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CONTROLLABILITY OF NONLINEAR DIFFERENTIAL EVOLUTION SYSTEMS IN A SEPARABLE BANACH SPACE

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Abstract. In this article, we study the controllability of semilinear evolution differential systems with nonlocal initial conditions in a separable Banach space. The results are obtained by using Hausdorff measure of noncompactness and a new calculation method.

1. Introduction

In various fields of science and engineering, many problems that are related to linear viscoelasticity, nonlinear elasticity and Newtonian or non-Newtonian fluid mechanics have mathematical models. Popular models essentially fall into two categories: the differential models and the integrodifferential models. A large class of scientific and engineering problems is modelled by partial differential equations, integral equations or coupled ordinary and partial differential equations which can be described as differential equations in infinite dimensional spaces using semigroups. In general functional differential equations or evolution equations serve as an abstract formulation of many partial integrodifferential equations which arise in problems connected with heat-flow in materials with memory and many other physical phenomena.

It is well known that the systems described by partial differential equations can be expressed as abstract differential equations [18]. These equations occur in various fields of study and each system can be represented by different forms of differential or integrodifferential equations in Banach spaces. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [18] and the nonlocal problem for the same equations has been first studied by Byszewskii [6, 7]. There have been appeared a lot of papers concerned with the existence of semilinear evolution equations with nonlocal conditions [14, 21].

Motivated by the fact that a dynamical system may evolve through an observable quantity rather than the state of the system, a general class of evolutionary equations is defined. This class includes standard ordinary and partial differential equations as well as functional differential equations of retarded and neutral type. In this way, the theory serves as a unifier of these classical problems. Included

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in this general formulation is a general theory for the evolution of temperature in a solid material. In the general case, temperature is transmitted as waves with a finite speed of propagation. Special cases include a theory of delayed diffusion. When physical problems are simulated, the model often takes the form of semilinear evolution equations. Such problems in the control fluid flow can be modelled by a semilinear evolution system in a Banach space. For actual flow, control problems leading to this kind of model and the resulting model equation are discussed in [11]. Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behaviour so as to accomplish a desired goal. In order to implement this influence, practitioners build devices and their interaction with the object being controlled is the subject of control theory. In control theory, one of the most important qualitative aspects of a dynamical system is controllability. Controllability is an important property of a control system and the controllability property plays a crucial role in many control problems such as stabilization of unstable systems by feedback or optimal control. Roughly the concept of controllability denotes the ability to move a system around in its entire configuration space using only certain admissible manipulations.

Controllability of linear and nonlinear systems represented by ordinary differential equations in finite-dimensional spaces has been extensively investigated. The problem of controllability of linear systems represented by differential equations in Banach spaces has been extensively studied by several authors [10]. Several papers have appeared on finite dimensional controllability of linear systems [13] and infinite dimensional systems in abstract spaces [9]. Of late the controllability of nonlinear systems in finite-dimensional spaces is studied by means of fixed point principles [1]. Several authors have extended the concept of controllability to infinite-dimensional spaces by applying semigroup theory [8, 18, 23, 24]. Controllability of nonlinear systems with different types of nonlinearity has been studied by many authors with the help of fixed point principles [2]. Naito [17] discussed the controllability of nonlinear Volterra integrodifferential systems and in [15, 16] he studied the controllability of semilinear systems whereas Yamamoto and Park [22] investigated the same problem for a parabolic equation with a uniformly bounded nonlinear term.

A standard approach is to transform the controllability problem into a fixed point problem for an appropriate operator in a function space. Most of the above mentioned works require the assumption of compactness of the semigroups. Balachandran and Kim [3] pointed out that controllability results are only true for ordinary differential systems in finite dimensional spaces if the corresponding semigroup is compact. However, controllability results maybe true for abstract differential systems in infinite dimensional spaces if the compactness of the corresponding operator semigroup is dropped.

Consider the semilinear evolution differential system with nonlocal conditions

$$x'(t) = A(t)x(t) + Bu(t) + f(t, x(t)), \quad t \in J, \tag{1.1}$$

$$x(0) = g(x), \tag{1.2}$$

where the state variable $x(\cdot)$ takes values in a separable Banach space X with norm $\|\cdot\|$, $A(t) : D_t \subset X \to X$ generates an evolution system $\{U(t,s)\}_{0\le s\le t\le b}$ on the separable Banach space X. The control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space and the interval

$J = [0, b]$. The functions $g : \mathcal{C}(J, X) \to X$ and $f : J \times X \to X$ are continuous and B is a bounded linear operator from U into X.

In this paper, we give conditions guaranteeing the controllability for nonlocal evolution system (1.1)–(1.2) without assumptions on the compactness of f, g and the evolution system $\{U(t, s)\}$ is strongly continuous. The results obtained are based on the new calculation method which employs the technique of a measure of noncompactness.

