemytski operator generated by $f$ is a continuous operator from $L^1(\mathbb{R}_+)$ into $L^1(\mathbb{R}_+)$. Thus the operator $F$ is continuous.

Step 2. There exists a positive number $r$ such that the operator $F$ takes the ball $B(\theta, r)$ into itself. Let $x \in L^1(\mathbb{R}_+)$. Then

$$
\begin{aligned}
\|Fx\| &= \int_0^\infty |(Fx)(t)|dt \\
&= \Big| \int_0^\infty f\big(t, \int_0^t u(t,s,x(s))ds, \int_0^\infty a_2(t)v(s,x(s))ds\big)dt \Big| \\
&\le \|a_1\| + B\int_0^\infty \Big( \int_0^t |u(t,s,x(s))|ds + \int_0^\infty |a_2(t)v(s,x(s))|ds \Big)dt \\
&\le \|a_1\| + B(K_1 + \|K_2\|\|x\|) + B\|a_2\|(\|n\| + b\|x\|),
\end{aligned}
\tag{4.13}
$$

where

$$
K_1 = \int_0^\infty \int_0^t k_1(t,s)\,ds\,dt.
$$

From (4.13) and assumption (v), one can deduce that for $r = \frac{\|a_1\|+B(K_1+\|a_2\|\|n\|)}{1-B(\|K_2\|+\|a_2\|b)}$, the operator $F$ takes $B_r = B(\theta, r)$ into itself.

Step 3. There exists a weakly compact subset $Y$ such that the operator $F$ maps $Y$ into itself. Let $X$ be a nonempty subset of $B_r$. Let $\varepsilon > 0$ be an arbitrary number and $D \subset \mathbb{R}_+$ be a measurable subset with $m(D) \le \varepsilon$. Then for $x \in X$ we have:

$$
\begin{aligned}
&\int_D |(Fx)(t)|dt \\
&\le \int_D a_1(t)dt + B\int_D \Big( \int_0^t |u(t,s,x(s))|ds + \int_0^\infty |a_2(t)v(s,x(s))|ds \Big)dt \\
&\le \int_D a_1(t)dt + B\int_D \int_0^t k_1(t,s)\,ds\,dt \\
&\quad + B\int_D |(K_2x)(t)|dt + B(\|n\| + br)\int_D |a_2(t)|dt \\
&\le \int_D a_1(t)dt + B\int_D \int_0^t k_1(t,s)\,ds\,dt \\
&\quad + B\|K_2\|_D \int_D |x(t)|dt + B(\|n\| + br)\int_D |a_2(t)|dt
\end{aligned}
\tag{4.14}
$$

Further, as a simple consequence of the fact that a single set in $L^1(\mathbb{R}_+)$ is weakly compact, for $\gamma(t) = a_1(t), \int_0^t k_1(t,s)ds$ or $a_2(t)$, we have:

- (C1) $\lim_{\varepsilon\to 0}\{\sup[\int_D |\gamma(t)|dt : D \subset \mathbb{R}_+,\ m(D) \le \varepsilon]\} = 0$,
- (C2) $\lim_{T\to\infty} \int_T^\infty |\gamma(t)|dt = 0$.

Then from (4.14) and (C1) we conclude that

$$
c(FX) \le B\|K_2\|c(X). \tag{4.15}
$$

By similar calculations we obtain:

$$
\begin{aligned}
\int_T^\infty |(Fx)(t)|dt \le \int_T^\infty a_1(t)dt + B\int_T^\infty \int_0^t k_1(t,s)ds\,dt \\
+ B\|K_2\| \int_T^\infty |x(t)|dt + B(\|n\| + br)\int_T^\infty |a_2(t)|dt.
\end{aligned}
\tag{4.16}
$$

Therefore, from (C2), we have

$$d(FX) \le B\|K_2\| d(X). \tag{4.17}$$

Hence by adding (4.15) and (4.17) we obtain

$$\mu(FX) \le B\|K_2\|\mu(X). \tag{4.18}$$

Assumption (v) implies that $B\|K_2\| < 1$. Thus inequality (4.18) yields that there exists a closed, convex and weakly compact set $X_\infty \subset B_r$ such that $FX_\infty \subset X_\infty$. Let $Y = \operatorname{conv} FX_\infty$. Obviously $FY \subset Y \subset X_\infty$. Thus $FY$ and $Y$ are relatively weakly compact.

