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SEMILINEAR ELLIPTIC EQUATIONS WITH DEPENDENCE ON THE GRADIENT

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Abstract. In this article we consider elliptic equations whose nonlinear term depends on the gradient of the unknown. We assume that the nonlinearity has a asymptotically linear growth at zero and at infinity with respect to the second variable. By applying Morse theory and an iterative method, we prove the existence of nontrivial solutions.

1. Introduction

In this article we consider the following elliptic equation with dependence on the gradient,

$$\begin{gathered} -\Delta u = f(x, u, \nabla u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary. Since the nonlinearity f depends on the gradient of the solution, solving (1.1) is not variational. In fact the well developed critical point theory cannot be applied directly. There have been several works on this equation, using sub and super-solution, topological degree, fixed point theorems and Galerkin method; see, for instance, [1, 2, 6, 11, 15, 16, 17].

In [5], de Figueiredo, Girardi and Matzeu developed a quite different method of variational type. Under the assumptions that f has a superlinear subcritical growth at zero and at infinity with respect to the second variable, they obtained the existence of a positive and a negative solutions of (1.1) by using the mountain pass theorem and iterative technique. Later, this method was applied to quasilinear elliptic equations [7, 8, 13], Hamiltonian systems [9] and impulsive differential equations [14].

In general the above papers which used mountain pass technique assume that the nonlinearity has a superlinear subcritical growth at zero and at infinity with respect to the second variable. Here we show that Morse theory and iterative method can be used to find solutions to (1.1) under the assumption that f has a asymptotically linear growth at zero and at infinity with respect to the second variable.

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Let $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots \le \lambda_k \le \dots$ be the eigenvalues associated with the eigenvectors $\varphi_1, \varphi_2, \varphi_3, \varphi_4, \dots$ of $-\Delta$ with Dirichlet boundary condition, and we make the following assumptions:

(H0) $f : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is continuous;
(H1) $f(x,t,\xi) = \lambda t + g_0(x,t,\xi)$, where $\lambda_j < \lambda < \lambda_{j+1}$, $g_0(x,t,\xi) = o(|t|)$ as $t \to 0$ uniformly for $x \in \overline{\Omega}, \xi \in \mathbb{R}^N$;
(H2) $f(x,t,\xi) = \mu t + g(x,t,\xi)$, where $\lambda_k < \mu < \lambda_{k+1}, k \ne j, k \ge 1, g(x,t,\xi) = o(|t|)$ as $|t| \to \infty$ uniformly for $x \in \overline{\Omega}, \xi \in \mathbb{R}^N$;
(H3) $f(x,t,\xi) = \mu t + g(x,t,\xi)$, where $\mu = \lambda_k$ and $\lambda_{k-l-1} < \lambda_{k-l} = \lambda_{k-l+1} = \dots = \lambda_{k-1} = \lambda_k < \lambda_{k+1}$, $k \ne j$, $k \ge l+1$, $|g(x,t,\xi)| \le C$ and $G(x,t,\xi) \to -\infty$ as $|t| \to \infty$ uniformly for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, where $C > 0$ is a constant, $G(x,t,\xi) = \int_0^t g(x,s,\xi)ds$;
(H4) For any $x \in \overline{\Omega}$, $t_1, t_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$, $f(x,t,\xi)$ satisfies the Lipschitz condition
$$|f(x,t_2,\xi_2) - f(x,t_1,\xi_1)| \le L(|t_2 - t_1| + |\xi_2 - \xi_1|),$$
where $L > 0$ is a constant.

By (H1), zero is a solution of (1.1), called trivial solution. The purpose of this article is to find nontrivial solutions. Our main results as as follows:

Theorem 1.1. *Assume that* (H0), (H1), (H2), (H4) *hold. If* $0 < \frac{L\sqrt{\lambda_1}}{\lambda_1 - L} < 1$*, then* (1.1) *has at least a nontrivial weak solution.*

Theorem 1.2. *Assume that* (H0), (H1), (H3), (H4) *hold. If* $0 < \frac{L\sqrt{\lambda_1}}{\lambda_1 - L} < 1$*, then* (1.1) *has at least a nontrivial weak solution.*

This article is organized as follows. In section 2 we give a simple revisit to Morse theory. In section 3 we prove Theorem 1.1 and Theorem 1.2 by using Morse theory and iterative method. An example will be given in section 4.