2. Preliminaries

In this section, we collect some definitions, notation, lemmas and results which are used later. Let $(X, \|\cdot\|)$ be a real Banach space with zero element θ. Denote by $\mathbb{B}(y, r)$ the closed ball in X centered at y and with radius r. The collections of all linear and bounded operators from X into itself will be denoted by $\mathcal{B}(X)$. If Y is a subset of X we write $\overline{Y}, \operatorname{Conv} Y$ to denote the closure and convex closure of Y respectively.

Moreover we denote by $\mathcal{F}_X$ the family of all nonempty and bounded subsets of X and by $\mathcal{G}_X$ its subfamily consisting of relatively compact sets.

Definition 2.1 ([5])**.** A function $\chi : \mathcal{F}_X \to \mathbb{R}_+$ is said to be a measure of noncompactness if it satisfies the following conditions:

(i) The family $\ker \chi = \{Y \in \mathcal{F}_X : \chi(Y) = 0\}$ is nonempty and $\ker \chi \subset \mathcal{G}_X$.
(ii) $Y \subset Z \Rightarrow \chi(Y) \le \chi(Z)$.
(iii) $\chi(\operatorname{Conv} Y) = \chi(Y)$.
(iv) $\chi(\lambda Y + (1-\lambda)Z) \le \lambda\chi(Y) + (1-\lambda)\chi(Z)$, for $\lambda \in [0, 1]$.
(v) If $\{Y_n\}_{n=1}^{\infty}$ is a sequence of nonempty, bounded and closed subsets of X such that $Y_{n+1} \subset Y_n$ $(n = 1, 2, \dots)$ and if $\lim_{n\to\infty} \chi(Y_n) = 0$, then the intersection $Y_\infty = \cap_{n=1}^{\infty} Y_n$ is nonempty and compact in X.

The family $\ker \chi$ defined in (i) is called the *kernel* of the measure of noncompactness χ.

Remark 2.2. Let us notice that the intersection set Y_∞ described in axiom (v) is a member of the kernel of the measure of noncompactness χ. In fact the inequality $\chi(Y_\infty) \le \chi(Y_n)$, for $n = 1, 2, \dots$ implies that $\chi(Y_\infty) = 0$. Hence $Y_\infty \in \ker \chi$. This property of the set Y_∞ will be important in our investigations.

Throughout this paper, $\{A(t) : t \in \mathbb{R}\}$ is a family of closed linear operators defined on a common domain $\mathcal{D}$ which is dense in X and we assume that the linear non-autonomous system

$$
\begin{gathered}
u'(t) = A(t)u(t), \quad s \le t \le b, \\
u(s) = x \in X,
\end{gathered} \tag{2.1}
$$

has associated evolution family of operators $\{U(t, s) : 0 \le s \le t \le b\}$. In the next definition, $\mathcal{L}(X)$ is a space of bounded linear operators from X into X endowed with the uniform convergence topology.

Definition 2.3. A family of operators $\{U(t, s) : 0 \le s \le t \le b\} \subset \mathcal{L}(X)$ is called a evolution family of operators for (3) if the following properties hold:

(a) $U(t, s)U(s, \tau) = U(t, \tau)$ and $U(t, t)x = x$, for every $s \le \tau \le t$ and all $x \in X$;
(b) For each $x \in X$, the function for $(t, s) \to U(t, s)x$ is continuous and $U(t, s) \in \mathcal{L}(X)$, for every $t \ge s$, and

(c) For $0 \le s \le t \le b$, the function $t \to U(t,s)$, for $(s,t] \in \mathcal{L}(X)$, is differentiable with $\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s)$.

The most frequently applied measure of noncompactness is defined in the following way

$$\beta(Y) = \inf\{r > 0 : Y \text{ can be covered by a finite number of balls with radii } r\}.$$

The measure β is called the Hausdorff measure of noncompactness.

In the sequel, we work in the space $\mathcal{C}(J,X)$ consisting of all functions defined and continuous on J with values in the Banach space X. The space $\mathcal{C}(J,X)$ is furnished with the standard norm

$$\|x\|_{\mathcal{C}} = \sup\{\|x(t)\| : t \in J = [0,b]\}.$$

To define the measure, let us fix a nonempty bounded subset Y of the space $\mathcal{C}(J,X)$ and a positive number $t \in J$. For $y \in Y$ and $\epsilon \ge 0$ denote by $\omega^t(y,\epsilon)$ the *modulus of continuity* of the function y on the interval $[0,t]$; that is,

$$\omega^t(y,\epsilon) = \sup\{\|y(t_2) - y(t_1)\| : t_1, t_2 \in [0,t],\ |t_2 - t_1| \le \epsilon\}.$$

