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GROWTH OF ENTIRE SOLUTIONS OF SINGULAR INITIAL-VALUE PROBLEM IN SEVERAL COMPLEX VARIABLES

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Abstract. In this article, we characterize the order, type, lower order, and lower type of entire function solutions to a class of singular initial-value problems, in terms of multinomials for $n \geq 2$.

1. Introduction

Let $z_j = x_j + iy_j$ denote a complex variable, $1 \leq j \leq n$. Let $z = (z_1, \dots, z_n)$, $z^{2k} = z_1^{2k_1}, \dots, z_n^{2k_n}$ where k is the vector $(k_1, \dots, k_n)$ with k_j a nonnegative integer $(j = 1, \dots, n)$ and let $\|k\| = k_1 + \dots + k_n$. Let $\phi(z)$ be an entire function of $z_1^2, \dots, z_n^2$ in a domain D that includes the origin and let $\Delta_j = D_{z_j}^2 + \frac{\alpha_j}{z_j} D_{z_j}, \alpha_j \geq 0, j = 1, \dots, n$. Also, let $a > -1$ and $\varepsilon_j = 1$ if $j = 1, \dots, m$ and $\varepsilon_j = -1$ if $j = m+1, \dots, n$. Now consider the representations of an entire function solutions of the problem

$$\Big(D_t^2 + \frac{a}{t} D_t\Big) u(z,t) = \sum_{j=1}^{n} \varepsilon_j \Delta_j u(z,t) \tag{1.1}$$

with initial data

$$u(z,0) = \phi(z), \quad u_t(z,0) = 0$$

in terms of a set of associated multinomials $\{R_k(z,t)\}$ throughout (z,t) space, t real. These multinomials are solutions of (1.1) corresponding to the choice of $\phi(z) = z^{2k}$ in (1.1).

Let G be a region in $\mathbb{R}^n$ (positive hyper octant) and let $G_R \subset \mathbb{C}^n$ denote the region obtained from G by a similarity transformation about the origin, with ratio of similitude R.

Definition 1.1. Let $\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z_n^{2k_n}$ be an entire function of several complex variables. Then $\phi(z)$ is of growth (ρ, T) if

$$T = \limsup_{\|k\| \to \infty} \frac{\|2k\|}{e\rho} [|a_k| d_k(G)]^{\rho/\|2k\|}, \quad (0 < \rho < \infty) \tag{1.2}$$

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where

$$d_k(G) = \max_{R\in G}(R^{2k}); \quad R^{2k} = R_1^{2k_1}, \dots, R_n^{2k_n}.$$

This implies the existence of a positive constant M such that

$$|\phi(z)| \le Me^{T|z^2|^\rho} \quad \forall z \in \mathbb{C}^n.$$

Using (1.2), for each $\varepsilon > 0$ there exists a positive integer k_0 such that if $k \ge k_0$, then

$$|a_k|d_k(G) \le \Big[\frac{e\rho(T+\varepsilon)}{\|2k\|}\Big]^{\|2k\|/rho}. \tag{1.3}$$

We can easily estimate, from [1, (4.14)], that

$$|R_k(z,t)| \le \Big(\frac{\|2k\|}{\rho T}\Big)^{\|2k\|/rho}\overline{M}(\rho,T)e^{-\|2k\|/\rho}e^{K|t|+\sum_{j=1}^n T_j|z_j^2|^{\rho_j}}, \tag{1.4}$$

where

$$\overline{M}(\rho,T) = \int_0^\infty e^{-\sigma+T|\sigma^2|^\rho}d\sigma,$$

and K is the sum of the absolute values of the coefficients of multinomial and $M(\rho,T)$ is a generic constant depending only on the $\rho_j^{'}s$ and $T_j^{'}s$.