2. Preliminaries about Morse theory

Let H be a real Hilbert space and $J \in C^1(H,\mathbb{R})$ be a functional satisfying the (PS) condition. Denote by $H_q(A,B)$ the q-th singular relative homology group of the topological pair with coefficients in a field G. Let u be an isolated critical point of J with $J(u) = c$. The group
$$C_q(J,u) := H_q(J^c, J^c \setminus \{u\}), \quad q \in \mathbb{Z},$$
is called the q-th critical group of J at u, where $J^c = \{u \in H \mid J(u) \le c\}$. Denote $K = \{u \in H \setminus J'(u) = 0\}$. Assume that K is a finite set. Take $a < \inf J(K)$. The critical groups of J at infinity are defined by
$$C_q(J,\infty) := H_q(H, J^a \setminus \{u\}), \quad q \in \mathbb{Z}.$$
The following result is important in proving the existence of nontrivial critical points.

Proposition 2.1 ([3, Proposition 3.6])**.** *Suppose J satisfies the (PS) condition. If $K = \emptyset$, then $C_q(J,\infty) \cong 0$, $q \in \mathbb{Z}$. If $K = \{u_0\}$, then $C_q(J,\infty) \cong C_q(J,u_0)$, $q \in \mathbb{Z}$.*

Let A_∞ and A_0 be bounded self-adjoint operators defined on H. According to their spectral decomposition, $H = H^+ \oplus H^0 \oplus H^-$, where H^+, H^0, H^- are invariant subspaces corresponding to the positive, zero and negative spectrum of A_∞, respectively, similarly, $H = H_0^+ \oplus H_0^0 \oplus H_0^-$, where H_0^+, H_0^0, H_0^- are invariant subspaces corresponding to the positive, zero and negative spectrum of A_0, respectively. Let $P_0 : H \to H^0$ be the orthogonal projector.

Consider the functionals

$$\Phi(u) = \frac{1}{2}\langle A_\infty u, u\rangle + \varphi(u), \quad \Phi_0(u) = \frac{1}{2}\langle A_0 u, u\rangle + \varphi_0(u).$$

We make the following assumptions:

(A1) $(A_\infty)_\pm := A_\infty|_{H^\pm}$ has a bounded inverse on $H^\pm$.
(A2) $\gamma := \dim(H^- \oplus H^0) < \infty$.
(A3) $\varphi \in C^1(H, \mathbb{R})$ has a compact gradient mapping $\nabla\varphi(u)$, and $\nabla\varphi(u) = o(\|u\|)$ as $\|u\| \to \infty$. In addition, if $\dim H_0 \neq 0$, we assume

$$\|\nabla\varphi(u)\| \le C, \forall u \in H, \quad \varphi(P_0 u) \to -\infty \quad \text{as } \|P_0 u\| \to \infty.$$

(A4) $(A_0)_\pm := (A_0)|_{H_0^\pm}$ has a bounded inverse on $H_0^\pm$.
(A5) $\beta := \dim(H_0^-) < \infty$, and $\dim H_0^0 = 0$.
(A6) $\varphi_0 \in C^1(H, \mathbb{R})$ has a compact gradient mapping $\nabla\varphi_0(u)$, and

$$\nabla\varphi_0(u) = o(\|u\|) \quad \text{as } \|u\| \to 0.$$

Also we use the following results.