Step 4. The set $FY$ obtained in the Step 3 is a relatively compact subset of $L^1(\mathbb{R}_+)$. Suppose $\{y_n\} \subset Y$, and fix an arbitrary $\varepsilon > 0$. Applying Theorem 3.2 for relatively weakly compact set $FY$ implies that there exists $T > 0$ such that for $m, n \in \mathbb{N}$

$$\int_T^\infty |(Fy_n)(t) - (Fy_m)(t)|dt \le \frac{\varepsilon}{2}. \tag{4.19}$$

Further, by using Lemma 4.1 for any $k \in \mathbb{N}$ there exists a closed set $D_k \subset [0,T]$ with $m([0,T] \setminus D_k) \le \frac{1}{k}$ such that $\{Fy_n\}$ is a relatively compact subset of $C(D_k)$. So for any $k \in \mathbb{N}$ there exists a subsequence $\{y_{k,n}\}$ of $\{y_n\}$ which is a Cauchy sequence in $C(D_k)$. Also these subsequences can be chosen such that $\{y_{k+1,n}\} \subseteq \{y_{k,n}\}$. Consequently the subsequence $\{y_{n,n}\}$ is a Cauchy sequence in each space $C(D_k)$ for any $k \in \mathbb{N}$ which for simplicity we call it again $\{y_n\}$.

From the relatively weakly compactness of $\{Fy_n\}$ we can find $\delta > 0$ such that for each closed subset $D_\delta$ with $m([0,T] \setminus D_\delta) \le \delta$ we obtain:

$$\int_{[0,T]\setminus D_\delta} |(Fy_n)(t) - (Fy_m)(t)|dt \le \frac{\varepsilon}{4}, \; m, n \in \mathbb{N}. \tag{4.20}$$

Considering the fact $\{Fy_n\}$ is Cauchy in $C(D_k)$ for each $k \in \mathbb{N}$ one can find $k_0$ such that $m([0,T] \setminus D_{k_0}) \le \delta$ and for $m, n \ge k_0$

$$\|(Fy_n) - (Fy_m)\|_{C(D_{k_0})} \le \frac{\varepsilon}{4(m(D_{k_0}) + 1)}, \tag{4.21}$$

consequently (4.20) and (4.21) imply that

$$\begin{aligned}
&\int_0^T |(Fy_n)(t) - (Fy_m)(t)|dt \\
&= \int_{D_{k_0}} |(Fy_n)(t) - (Fy_m)(t)|dt + \int_{[0,T]\setminus D_{k_0}} |(Fy_n)(t) - (Fy_m)(t)|dt \\
&\le \frac{\varepsilon}{2},
\end{aligned} \tag{4.22}$$

for $m, n \ge k_0$. Now by considering (4.19) and (4.22) for $m, n \ge k_0$ we obtain the inequality

$$\|(Fy_n) - (Fy_m)\|_{L^1} = \int_0^\infty |(Fy_n)(t) - (Fy_m)(t)|dt \le \varepsilon, \tag{4.23}$$

which shows that the sequence $\{Fy_n\}$ is a Cauchy sequence in the Banach space $L^1(\mathbb{R}_+)$. Then $\{Fy_n\}$ has a convergent subsequence which implies that $FY$ is a relatively compact subset of $L^1(\mathbb{R}_+)$.

Step 5. By the Step 4 there exists a bounded, closed, convex set $Y \subset L^1(\mathbb{R}_+)$ such that the operator $F : Y \to Y$ is continuous and compact. Then Schauder fixed point theorem completes the proof. $\square$

Next, by applying our theorem we prove the existence of solutions for some integral equations.