Now let

$$u(z,t) = \sum_{\|k\|=0}^\infty a_k R_k(z,t)$$

or

$$|u(z,t)| \le \sum_{\|k\|=0}^N |a_k||R_k(z,t)| + \sum_{\|k\|=N+1}^\infty |a_k||R_k(z,t)|. \tag{1.5}$$

Using the bound (1.5) on $|R_k(z,t)|$ and the estimate on $|a_k|d_k(G)$ from (1.3), we see that the bound on second sum in (1.5) is given by

$$\frac{K(\rho,T)}{d_k(G)}e^{K|t|+\sum_{j=1}^n T_j|z_j^2|^{\rho_j}} \sum_{\|k\|=N+1}^\infty \Big(\frac{T+\varepsilon}{T}\Big)^{\|2k\|/\rho}.$$

Since the series of constants in (1.5) converges, it follows that the series $u(z,t) = \sum_{\|k\|=0}^\infty a_k R^k(z,t)$ converges for all n complex variables $(z_1,\dots,z_n)$ and real t and uniformly so in compact subsets of (z,t) space.

Now we can establish a theorem.

Theorem 1.2. *Let $\phi(z) = \sum_{\|k\|=0}^\infty a_k z_1^{2k_1},\dots,z_n^{2k_n}$ be entire in $(z_1^2,\dots,z_n^2)$ and converge in a domain $G_r : z \in \mathbb{C}^n; |z|^2 = \max_{i\le j\le n}|z_j|^2 < R^2, R > 0$ is a fixed positive real. Then the series $u(z,t) = \sum_{\|k\|=0}^\infty a_k R_k(z,t)$ converges for all n-complex variables $(z_1,\dots,z_n)$ and real t and uniformly so in compact subsets of (z,t) space.*

Bragg and Dettman [2] proved the following theorem.

Theorem 1.3. *Let $\phi(x) = \sum_{\|k\|=0}^\infty a_k x^{2k}$ be analytic in $(x_1^2,\dots,x_n^2)$ and converge in a domain D that includes the origin. Then the series $\sum_{\|k\|=0}^\infty a_k P_k(x,t)$ converges to an analytic solution of the problem (1.1) replacing z by x, at least in region S where S is defined by $(x,t) \in S$ if and only if*

$$|x_1| + |t|, \dots, |x_m| + |t|, \big(x_{m+1}^2 + t^2\big)^{1/2}, \dots, \big(x_n^2 + t^2\big)^{1/2} \in D. \tag{1.6}$$

We shall proceed to the complex transformation of above Theorem A in the following manner.

Let $(z_1, \dots, z_n)$ be an element of $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, the space of real coordinates. The transformation from real to the complex coordinates are given by $x_k = \frac{z_k + \bar{z}_k}{2}$, $y_k = \frac{z_k - \bar{z}_k}{2_i}$. We equip $\mathbb{C}^n$ with the Euclidean metric of $\mathbb{R}^{2n}$;

$$ds^2 = \sum_{k=1}^{n} (dx_k^2 + dy_k^2) = \sum_{k=1}^{n} dz_k . d_{\bar{z}_k}.$$

Let z_k be a point on the domain G_R for which $|a_k R_k(z_k, 0)| = \sup_{z_k \in G_R} |a_k R_k (z_k, 0)| = C_k$. By a rotation, we can assume that $z_k^2 = (x_k^2, 0, \dots, 0)$. If $\widetilde{f}(w) = f(w^2, 0, \dots, 0)$ and $\widetilde{f}(w) = \sum_{l=0}^{\infty} a_l w^{2l}$ is the Taylor series expansion of $\widetilde{f}$ at the origin, then $\left|a_k x_k^{2k}\right| = C_k$ and therefore we have the following theorem.

Theorem 1.4. *Let $\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k}$ be entire in $(z_1^2, \dots, z_n^2)$ and converge in a domain G_R that includes the origin. Then the series $u(z,t) = \sum_{\|k\|+0}^{\infty} a_k R_k(z,t)$ converges to an entire solution of the problem* (1.1) *at least in a region S where S is defined by $(z,t) \in S$ if and only if*

$$|z_1| + |t|, \dots, |z_m + |t|,\ (z_{m+1}^2 + t^2)^{1/2}, \dots, (z_n^2 + t^2)^{1/2} \in G_R.$$

Let $\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k}$ be the power series expansion of the function $\phi(z)$. Then the maximum modulus of $u(z,t)$ and $\phi(z)$ are defined as in complex function theory [15, pp. 129, 132],

$$M_{f,G}(R) = \max_{z \in G_R} |f(z)|,$$
$$M_{u,S}(R) = \max_{(z,t) \in S} |u(z,t)|.$$