Further let us put

$$\omega^t(Y,\epsilon) = \sup\{\omega^t(y,\epsilon) : y \in Y\}, \quad \omega_0^t(Y) = \lim_{\epsilon \to 0+} \omega^t(Y,\epsilon).$$

Apart from this, put

$$\overline{\beta}(Y) = \sup\{\beta(Y(t)) : t \in J\},$$

where β denotes the Hausdroff measure of noncompactnesss in X. Finally we define the function χ on the family $\mathcal{F}_{\mathcal{C}(J,X)}$ by putting

$$\chi(Y) = \omega_0^t(Y) + \overline{\beta}(Y).$$

It may be shown that the function χ is the measure of noncompactness in the space $\mathcal{C}(J,X)$ (see [4, 5]). The kernel $\ker\chi$ is the family of all nonempty and bounded subsets Y such that functions belonging to Y are equicontinuous on J and the set $Y(t)$ is relatively compact in X, for $t \in J$.

Next, for a given set $Y \in \mathcal{F}_{\mathcal{C}(J,X)}$, let us denote

$$\int_0^t Y(s)ds = \Big\{ \int_0^t y(s)ds : y \in Y \Big\}, \quad t \in J,$$
$$Y([0,t]) = \{y(s) : y \in Y,\ s \in [0,t]\}.$$

Lemma 2.4 ([12]). *If the Banach space X is* separable *and a set $Y \subset \mathcal{C}(J,X)$ is bounded, then the function $t \to \beta(Y(t))$ is measurable and*

$$\beta\Big(\int_0^t Y(s)ds\Big) \le \int_0^t \beta(Y(s))ds, \quad \text{for each } t \in J.$$

Remark 2.5. Observe that in the above lemma we do not require the equicontinuity of functions from the set Y.

Lemma 2.6. *Assume that a set $Y \subset \mathcal{C}(J,X)$ is bounded. Then*

$$\beta(Y([0,t])) \le \omega_0^t(Y) + \sup_{s \le t} \beta(Y(s)), \quad \text{for } t \in J. \tag{2.2}$$

Proof. Let $\delta > 0$ be arbitrary. Then there exists $\epsilon > 0$ such that

$$\omega^t(Y, \epsilon) \leq \omega_0^t(Y) + \delta/2. \tag{2.3}$$

Let us take a partition $0 = t_0 < t_1 < \dots < t_k = t$ such that $t_i - t_{i-1} \leq \epsilon$, for $i = 1, 2, \dots, k$. Then, for each $t' \in [t_{i-1}, t_i]$ and $y \in Y$, the following inequality is satisfied

$$\|y(t') - y(t_i)\| \leq \omega_0^t(Y) + \delta/2. \tag{2.4}$$

Let us notice that, for each $i = 1, 2, \dots, k$, there are points $z_{ij} \in X (j = 1, 2, \dots, n_i)$ such that

$$Y(t_i) \subset \cup_{j=1}^{n_i} B(z_{ij}, \sup_{s \leq t} \beta(Y(s)) + \delta/2). \tag{2.5}$$

We show that

$$Y([0, t]) = \cup_{i=1}^{k} \cup_{j=1}^{n_i} B(z_{ij}, \sup_{s \leq t} \beta(Y(s)) + \omega_0^t(Y) + \delta). \tag{2.6}$$

Let us choose an arbitrary element $q \in Y([0, t])$. Then we can find $t' \in [0, t]$ and $y \in Y$, such that $q = y(t')$. Choosing i such that $t' \in [t_{i-1}, t_i]$ and j such that $B(z_{ij}, \sup_{s \leq t} \beta(Y(s)) + \delta/2)$, we obtain, from (2.4) and (2.5),

$$\begin{aligned} \|q - z_{ij}\| = \|y(t') - z_{ij}\| &\leq \|y(t') - y(t_i)\| + \|y(t_i) - z_{ij}\| \\ &\leq \omega_0^t(Y) + \sup_{s \leq t} \beta(Y(s)) + \delta \end{aligned}$$

and this verifies (2.6). Condition (2.6) yields

$$\beta(Y([0, t])) \leq \omega_0^t(Y) + \sup_{s \leq t} \beta(Y(s)) + \delta.$$

Letting $\delta \to 0+$, we obtain (2.2). $\square$

Definition 2.7. A function $x(\cdot) \in \mathcal{C}([0, b], X)$ is said to be a mild solution of (1.1)–(1.2) if $x(s) = g(x)$, for $s \in [0, b]$, and the following integral equation is satisfied.

$$x(t) = U(t, 0)g(x) + \int_0^t U(t, s)Bu(s)ds + \int_0^t U(t, s)f(s, x(s))ds, \quad t \in J.$$

To study the controllability problem, we assume the following hypotheses:

(H1) $A(t)$ generates a strongly continuous semigroup of a family of evolution operators $U(t, s)$ and there exist constants $N_1 > 0,\ N_0 > 0$ such that

$$\|U(t, s)\| \leq N_1, \quad \text{for } \ 0 \leq s \leq t \leq b,$$

and $N_0 = \sup\{\|U(s, 0)\| : 0 \leq s \leq t\}$.