Theorem 2.2 ([4, Lemma 5.1]). *Assume that* (A1)–(A3) *hold, then* Φ *satisfies the (PS) condition, and*

$$C_q(\Phi, \infty) = \begin{cases} G, & q = \gamma, \\ 0, & q \neq \gamma. \end{cases}$$

Theorem 2.3 ([4, Theorem 4.1]). *Assume that* (A4)–(A6) *hold, then*

$$C_q(\Phi_0, 0) = \begin{cases} G, & q = \beta, \\ 0, & q \neq \beta. \end{cases}$$

3. Proof of Theorems 1.1 and 1.2

Let $H_0^1(\Omega)$ be the usual Sobolev space with the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx, \quad \forall u, v \in H_0^1(\Omega).$$

For $w \in H_0^1(\Omega)$, consider the problem

$$\begin{gathered} -\Delta u = f(x, u, \nabla w), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega \end{gathered} \tag{3.1}$$

and the associated functional $I_w : H_0^1(\Omega) \to \mathbb{R}$,

$$I_w(v) = \frac{1}{2}\int_\Omega |\nabla v(x)|^2\, dx - \int_\Omega F(x, v(x), \nabla w(x))\, dx.$$

By (H0) (H2) or (H0) (H3), $I_w \in C^1(H_0^1(\Omega), \mathbb{R})$, and the weak solutions of the problem (3.1) corresponds to the critical points of the functional I_w, see [12].

Define the operators $L_\infty, L_0 : H_0^1(\Omega) \to H_0^1(\Omega)$ by $L_\infty u = u - \mu(-\triangle)^{-1}u$ and $L_0 u = u - \lambda(-\triangle)^{-1}u$. Obviously, L_∞ and L_0 are bounded self-adjoint operators. Let

$$\phi_w(u) = \int_\Omega g(x, u(x), \nabla w(x))\,dx, \quad \phi_{w0}(u) = \int_\Omega g_0(x, u(x), \nabla w(x))\,dx.$$

It is well known that $\nabla\phi_w, \nabla\phi_{w0}$ are compact mappings. By (H1), (H2) or (H1), (H3), $\nabla\phi_w(u) = o(\|u\|)$ as $\|u\| \to \infty$, and $\nabla\phi_{w0} = o(\|u\|)$ as $\|u\| \to 0$. We can rewrite the functional I_w by

$$I_w(u) = \frac{1}{2}\langle L_\infty u, u\rangle - \phi_w(u) = \frac{1}{2}\langle L_0 u, u\rangle - \phi_{w0}(u).$$

According to the spectral decomposition of the operator L_∞, $H_0^1(\Omega) = H^+ \oplus H^0 \oplus H^-$, where H^+, H^0, H^- are invariant subspaces corresponding to the positive, zero and negative spectrum of L_∞ respectively.

If (H2) holds, then $H^- = \operatorname{span}\{\varphi_1, \varphi_2, \dots \varphi_k\}$, $H^+ = (H^-)^\perp$.

If (H3) holds, then $H^- = \operatorname{span}\{\varphi_1, \varphi_2, \dots \varphi_{k-l-1}\}$, $H^0 = \operatorname{span}\{\varphi_{k-l}, \dots \varphi_k\}$, $H^+ = (H^0 \oplus H^-)^\perp$.

Similarly, according to the spectral decomposition of the operator L_0, $H_0^1(\Omega) = H_0^+ \oplus H_0^0 \oplus H_0^-$, where H_0^+, H_0^0, H_0^- are invariant subspaces corresponding to the positive zero and negative spectrum of L_0 respectively. If (H1) holds, then L_0 is invertible and $H_0^- = \operatorname{span}\{\varphi_1, \varphi_2, \dots \varphi_j\}$, $H_0^+ = (H_0^-)^\perp$.