Following the usual definitions of order and type of an entire function of n-complex variables $(z_1^2, \dots, z_n^2)$, the order ρ and type T of $u(z,t)$ are defined as in [4]

$$\rho(u) = \limsup_{R \to \infty} \frac{\log \log M_{u,S}(R)}{\log R}, \tag{1.7}$$

$$T(u) = \limsup_{R \to \infty} \frac{\log M_{u,S}(R)}{R^{\rho(u)}}. \tag{1.8}$$

In this paper we characterize the order, lower order, type and lower type of entire function solutions of problem (1.1) in terms of a set $\{R_k(z,t)\}$ of multinomials for $n \geq 2$. Multinomials of this type have been constructed by Miles and Yong [12] when $z = x$ and $m = n$ or $m = 0$. In these cases (1.1) reduces to either the generalized Euler-Poisson-Darboux or the generalized Beltrami equation. Gilbert and Howard [5, 6] discussed analyticity properties of solutions of special cases of (1.1). Bragg and Dettman obtained representation of analytic solutions of problem (1.1) for $z = x$ in terms of these multinomials for $n \geq 2$ [2] and for $n = 1$ in [3]. It has been found [2] that $R_k(x,t), n \geq 2$, can be expressed as a convolution of n polynomials $R_{k_j}(x_j, t), j = 1, \dots, n$. For $n = 1$ the corresponding $R_k(x,t)$ are defined in terms of Jacobi polynomials. The Growth estimates for the solutions of (1.1) in terms of multinomials $R_k(z,t)$ for $n \geq 2$ then permit the obtaining of global region of convergence from acknowledge of singularities of the given data function $\phi(z)$. It should be noted that the function $\phi(z)$ is the analytic continuation of its restriction to the axis of symmetry; i.e., $\phi(z) = u(z,0)$. Using various techniques, the characterizations of order and type of entire function solutions of similar problems were obtained by McCoy [13, 14] Kumar [8, 9, 10] and others for $n = 1$. However, non of them have considered the case for $n \geq 2$.

2. Auxiliary Results

In this section we shall prove some auxiliary results which will be used in the sequel.

Lemma 2.1. *If $u(z,t) = \sum_{\|k\|=0}^{\infty} a_k R_k(z,t)$ is an entire function solution of problem (1.1) in terms of a set $\{R_k(z,t)\}$ of multinomials corresponding to given data function $\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k}$ in (1.1) then ϕ and ϕ^* are also entire functions of $n-$complex variables $(z_1^2, \dots, z_n^2)$. Further,*

$$[N(\varepsilon)]^{-1} M_{\phi,G}(R) \leq M_{u,S}(\varepsilon^{-1}R) \leq C M_{\phi^*,G}(R) \tag{2.1}$$

where

$$\phi^*(z) = \sum_{\|k\|=0}^{\infty} |a_k| \Big\{ \prod_{j=1}^{n} k_j^{p_j} \Big\} z_1^{2k_1}, \dots, z_n^{2k_n},$$

$$N(\varepsilon) = \sup\{N(\varepsilon e^{i\theta}, \xi) : 0 \leq \theta \leq 2\pi, -1 \leq \xi \leq 1, 0 < \varepsilon < 1\}$$

and C is a constant.

Proof. From Theorem 1.1 and 1.2, bearing in mind with the relation of [2, (3.1)], we obtain

$$\begin{aligned} |u(z,t)| \leq \sum_{\|k\|=0}^{\infty} |a_k| \Big\{ \Gamma\left(\frac{a+1}{2n}\right) \Big\}^n \frac{2^m K^{n-m}}{\pi^{m/2}} \Big\{ \prod_{j=1}^{m} k_j \frac{\Gamma(k_j + (\alpha_j+1)/2)}{\Gamma(k_j - 1/2)} \Big\} \\ \times \{|z_j| + |t|\}^{2k_j} \Big\{ \prod_{j=m+1}^{n} \frac{k_j^{q_j} k_j!}{\Gamma((k_j) + (a+1)/2n)} (z_j^2 + t^2)^{k_j} \Big\} \end{aligned}$$

where $q_j = \max((\alpha_j - 1)/2, ((a+1)/2n) - 1, -1/2)$, $j = m+1, \dots, n$.