(H2) The linear operator $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^b U(b, s)Bu(s)ds$$

has an inverse operator W^{-1} which takes values in $L^2(J, U)/\ker W$ and there exists a positive constant K_1 such that $\|BW^{-1}\| \leq K_1$.

(H3) (i) The mapping $f : J \times X \to X$ satisfies the Carathéodory condition, that is, $f(\cdot, x)$ is measurable for $x \in X$ and $f(t, \cdot)$ is continuous for a.e. $t \in J$.

(ii) The mapping f is bounded on bounded subsets of $\mathcal{C}(J, X)$.

(iii) There exists a constant $m_f > 0$ such that, for any bounded set $Y \subset \mathcal{C}(J,X)$, the inequality
$$\beta(f([0,t] \times Y)) \le m_f \beta(Y([0,t]))$$
holds for $t \in J$, where $f([0,t] \times Y) = \{f(s,x(s)) : 0 \le s \le t,\ x \in Y\}$.

(H4) The function $g : \mathcal{C}(J,X) \to X$ is continuous and there exists a constant $m_g \ge 0$ such that
$$\beta(g(Y)) \le m_g \beta(Y(J)),$$
for each bounded set $Y \subset \mathcal{C}(J,X)$.

(H5) There exists a constant $r > 0$ such that
$$(1 + bN_1K_1)\Big[N_0 \sup_{x \in \mathbb{B}(\theta,r)} \|g(x)\| + N_1 \sup_{x \in \mathbb{B}(\theta,r)} \int_0^b \|f(\tau, x(\tau))\| d\tau\Big] + bN_1K_1\|x_1\| \le r,$$
for $t \in J$, where $\mathbb{B}(\theta,r)$ is a closed ball in $\mathcal{C}(J,X)$ centered at θ and with radius r.

(H6)
$$\begin{aligned}&\min\{3m_gN_0(b) + 3m_f bN_0(b) + 3bm_gN_0(b)N_1(b)K_1\\ &+ 2m_f b^2 N_0(b)N_1^2(b)K_1\} < 1.\end{aligned}$$

Definition 2.8 ([19, 20])**.** System (1.1)–(1.2) is said to be *controllable* on the interval J, if for every initial functions $x_0 \in X$ and $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the solution $x(\cdot)$ of (1.1)–(1.2) satisfies $x(0) = x_0$ and $x(b) = x_1$.

3. Controllability Result

Mathematical control theory is the area of application oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object means to influence its behavior so as to achieve a desired goal. In this section, we study the controllability results for the semilinear differential system (1.1)–(1.2).

Using (H2) for an arbitrary function $x(\cdot) \in \mathcal{C}(J,X)$, we define the control
$$u(t) = W^{-1}\Big[x_1 - U(b,0)g(x) - \int_0^b U(b,s)f\big(s,x(s)\big)ds\Big](t). \tag{3.1}$$
Consider the Banach space $\mathcal{Z} = \mathcal{C}(J,X)$ with norm $\|x\| = \sup\{|x(t)| : t \in J\}$.

We shall show that when using the control $u(t)$, the operator $\Psi : \mathcal{Z} \to \mathcal{Z}$ defined by
$$\begin{aligned}(\Psi x)(t) &= U(t,0)g(x) + \int_0^t U(t,s)f\big(s,x(s)\big)ds\\ &\quad + \int_0^t U(t,s)BW^{-1}\Big[x_1 - U(b,0)g(x) - \int_0^b U(b,s)f\big(s,x(\tau)\big)d\tau\Big](s)ds\end{aligned}$$
has a fixed point $x(\cdot)$. This fixed point is a mild solution of the system (1.1)–(1.2) and this implies that the system is controllable on J.

Next consider the operators $v_1,\ v_2,\ v_3 : \mathcal{C}(J,X) \to \mathcal{C}(J,X)$ defined by
$$(v_1x)(t) = U(t,0)g(x),$$

$$(v_2x)(t)=\int_0^t U(t,s)f\big(s,x(s)\big)ds,$$
$$(v_3x)(t)=\int_0^t U(t,s)BW^{-1}\Big[x_1-U(b,0)g(x)-\int_0^b U(b,\tau)f\big(\tau,x(\tau)\big)d\tau\Big](s)ds.$$

Lemma 3.1. *Assume that* (H1), (H3) *are satisfied and a set* $Y\subset\mathcal{C}(J,X)$ *is bounded. Then*
$$\omega_0^t(v_2Y)\le 2bN_1\beta(f([0,b]\times Y)),\quad \textit{for } t\in J.$$