Lemma 3.1. *Assume that* (H0)–(H2) *hold. Then for any* $w \in H_0^1(\Omega)$*,* (3.1) *has at least a nontrivial weak solution.*

Proof. By (H2), $\dim H^0 = 0$, $L_\infty|_{H^\pm}$ has a bounded inverse on $H^\pm$ and $\dim H^- = k$, thus (A1), (A2) and (A3) hold. So by Theorem 2.2, I_w satisfies (PS) condition and

$$C_q(I_w, \infty) = \begin{cases} G, & q = k, \\ 0, & q \neq k. \end{cases}$$

By (H1), $\dim H^0 = 0$, $L_0|_{H_0^\pm}$ has a bounded inverse on $H_0^\pm$ and $\dim H^- = j$, thus (A4), (A5) and (A6) hold. By Theorem 2.3

$$C_q(I_w, 0) = \begin{cases} G, & q = j, \\ 0, & q \neq j. \end{cases}$$

Since $k \neq j$, $C_q(I_w, \infty) \neq C_q(I_w, 0)$ for some $q \in \mathbb{Z}$, hence by Proposition 2.1, I_w has at least a nontrivial critical point and (3.1) has at least a nontrivial weak solution. $\square$

Lemma 3.2. *Assume that* (H0), (H1), (H3) *hold. Then for any* $w \in H_0^1(\Omega)$*,* (3.1) *has at least a nontrivial weak solution.*

Proof. By (H3), $L_\infty|_{H^\pm}$ has a bounded inverse on $H^\pm$, and $\dim(H^- \oplus H^0) = k$, so (A1), (A2) hold. On the other hand, $\dim H^0 > 0$, but it can be checked that $\|\nabla\phi_w(u)\| \le C'$ for any $u \in H_0^1(\Omega)$ and a constant $C' > 0$, and $\phi_w(u) \to -\infty$ with $u \in H^0$ as $\|u\| \to \infty$. Indeed, by (H3), Hölder inequality and Sobolev inequality, for any $u, v \in H_0^1(\Omega)$, we have

$$|\langle\nabla\phi_w(u), v\rangle| \le \int_\Omega |g(x, u, \nabla w)||v|\,dx \le C(\int_\Omega |v|^2)^{1/2} \le C'\|v\|,$$

where $C, C' > 0$ are constants, this implies $\|\nabla\phi_w(u)\| \le C'$ for all $u \in H_0^1(\Omega)$.

We claim that $\phi_w(u) \to -\infty$ with $u \in H^0$ as $\|u\| \to \infty$. If this is not true, then there exists a sequence $\{u_n\}$ and constant $M > 0$ such that $u_n \in H^0$, $\|u_n\| \to \infty$ as $n \to \infty$, and $\phi_w(u_n) \ge -M$. Let $\tilde{u}_n = \frac{u_n}{\|u_n\|}$, then $\tilde{u}_n \in H^0$ and $\|\tilde{u}_n\| = 1$. By $\dim H^0 < \infty$, there exists a subsequence of $\{\tilde{u}_n\}$ still denoted by $\{\tilde{u}_n\}$, and $\tilde{u}$ such that $\tilde{u}_n$ converges strongly to $\tilde{u} \in H^0$ as $n \to \infty$, then $\tilde{u}$ satisfies the equation

$$
\begin{gathered}
-\Delta\tilde{u} = \lambda_k\tilde{u}, \quad \text{in } \Omega, \\
\tilde{u} = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}
$$

Since $\tilde{u} \neq 0$, by the unique continuation property as in [10], $\tilde{u} \neq 0$ a.e. in Ω, which implies $u_n \to \infty$ a.e. in Ω. Hence by (H3), $G(x, u_n(x), \nabla w(x)) \to -\infty$ a.e. in Ω, then

$$
\phi_w(u_n) = \int_\Omega G(x, u_n(x), \nabla w(x))\,dx \to -\infty
$$

as $n \to \infty$, we obtain a contradiction. Therefore, (A3) holds.