Using the relation $\Gamma(x+a)/\Gamma x \sim x^a$ as $x \to \infty$, we have

$$\frac{\Gamma(k_j + (\alpha_j+1)/2)}{\Gamma(k_j - 1/2)} \sim (k_j - 1/2)^{(\alpha_j+2)/2}, \quad \frac{k_j^{q_j} k_j!}{\Gamma k_j + \frac{(a+1)}{2n}} \sim k_j^{q_j+1} (k_j)^{(a+1)/2n}$$

and we see that there exist constants $C, p_1, \dots, p_n$ with $p_j = p_j(\alpha_j)$, $j = 1, \dots, m$ and $p_j = p_j(\alpha_j, a, n)$ for $j = m+1, \dots, n$ such that

$$\begin{aligned} |u(z,t)| \leq \sum_{\|k\|=0}^{\infty} |a_j| C \Big\{ \prod_{j=1}^{n} k_j^{p_j} \Big\} (|z_1| + |t|)^{2k_1} \dots (|z_m| + |t|)^{2k_m} \\ \times (z_{m+1}^2 + t^2)^{k_{m+1}} \dots (z_n^2 + t^2)^{k_n}. \end{aligned} \tag{2.2}$$

Now, $|\phi(z)| \leq \sum_{\|k\|=0}^{\infty} |a_k| |z_1|^{2k_1} \dots |z_n|^{2k_n}$, the series (2.2) converges for $z \in G_R$. But for $z \in G_R$, the series

$$\sum_{\|k\|=0}^{\infty} |a_k| \Big\{ \prod_{j=1}^{n} k_j^{p_j} \Big\} |z_1|^{2k_1} \dots |z_n|^{2k_n}$$

also converges. By Theorem 1.2, if $\phi(z)$ is entire in $(z_1^2,\dots,z_n^2)$, then $u(z,t)$ converges to an entire solution of problem (1.1). We see that

$$\lim_{\|k\|\to\infty}\Big[|a_k|\prod_{j=1}^n k_j^{p_j}\Big]^{1/\|2k\|}=\lim_{\|k\|\to\infty}|a_k|^{\frac{1}{\|2k\|}}=0.$$

Hence both ϕ and ϕ^* are entire.

Using (2.2) we obtain

$$M_{u,S}(R)\le C\sum_{\|k\|=0}^{\infty}|a_k|\Big\{\prod_{j=1}^n k_j^{p_j}\Big\}R^{2k_n}=CM_{\phi^*,G}(R) \tag{2.3}$$

where

$$\phi^*(z)=\sum_{\|k\|=0}^{\infty}|a_k|\Big\{\prod_{j=1}^n k_j^{p_j}\Big\}z_1^{2k_1},\dots,z_n^{2k_n}.$$

Now for reverse relation, we have

$$\begin{aligned}\phi(z)&=\sum_{\|k\|=0}^{\infty}a_k z_1^{2k_1}\dots z_n^{2k_n}\\
|\phi(z)|&\le\sum_{\|k\|=0}^{\infty}|a_k|\Big\{\prod_{j=1}^n k_j^{p_j}\Big\}|z_1|^{2k_1}\dots|z_n|^{2k_n}\\
&=\sum_{\|k\|=0}^{\infty}|a_k|\Big\{\prod_{j=1}^n k_j^{p_j}\Big\}[|z_1+|t|]^{2k_1}\dots\{|z_m|+|t|\}^{2k_m}\\
&\quad\times\Big[\big(z_{m+1}^2+t^2\big)^{1/2}\Big]^{2k_{m+1}}\dots\Big[(z_n^2+t^2)^{1/2}\Big]^{2k_n}\\
&\quad\times\Big[\frac{|z_1|}{|z_1|+|t|}\Big]^{2k_1}\dots\Big[\frac{|z_m|}{|z_m|+|t|}\Big]^{2k_m}\\
&\quad\times\Big[\frac{|z_{m+1}|}{(z_{m+1}^2+t^2)^{1/2}}\Big]^{2k_{m+1}}\dots\Big[\frac{|z_n|}{(z_n^2+t^2)^{1/2}}\Big]^{2k_n}.\end{aligned}$$