Proof. Fix $t\in J$ and denote $Q=f([0,t]\times Y)$,
$$q^t(\epsilon)=\sup\Big\{\|(U(t_2,s)-U(t_1,s))q\|:0\le s\le t_1\le t_2\le t,\ t_2-t_1\le\epsilon,\ q\in Q\Big\}.$$
At the beginning, we show that
$$\lim_{\epsilon\to 0+}q^t(\epsilon)\le 2N_1\beta(Q). \tag{3.2}$$
Suppose the contrary. Then there exists a number d such that
$$\lim_{\epsilon\to 0+}q^t(\epsilon)>d>2N_1\beta(Q). \tag{3.3}$$
Fix $\delta>0$ such that
$$\lim_{\epsilon\to 0+}q^t(\epsilon)>d+\delta>d>2N_1(\beta(Q)+\delta). \tag{3.4}$$
Condition (3.3) yields that there exist sequences $\{t_{2,n}\},\{t_{1,n}\},\{s_n\}\subset J$ and $\{q_n\}\subset Q$ such that $t_{2,n}\to t'$, $t_{1,n}\to t'$, $s_n\to s$ and
$$\|(U(t_{2,n},s_n)-U(t_{1,n},s_n))q_n\|>d. \tag{3.5}$$
Let the points $z_1,z_2,\dots,z_k\in X$ be such that $Q\subset\cup_{i=1}^k B(z_i,\beta(Q)+\delta)$. Then there exists a point z_i and a subsequence $\{q_n\}$ such that $\{q_n\}\in B(z_i,\beta(Q)+\delta)$; that is,
$$\|z_j-q_n\|\le\beta(Q)+\delta,\ \text{for } n=1,2,\dots$$
Further we obtain
$$\begin{aligned}
&\|U(t_{2,n},s_n)q_n-U(t_{1,n},s_n)q_n\|\\
&\le\|U(t_{2,n},s_n)q_n-U(t_{1,n},s_n)z_j\|+\|U(t_{2,n},s_n)z_j-U(t_{1,n},s_n)z_j\|\\
&\quad\times\|U(t_{2,n},s_n)z_j-U(t_{1,n},s_n)q_n\|\\
&\le N_1\|q_n-z_j\|+N_1\|z_j-q_n\|+\|U(t_{2,n},s_n)z_j-U(t_{1,n},s_n)z_j\|\\
&\le 2N_1(\beta(Q)+\delta)+\|U(t_{2,n},s_n)z_j-U(t_{1,n},s_n)z_j\|.
\end{aligned}$$
Letting $n\to\infty$ and using the properties of the evolution system $\{U(t,s)\}$ we obtain
$$\limsup_{n\to\infty}\|U(t_{2,n},s_n)q_n-U(t_{1,n},s_n)q_n\|\le 2N_1(\beta(Q)+\delta).$$
This contradicts (3.3) and (3.4).

Now fix $\epsilon>0$ and $t_1,\ t_2\in[0,t]$ such that $0\le t_2-t_1\le\epsilon$. Applying (H3), we obtain
$$\begin{aligned}
&\|(v_2x)(t_2)-(v_2x)t_1\|\\
&\le\int_0^{t_1}\|(U(t_2,s)-U(t_1,s))f(s,x(s))\|ds+\int_{t_1}^{t_2}\|U(t_2,s)f(s,x(s))\|ds\\
&\quad+\int_0^t\|(U(t_2,s)-U(t_1,s))f(s,x(s))\|ds+\epsilon N_1\sup\{\|f(s,x(s))\|:x\in Y\}.
\end{aligned}$$

Hence we derive the inequality

$$\omega^t(v_1Y,\epsilon) \le \sup\Big\{ \int_0^t \|(U(t_2,s)-U(t_1,s))f(s,x(s))\|ds : t_1,t_2\in[0,t],\\ 0\le t_2-t_1\le\epsilon,\ x\in Y\Big\} + \epsilon N_1 \sup\{\|f(s,x(s))\| : x\in Y\}.$$

Letting $\epsilon\to 0+$, we obtain the result. □

Lemma 3.2. *Assume that the assumptions* (H1), (H4) *are satisfied and a set* $Y\subset \mathcal{C}(J,X)$ *is bounded. Then*

$$\omega_0^t(v_1Y) \le 2N_0(t)\beta(g(Y)), \quad \text{for } t\in J.$$

The proof of the above lemma simple and is omitted.