Next by using the argument used in the proof of Lemma 3.1 we complete the proof. $\square$

Lemma 3.3. *There exists a constant $c_1 > 0$ independent of w such that $\|u_w\| \ge c_1$ for all solutions u_w obtained in Lemma 3.1 or Lemma 3.2.*

Proof. First we decompose u_w as $u_w = u_w^+ + u_w^- \in H_0^+ \oplus H_0^-$. Since u_w is a weak solution of the problem (3.1), one has

$$
\int_\Omega \nabla u_w \cdot \nabla\phi\,dx = \int_\Omega (\lambda u_w + g_0(x, u_w, \nabla w))\phi\,dx, \quad \forall \phi \in H_0^1(\Omega). \tag{3.3}
$$

Particularly, take $\phi = u_w^+ - u_w^-$ into (3.3), we have

$$
\int_\Omega \nabla u_w \cdot \nabla(u_w^+ - u_w^-) - \lambda u_w(u_w^+ - u_w^-)\,dx = \int_\Omega g_0(x, u_w, \nabla w)(u_w^+ - u_w^-)\,dx. \tag{3.4}
$$

By (H1), $\lambda_{j-1} < \lambda < \lambda_j$, then we have

$$
\begin{aligned}
&\int_\Omega \nabla u_w(x) \cdot \nabla(u_w^+ - u_w^-) - \lambda u_w(x)(u_w^+ - u_w^-)\,dx \\
&= \int_\Omega (|\nabla u_w^+|^2 - \lambda|u_w^+|^2) - (|\nabla u_w^-|^2 - \lambda|u_w^-|^2)\,dx \\
&\ge (1 - \frac{\lambda}{\lambda_j}) \int_\Omega |\nabla u_w^+|^2\,dx + (\frac{\lambda}{\lambda_{j-1}} - 1) \int_\Omega |\nabla u_w^-|^2\,dx \\
&\ge m \int_\Omega |\nabla u_w|^2\,dx,
\end{aligned} \tag{3.5}
$$

where $m = \min\{(1 - \frac{\lambda}{\lambda_j}), (\frac{\lambda}{\lambda_{j-1}} - 1)\} > 0$. Fix $(N+2)/(N-2) > p > 1$, by (H1) (H2) or (H1) (H3), for any $\epsilon > 0$, there exists constant $k_\epsilon > 0$ such that

$|g_0(x,t,\xi)| \le \epsilon|t| + k_\epsilon|t|^p$. By Hölder inequality and Sobolev inequality

$$
\begin{aligned}
&\int_\Omega g_0(x,u_w(x),\nabla w(x))(u_w^+ - u_w^-)\,dx \\
&\le \int_\Omega (\epsilon|u_w(x)| + k_\epsilon|u_w(x)|^p)(|u_w^+| + |u_w^-|)\,dx \\
&\le \epsilon\|u_w\|_{L^2(\Omega)}\|u_w^+\|_{L^2(\Omega)} + \epsilon\|u_w\|_{L^2(\Omega)}\|u_w^-\|_{L^2(\Omega)} \\
&\quad + k_\epsilon\|u_w\|^p_{L^{p+1}(\Omega)}\|u_w^+\|_{L^{p+1}(\Omega)} + k_\epsilon\|u_w\|^p_{L^{p+1}(\Omega)}\|u_w^-\|_{L^{p+1}(\Omega)} \\
&\le \frac{\epsilon}{\lambda_1}\|u_w\|\|u_w^+\| + \frac{\epsilon}{\lambda_1}\|u_w\|\|u_w^-\| + Ck_\epsilon\|u_w\|^p\|u_w^+\| + Ck_\epsilon\|u_w\|^p\|u_w^-\| \\
&\le \frac{2\epsilon}{\lambda_1}\|u_w\|^2 + 2Ck_\epsilon\|u_w\|^{p+1}.
\end{aligned}
\tag{3.6}
$$

Combining (3.4), (3.5) and (3.6) we obtain $(m - \frac{2\epsilon}{\lambda_1})\|u_w(x)\|^2 \le 2Ck_\epsilon\|u_w(x)\|^{p+1}$. Since $m > 0$, we can take $\epsilon > 0$ sufficiently small such that $m - \frac{2\epsilon}{\lambda_1} > 0$, note that $p+1 > 2$, thus there exists a constant $c_1 > 0$ independent of w such that $\|u_w\| \ge c_1$. □