This relation is valid globally, and leads to the estimates

$$\begin{gathered}|\phi(z)|\le M_{u,S}(R)N(\varepsilon),\varepsilon=(|z|/R)^2=\max_{1\le j\le n}\Big(\frac{|z_j|}{R_j}\Big)^2,\\
N(\varepsilon)=\sup\{|N(\varepsilon e^{i\theta},\xi)|:0\le\theta\le 2\pi,-1\le\xi\le 1\}.\end{gathered}$$

For $z=\varepsilon Re^{i\theta}(\varepsilon$ real, $0<\varepsilon<1\}$, we have

$$M_{\phi,G}(\varepsilon R)\le M_{u,S}(R)N(\varepsilon)$$

or

$$[N(\varepsilon)]^{-1}M_{\phi,G}(R)\le M_{u,S}(\varepsilon^{-1}R). \tag{2.4}$$

Combining (2.3) and (2.4) we obtain (2.1). □

Lemma 2.2. *Let $u(z,t)$ be an entire function solution of (1.1) in terms of a set $\{R_k(z,t)\}$ of multinomials corresponding to given data function $\phi(z)$ in (1.1). Then the orders and types of $u(z,t)$ and ϕ respectively are identical.*

Proof. Let $\phi(z) = \sum_{\|k\|=0}^{\infty} a_k z_1^{2k_1} \dots z_n^{2k_n}$ be an entire function of order $\rho(\phi)$ and type $T(\phi)$. Then it is well known [7, Thm. 1] that

$$\rho(\phi) = \limsup_{\|k\|\to\infty} \Big\{ \frac{\|2k\| \log \|k\|}{-\log |a_k|} \Big\}, \tag{2.5}$$

$$(e\rho(\phi)T(\phi))^{1/\rho(\phi)} = \limsup_{\|k\|\to\infty} \big\{ \|2k\|^{1/\rho(\phi)} [|a_k| d_k(G)]^{1/\|2k\|} \big\}. \tag{2.6}$$

Hence for the function $\phi^*(z) = \sum_{\|k\|=0}^{\infty} |a_k| \left\{ \prod_{j=1}^{n} k_j^{p_j} \right\} z_1^{2k_1} \dots z_n^{2k_n}$, we have

$$\begin{aligned}
\frac{1}{\rho(\phi^*)} &= \liminf_{\|k\|\to\infty} \frac{\log[|a_k| \prod_{j=1}^{n} k_j^{p_j}]^{-1}}{2\|k\| \log \|k\|} \\
&= \liminf_{\|k\|\to\infty} \frac{\log |a_k|^{-1} - \log[\prod_{j=1}^{n} k_j^{p_j}]}{2\|k\| \log \|k\|} \\
&= \liminf_{\|k\|\to\infty} \frac{\log |a_k|^{-1}}{2\|k\| \log \|k\|}.
\end{aligned}$$

Hence $\rho(\phi) = \rho(\phi^*)$. Since ϕ and ϕ^* have same order, using (2.6) we can easily show that $T(\phi) = T(\phi^*)$.

Now using the relation (2.1) with the definitions of order and type given by (1.7) and (1.8), the proof is complete. □

Lemma 2.3. *If $|a_k|/|a_{k'}|$, $\|k'\| = \|k\| + 1$, forms a non-decreasing function of k then $|\beta_k|/|\beta_{k'}|$ also forms a non-decreasing function of k, where*

$$\begin{aligned}
\beta_k = a_k &\Big\{ \Gamma\left(\frac{a+1}{2n}\right) \Big\}^n \frac{2^n K^{n-m}}{\pi^{m/2}} \Big\{ \prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j+2)/2} \Big\} \\
&\times \Big\{ \prod_{j=m+1}^{n} k_j^{(q_j+1+(a+1)/2n)} \Big\}.
\end{aligned} \tag{2.7}$$