Lemma 3.3. *Assume that the assumptions* (H1)–(H4) *are satisfied and a set* $Y\subset \mathcal{C}(J,X)$ *is bounded. Then*

$$\omega_0^t(v_3Y) \le 2bN_1K_1\Big(\|x_1\| + N_0\beta(g(Y)) + bN_1\beta(f(Q))\Big), \quad \text{for } t\in J.$$

Proof. As in the Lemmas 3.1 and 3.2, also fix $\epsilon>0$ and $t_1,t_2\in[0,t]$, $0\le t_2-t_1\le\epsilon$. Applying (H3) and (H4), we obtain

$$\begin{aligned}
&\|(v_3x)(t_2)-(v_3x)(t_1)\|\\
&\le \int_0^{t_1}\Big\|(U(t_2,s)-U(t_1,s))BW^{-1}\Big[x_1-U(b,0)g(x)-\int_0^b U(b,\tau)f(\tau,x(\tau))d\tau\Big]\Big\|ds\\
&\quad+\int_{t_1}^{t_2}\Big\|U(t_2,s))BW^{-1}\Big[x_1-U(b,0)g(x)-\int_0^b U(b,\tau)f(\tau,x(\tau))d\tau\Big]\Big\|ds\\
&\le K_1\int_0^{t_1}\|U(t_2,s)-U(t_1,s)\|\Big[\|x_1\|+\|U(b,0)g(x)\|\\
&\quad+\int_0^b\|U(b,\tau)f(\tau,x(\tau))d\tau\|\Big]ds\\
&\quad+\epsilon K_1N_1\Big[\|x_1\|+N_0\sup\{\|g(x)\|:x\in Y\}+N_1\sup\{\|f(s,x(s))\|:x\in Y\}\Big].
\end{aligned}$$

Hence we derive the inequality

$$\begin{aligned}
&\omega_0^t(v_3Y)\\
&\le \sup\Big\{K_1\int_0^{t_1}\|U(t_2,s)-U(t_1,s)\|\Big[\|x_1\|+\|U(b,0)g(x)\|\\
&\quad+\int_0^b\|U(b,\tau)f(\tau,x(\tau))d\tau\|\Big]ds : t_1,t_2\in[0,b],\ 0\le t_2-t_1\le\epsilon, x\in Y\Big\}\\
&\quad+\epsilon K_1N_1\Big[\|x_1\|+N_0\sup\{\|g(x)\|:x\in Y\}+N_1\sup\{\|f(s,x(s))\|:x\in Y\}\Big].
\end{aligned}$$

Letting $\epsilon\to 0+$, we obtain

$$\omega_0^t(v_3Y) \le 2bN_1K_1\Big(\|x_1\| + N_0\beta(g(Y)) + bN_1\beta(f(Q))\Big).$$

The proof is complete. □

Our main result is as follows.

Theorem 3.4. *If the Banach space X is separable and assumptions* (H1)-(H4) *are satisfied then system* (1.1)–(1.2) *is controllable on J.*

Proof. Consider the operator $\mathcal{P}$ defined by

$$\begin{aligned}(\mathcal{P}x)(t) &= U(t,0)g(x) + \int_0^t U(t,s)f\big(s,x(s)\big)ds \\ &\quad + \int_0^t U(t,s)BW^{-1}\Big[x_1 - U(b,0)g(x) - \int_0^b U(b,s)f\big(s,x(\tau)\big)d\tau\Big](s)ds.\end{aligned}$$

For an arbitrary $x \in \mathcal{C}(J,X)$ and $t \in J$, we obtain

$$\begin{aligned}\|(\mathcal{P}x)(t)\| &\le N_0\|g(x)\| + N_1\int_0^t \|f(s,x(s))\|ds \\ &\quad + N_1K_1\int_0^t \Big[\|x_1\| + N_0\|g(x)\| + N_1\int_0^b \|f(\tau,x(\tau))\|d\tau\Big]ds \\ &\le (1+bN_1K_1)\Big[N_0\|g(x)\| + N_1\int_0^b \|f(\tau,x(\tau))\|d\tau\Big] + bN_1K_1\|x_1\|.\end{aligned}$$

From the above estimate and assumption (H5) we infer that there exists a constant $r > 0$ such that the operator $\mathcal{P}$ transforms closed ball $\mathbb{B}$ into itself.

Now we prove that the operator $\mathcal{P}$ is continuous on $\mathbb{B}(\theta,r)$.

Let us fix $x \in \mathbb{B}(\theta,r)$ and take an arbitrary sequence $\{x_n\} \in \mathbb{B}(\theta,r)$ such that $x_n \to x$ in $\mathcal{C}(J,X)$. Next we have

$$\begin{aligned}&\|\mathcal{P}x_n - \mathcal{P}x\| \\ &\le N_0\|g(x_n) - g(x)\| + N_1\int_0^t \|f(s,x_n(s)) - f(s,x(s))\|ds \\ &\quad + K_1\int_0^t \|U(t,s)\|\Big[N_0\|g(x_n)-g(x)\| + N_1\int_0^b \|f(\tau,x_n(\tau)) - f(\tau,x(\tau))\|d\tau\Big]ds \\ &\le (1+bN_1K_1)\Big[N_0\|g(x_n)-g(x)\| + N_1\int_0^b \|f(\tau,x_n(\tau)) - f(\tau,x(\tau))\|d\tau\Big].\end{aligned}$$

Applying Lebesgue dominated convergence theorem, we derive that $\mathcal{P}$ is continuous on $\mathbb{B}(\theta,r)$.