Proof of Theorem 1.1. First take $u_0 \in H_0^1(\Omega)$, by Lemma 3.1 we can construct a sequence $\{u_n\}$ such that for $n \ge 1$ u_n is a nontrivial solution of the equation

$$
\begin{gathered}
-\Delta u_n = f(x,u_n,\nabla u_{n-1}), \quad \text{in } \Omega, \\
u_n = 0, \quad \text{on } \partial\Omega.
\end{gathered}
\tag{3.7}
$$

From (3.7), u_{n+1} and u_n satisfy

$$
\int_\Omega \nabla u_{n+1}(\nabla u_{n+1} - \nabla u_n) = \int_\Omega f(x,u_{n+1},\nabla u_n)(u_{n+1} - u_n), \tag{3.8}
$$

$$
\int_\Omega \nabla u_n(\nabla u_{n+1} - \nabla u_n) = \int_\Omega f(x,u_n,\nabla u_{n-1})(u_{n+1} - u_n), \tag{3.9}
$$

By (3.8), (3.9), (H4), Sobolev inequality, and Hölder inequality, we obtain

$$
\begin{aligned}
&\|u_{n+1} - u_n\|^2 \\
&= \int_\Omega (f(x,u_{n+1},\nabla u_n) - f(x,u_n,\nabla u_{n-1}))(u_{n+1} - u_n)\,dx \\
&\le \int_\Omega L(|u_{n+1} - u_n| + |\nabla u_n - \nabla u_{n-1}|)|u_{n+1} - u_n|\,dx \\
&\le \int_\Omega \frac{L}{\lambda_1}|\nabla(u_{n+1} - u_n)|^2\,dx + L(\int_\Omega |\nabla(u_n - u_{n-1})|^2\,dx)^{1/2}(\int_\Omega |u_{n+1} - u_n|^2\,dx)^{1/2} \\
&\le \frac{L}{\lambda_1}\|u_{n+1} - u_n\|^2 + \frac{L}{\sqrt{\lambda_1}}\|u_n - u_{n-1}\|\|u_{n+1} - u_n\|;
\end{aligned}
$$

thus

$$
\|u_{n+1} - u_n\| \le \frac{L\sqrt{\lambda_1}}{\lambda_1 - L}\|u_n - u_{n-1}\|.
$$

Since $0 < \frac{L\sqrt{\lambda_1}}{\lambda_1 - L} < 1$, $\{u_n\}$ is a Cauchy sequence in $H_0^1(\Omega)$, so $\{u_n\}$ converges strongly to some $u^* \in H_0^1(\Omega)$.

We claim that u^* is a weak solution of (1.1). Indeed for any $\phi \in C_0^\infty(\Omega)$, by (H4),

$$\begin{aligned}
&\int_\Omega (f(x,u_n,\nabla u_{n-1}) - f(x,u^*,\nabla u^*))\phi\, dx \\
&\le L\|\phi\|_{L^\infty(\Omega)} \int_\Omega (|u_n - u^*| + |\nabla u_{n-1} - \nabla u^*|)\, dx \\
&\le C_\phi(\|u_n - u^*\| + \|u_{n-1} - u^*\|) \to 0
\end{aligned}$$

as $n \to \infty$; thus by Lemma 3.1,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_\Omega \nabla u_n \nabla\phi - f(x,u_n,\nabla u_{n-1})\phi\, dx \\
&= \int_\Omega \nabla u^* \nabla\phi - f(x,u^*,\nabla u^*)\phi\, dx.
\end{aligned}$$

Hence the claim is proved. By Lemma 3.3, $\|u_n\| \ge c_1$; thus $\|u^*\| \ge c_1$. Therefore, u^* is a nontrivial weak solution of the problem (1.1). $\square$