**Example 4.3.** Consider the Fredholm integral equation

$$x(t) = \frac{t^{2/3}}{t^3+1} + \int_0^\infty a_2(t)\tanh\Big(\frac{s+|x(s)|}{(1+s^2)^2}\Big)ds, \quad t \geq 0, \tag{4.24}$$

where

$$a_2(t) = \frac{t\pi}{8}\chi_{[0,1]} + \frac{1}{4(1+t^2)}\chi_{(1,\infty)},$$

in which for $A \subset \mathbb{R}_+$ and $\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \mathbb{R}_+ \setminus A. \end{cases}$ Put

$$f(t,x,y) = \frac{t^{2/3}}{1+t^3} + y, \quad u(t,s,x) = 0,$$
$$v(s,x) = \tanh\Big(\frac{s+|x|}{(1+s^2)^2}\Big), \quad a_1(t) = \frac{t^{2/3}}{1+t^3},$$
$$n(s) = \frac{s}{(1+s^2)^2}, \quad B = 1,\ b = 1.$$

We know that $\tanh(\alpha) \leq \alpha$, for $\alpha > 0$. Then

$$|v(s,x)| \leq n(s) + b|x|,$$
$$|f(t,x,y)| \leq \frac{t^{2/3}}{1+t^3} + B(|x|+|y|),$$

Further

$$\|a_2\| = \int_0^\infty |a_2(t)|dt = \frac{\pi}{8}.$$

Since $u = 0$ we can choose $k_1 = k_2 = 0$ and then $\|K_2\| = 0$. Thus, $B(b\|a_2\| + \|K_2\|) = \frac{\pi}{8} < 1$. It is easy to see that assumptions (i)-(v) are fulfilled. Consequently Theorem 4.2 ensures that the equation (4.24) has at least one solution in $L^1(\mathbb{R}_+)$.

**Example 4.4.** Consider the mixed Volterra-Fredholm integral equation

$$\begin{aligned} x(t) = \frac{1+t^2}{\cosh(t)} &+ \int_0^t \frac{\lfloor t+s^2 \rfloor}{2}\exp(-t)\sin(x(s))ds \\ &+ \int_0^\infty \frac{t\ln(1+sx^2(s))}{3(1+t^2)^2(s+1)}ds, \quad t \geq 0, \end{aligned} \tag{4.25}$$

where the symbol $\lfloor z \rfloor$ means the largest integer less than or equal to $z$. Let

$$f(t,x,y) = \frac{1+t^2}{\cosh(t)} + x + y,$$
$$u(t,s,x) = \frac{\lfloor t+s^2 \rfloor}{2}\exp(-t)\sin(x),$$
$$v(s,x) = \frac{\ln(1+sx^2)}{s+1}, \quad a_2(t) = \frac{t}{3(1+t^2)^2},$$

$$k_2(t,s) = \frac{\lfloor t+s^2 \rfloor}{2} \exp(-t), \quad B = 1, \quad b = 1.$$

We know that $\ln(1+\alpha^2) \leq |\alpha|$ for $\alpha \in \mathbb{R}$. Then

$$|f(t,x,y)| \leq \frac{1+t^2}{\cosh(t)} + B(|x|+|y|),$$

$$|u(t,s,x)| \leq k_2(t,s)|x|, \quad |v(s,x)| \leq b|x|,$$

$$\|a_2\| = \int_0^\infty |a_2(t)|dt = \int_0^\infty \frac{t}{3(1+t^2)^2} dt = \frac{1}{6},$$

$$\int_s^\infty |k_2(t,s)|dt = \int_s^\infty \frac{\lfloor t+s^2 \rfloor}{2} \exp(-t)dt \leq \frac{(s^2+s+1)}{2} \exp(-s) \leq \frac{3}{2} \exp(-1),$$

for $s, t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Therefore, from Theorem 2.4, we have that $B\|K_2\| \leq \frac{3}{2}\exp(-1)$ and then $B(b\|a_2\| + \|K_2\|) \leq \frac{1}{6} + \frac{3}{2}\exp(-1) < 1$. Using Theorem 4.2 we deduce that the equation (4.25) has at least one solution in $L^1(\mathbb{R}_+)$.

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