Now we consider the sequence of sets $\{\Omega_n\}$ defined by induction as follows:

$$\Omega_0 = \mathbb{B}(\theta,r),\ \Omega_{n+1} = \operatorname{Conv}(\mathcal{P}\Omega_n), \quad \text{for } n = 1,2,\dots.$$

This sequence is decreasing; that is, $\Omega_n \supset \Omega_{n+1}$, for $n = 1,2,\dots$.

Further let us put

$$v_n(t) = \beta(\Omega_n([0,t])), \quad w_n(t) = \omega_0^t(\Omega_n).$$

Observe that each of the functions $v_n(t)$ and $w_n(t)$ is nondecreasing, while sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are non-increasing at any fixed $t \in J$. Put

$$v_\infty(t) = \lim_{n\to\infty} v_n(t), \quad w_\infty(t) = \lim_{n\to\infty} w_n(t), \quad \text{for } t \in J.$$

Using Lemmas 2.6, 3.2 and (H4), we obtain

$$\begin{aligned}\beta(v_1\Omega_n([0,t])) &\le \omega_0^t(v_1\Omega_n) + \sup_{s\le t}\beta(v_1\Omega_n(s)) \\ &\le 2N_0(t)\beta(g(\Omega_n)) + \sup_{s\le t} N_0(s)\beta(g(\Omega_n))\end{aligned}$$

$$
\begin{aligned}
&\le 3N_0(t)\beta(g(\Omega_n)) \\
&\le 3m_gN_0(t)\beta(\Omega_n([0,b])) \\
&= 3m_gN_0(t)v_n(b);
\end{aligned}
$$

that is,

$$
\beta(v_1\Omega_n([0,t])) \le 3m_gN_0(t)v_n(b). \tag{3.6}
$$

Moreover,

$$
\begin{aligned}
\beta(v_2\Omega_n([0,t])) &\le \omega_0^t(v_2\Omega_n) + \sup_{s\le t}\beta(v_2\Omega_n(s)) \\
&\le 2bN_1(t)\beta(f([0,t]\times\Omega_n)) + \sup_{s\le t}\beta\Big(\int_0^s U(s,\tau)f(\tau,\Omega_n(\tau))d\tau\Big) \\
&\le 2m_fbN_1(t)\beta(\Omega_n([0,t])) + \sup_{s\le t}N_1(t)\int_0^s\beta(f(\tau,\Omega_n(\tau)))d\tau \\
&\le 2m_fbN_1(t)v_n(t) + m_fN_1(t)\int_0^t v_n(\tau)d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta(v_3\Omega_n([0,t])) \\
&\le \omega_0^t(v_3\Omega_n) + \sup_{s\le t}\beta(v_3\Omega_n(s)) \\
&\le 2bN_1(t)K_1\Big(\|x_1\| + N_0\beta(g(\Omega_n)) + bN_1\beta(f(Q))\Big) \\
&\quad + \sup_{s\le t}\beta\Big\{\int_0^t U(t,s)BW^{-1}\Big[x_1 - U(b,0)g(\Omega_n) \\
&\quad - \int_0^b U(b,\tau)f\big(\tau,\Omega_n(\tau)\big)d\tau\Big](s)ds\Big\} \\
&\le 2bN_1(t)K_1\Big(\|x_1\| + N_0(t)\beta(g(\Omega_n)) + bN_1(t)\beta(f([0,t]\times\Omega_n)))\Big\} \\
&\quad + \sup_{s\le t}bN_1(s)K_1\Big\{\|x_1\| + N_0\beta(g(\Omega_n)) + N_1(t)\int_0^s\beta(f(\tau,\Omega_n(\tau)))d\tau\Big\} \\
&\le 3bN_1(t)K_1\Big(\|x_1\| + m_gN_0(t)v_n(b)\Big) + bm_fN_1(t)K_1\Big(2bN_1(t)v_n(t) \\
&\quad + N_1\int_0^t v_n(\tau)d\tau\Big).
\end{aligned}
$$