Proof of Theorem 1.2. By Lemma 3.2 and using the argument as used in the proof of Theorem 1.1 we complete the proof. $\square$

4. Examples

Consider the equation

$$\begin{gathered}
-\Delta u = mu - \frac{au}{1+u^2} - \frac{u^3}{1+u^4}\sin^2|\nabla u|, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N (N \ge 3)$ is a bounded domain with smooth boundary. Suppose that $a > 0$, $m - a < \lambda_k \le m$ for some $k \ge 1$, $m - a \ne \lambda_j$ for any $j \in \mathbb{N}$, and $0 < \frac{(m+a+3)\sqrt{\lambda_1}}{\lambda_1 - (m+a+3)} < 1$. We will show that (4.1) has at least one nontrivial weak solution. Let

$$f(t,\xi) = mt - \frac{at}{1+t^2} - \frac{t^3}{1+t^4}\sin^2|\xi|.$$

Then $f \in C(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$, so (H0) holds. Let

$$g_0(t,\xi) = at(1 - \frac{1}{1+t^2}) - \frac{t^3}{1+t^4}\sin^2|\xi|.$$

Then $f(t,\xi) = (m-a)t + g_0(t,\xi)$. It is not difficult to see that $g_0(t,\xi) = o(|t|)$ as $t \to 0$. Since $m - a \ne \lambda_j$ for any $j \in \mathbb{N}$, then (H1) holds. Let

$$g(t,\xi) = -\frac{at}{1+t^2} - \frac{t^3}{1+t^4}\sin^2|\xi|.$$

Then $f(t,\xi) = mt + g(t,\xi)$. It is not difficult to see that $g(t,\xi) = o(|t|)$ as $|t| \to \infty$. Note that $m - a < \lambda_k \le m$ for some $k \ge 1$. If $m \ne \lambda_{k+l}$ for any $l \ge 0$, then (H2) holds. If $m = \lambda_{k+l}$ for some $l \ge 0$, then (H3) holds, in fact, since $a > 0$, we have

$$|g(t,\xi)| \le |\frac{at}{1+t^2}| + |\frac{t^3}{1+t^4}\sin^2|\xi|| \le \frac{a}{2} + 1$$

and

$$G(t,\xi) = -\int_0^t \frac{as}{1+s^2}ds - \int_0^t \frac{s^3}{1+s^4}\sin^2|\xi| ds$$

$$= -\frac{a}{2}\ln(1+t^2) - \frac{1}{4}\ln(1+t^4)\sin^2|\xi| \to -\infty$$

as $|t| \to \infty$, uniformly in $\xi \in \mathbb{R}^n$.

Finally, we show that (H4) holds. By Lagrange mean value theorem, for any $t_1, t_2 \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^N$ we have

$$\begin{aligned}
&|f(t_2,\xi_2) - f(t_1,\xi_1)| \\
&= |(mt_2 - \frac{at_2}{1+t_2^2} - \frac{t_2^3}{1+t_2^4}\sin^2|\xi_2|) - (mt_1 - \frac{at_1}{1+t_1^2} - \frac{t_1^3}{1+t_1^4}\sin^2|\xi_1|)| \\
&\le m|t_2 - t_1| + |\frac{at_2}{1+t_2^2} - \frac{at_1}{1+t_1^2}| + |(\frac{t_2^3}{1+t_2^4} - \frac{t_1^3}{1+t_1^4})\sin^2|\xi_2|| \\
&\quad + |\frac{t_1^3}{1+t_1^4}(\sin^2|\xi_2| - \sin^2|\xi_1|)| \\
&\le m|t_2 - t_1| + a|t_2 - t_1| + 3|t_2 - t_1| + 2|\xi_2 - \xi_1| \\
&\le (m + a + 3)(|t_2 - t_1| + |\xi_2 - \xi_1|),
\end{aligned}$$

so (H4) holds. By Theorem 1.1 and Theorem 1.2, Equation (4.1) has at least one nontrivial weak solution.

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