Linking this estimate with (3.5), we obtain

$$
\begin{aligned}
v_{n+1}(t) &= \beta(\Omega_{n+1}([0,t])) \\
&= \beta(\mathcal{P}\Omega_n([0,t])) \\
&\le \beta(v_1\Omega_n([0,t])) + \beta(v_2\Omega_n([0,t])) + \beta(v_3\Omega_n([0,t])) \\
&\le 3m_gN_0(t)v_n(b) + 2m_fbN_1(t)v_n(t) + m_fN_1(t)\int_0^t v_n(\tau)d\tau \\
&\quad + 3bN_1(t)K_1\Big(\|x_1\| + m_gN_0(t)v_n(b)\Big) \\
&\quad + bm_fN_1(t)K_1\Big(2bN_1(t)v_n(t) + N_1\int_0^t v_n(\tau)d\tau\Big).
\end{aligned}
$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} v_\infty(t) &\le 3m_g N_0(t) v_\infty(b) + 2m_f b N_1(t) v_\infty(t) + m_f N_1(t)\int_0^t v_\infty(\tau)d\tau \\ &\quad + b m_f N_1(t) K_1\Big(2bN_1(t)v_\infty(t) + N_1\int_0^t v_\infty(\tau)d\tau\Big) \\ &\quad + 3b m_g N_0(t) N_1(t) K_1 v_\infty(b). \end{aligned}$$

Hence putting $t = b$, in view of (H6), we obtain

$$v_\infty(b) = 0. \tag{3.7}$$

Furthermore, applying Lemmas 3.1, 3.2, 3.3, we have

$$\begin{aligned} & w_{n+1}(t) \\ &= \omega_0^t(\Omega_{n+1}) \\ &= \omega_0^t(\mathcal{P}\Omega_n) \\ &\le \omega_0^t(v_1\Omega_n) + \omega_0^t(v_2\Omega_n) + \omega_0^t(v_3\Omega_n) \\ &\le 2m_g N_0 v_n(b) + 2m_f b N_1 v_n(t) + 2bN_1K_1\Big(\|x_1\| + N_0\beta(g(Y)) + bN_1\beta(f(Q))\Big) \\ &\le 2m_g N_0 v_n(b) + 2m_f b N_1 v_n(t) + 2bN_1K_1\Big(\|x_1\| + m_g N_0 v_n(b) + b m_f N_1 v_n(t)\Big) \\ &\le (2 + bN_1K_1)[m_g N_0 v_n(b) + m_f b N_1 v_n(t)]. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$w_\infty(t) \le (2 + bN_1K_1)[m_g N_0 v_\infty(b) + m_f b N_1 v_\infty(t)].$$

Putting $t = b$ and applying (3.7), we conclude that $w_\infty(b) = 0$. This fact together with (3.7) implies that $\lim_{n\to\infty}\chi(\Omega_n) = 0$. Hence, in view of the Remark 2.2, we deduce that the set $\Omega_\infty = \cap_{n=0}^{\infty}\Omega_n$ is nonempty, compact and convex. Finally. linking the above obtained facts concerning the set Ω_∞ and the operator $\mathcal{P} : \Omega_\infty \to \Omega_\infty$ and using the classical Schauder fixed point theorem, we infer that the operator $\mathcal{P}$ has at least one fixed point x in the set Ω_∞. Obviously the function $x = x(t)$ is a mild solution of (1.1)–(1.2) satisfying $x(b) = x_1$. Hence the given system is controllable on J. □

Remark 3.5. Let us consider the case when the mapping g is

$$g(x) = \sum_{i=1}^{n} d_i x(t_i),$$

where $0 \le t_1 < t_2 < \dots < t_n \le b$, $d_1, d_2, \dots, d_n$ are given constants. For a bounded set $Y \subset \mathcal{C}(J, X)$ we obtain

$$\beta(g(Y)) \le \sum_{i=1}^{n}|d_i|\beta(Y(t_i)) \le \sum_{i=1}^{n}|d_i|\beta(Y(J)).$$

Similarly,

$$\beta(g(Y)) \le \sum_{i=1}^{n}|d_i|\beta(Y(t_i)) \le \sum_{i=1}^{n}|d_i|.\sup_{t\in J}\beta(Y(t)).$$

These inequalities imply that the constant m_g from assumption (H4) satisfies the estimate

$$m_g \leq \sum_{i=1}^{n} |d_i|.$$

Now let us consider the case, when the mapping g is of the form

$$g(x) = \int_0^b h(t, x(t))dt,$$

where the mapping $h : J \times X \to X$ satisfies the Carathéodory condition, and moreover

$$\beta(h(t, W)) \leq m(t)\beta(W)$$

hold, for a.e. $t \in J$ and $W \subset X$, where the function $m : J \to \mathbb{R}+$ is integrable. Then, for a bounded set $Y \subset \mathcal{C}(J, X)$, we have

$$\beta(g(Y)) \leq \beta\Big(\int_0^b h(t, Y(t))dt\Big) \leq \int_0^b m(t)\beta(Y(t))dt \leq \int_0^b m(t). \sup_{t\in J} \beta(Y(t))dt$$

and therefore the constant m_g from (H4) satisfies the estimate $m_g \leq \int_0^b m(t)$.

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