Proof. We have

$$\begin{aligned}
\frac{|\beta_k|}{|\beta_{k'}|} &= a_k \Big\{ \Gamma\left(\frac{a+1}{2n}\right) \Big\}^2 \frac{2^n K^{n-m}}{\pi^{m/2}} \{ \prod_{j=1}^{m} k_j (k_j - 1/2)^{(\alpha_j+2)/2} \} \\
&\quad\times \frac{\{ \prod_{j=m+1}^{n} k_j^{q_j+1+(a+1)/2n} \}}{a_{k+1} \{ \Gamma\left(\frac{a+1}{2n}\right) \}^n \frac{2^n K^{n-m}}{\pi^{m/2}} \{ \prod_{j=1}^{m} (k_j+1) \left(k_j + \frac{1}{2}\right)^{(\alpha_j+2)/2} \}} \\
&\quad\times \frac{1}{\{ \prod_{j=m+1}^{n} k_j^{q_j+1+(a+1)/2n} \}} \\
&= \frac{a_k}{a_{k+1}} \frac{\prod_{j=1}^{m} k_j \left(k_j - \frac{1}{2}\right)^{(\alpha_j+2)/2} \{ \prod_{j=m+1}^{n} k_j^{(q_j+1+(a+1)/2n)} \}}{\prod_{j=1}^{m} (k_j+1)(k_j+1/2)^{(\alpha_j+2)/2}, \prod_{j=m+1}^{n} (k_j+1)^{(q_j+1+(a+1)/2n)}}.
\end{aligned}$$

Let

$$G(x) = \frac{\prod_{j=1}^{m} x_j (x_j - \frac{1}{2})^{(\alpha_j+2)/2} \prod_{j=m+1}^{n} x_j^{(q_j+1+(a+1)/2n)}}{\prod_{j=1}^{m} (x_j+1)(x_j+\frac{1}{2})^{(\alpha_j+2)/2} \prod_{j=m+1}^{n} (x_j+1)^{(q_j+1+(a+1)/2n)}}$$

$$
\begin{aligned}
\log G(x) &= \sum_{j=1}^{m} \log[x_j(x_j-1/2)^{(\alpha_j+2)/2}] + \sum_{j=m+1}^{n} \log x_j^{(q_j+1+(a+1)/2n)} \\
&\quad - \sum_{j=1}^{m} \log(x_j+1)(x_j+\frac{1}{2})^{(\alpha_j+2)/2} - \sum_{j=m+1}^{n} \log(x_j+1)^{(q_j+1+(a+1)/2n)}
\end{aligned}
$$

By logarithmic differentiation, we obtain

$$
\begin{aligned}
\frac{G'(x)}{G(x)} &= \sum_{j=1}^{m}\Big(\frac{1}{x_j} + \frac{(\alpha_j+2)}{2(x_j-\frac{1}{2})}\Big) + \sum_{j=m+1}^{n} \frac{q_j+1+\frac{(a+1)}{2n}}{x_j} \\
&\quad - \sum_{j=1}^{m} \frac{1}{x_j+1}\frac{(\alpha_j+2)}{2(x_j+\frac{1}{2})} - \sum_{j=m+1}^{n} \frac{q_j+1+\frac{(a+1)}{2n}}{x_j+1}.
\end{aligned}
$$

Let

$$
t(x_j) = \sum_{j=1}^{m} \frac{1}{x_j} + \frac{(\alpha_j+2)}{2(x_j-\frac{1}{2})} + \sum_{j=m+1}^{n} \frac{q_j+1+\frac{(a+1)}{2n}}{x_j}.
$$

Then $t(x_j) - t(x_{j+1}) > 0$ for any $x_j > 0$. Hence $t(x_j)$ is decreasing function and subsequently $G^{'}(x_j) > 0$ for $x_j > 0$. Hence $|\beta_k|/|\beta_{k'}|$ is non-decreasing if $|a_k|/|a_{k'}|$ is non-decreasing. ☐

3. Main Results

Theorem 3.1. *Let $u(z,t)$ be an entire function converges to solution of problem* (1.1) *corresponding to given data function $\phi(z)$ in* (1.1) *having order $\rho(u)$. Then*

$$
\rho(u) = \limsup_{\|k\|\to\infty} \frac{\|2k\| \log \|k\|}{-\log|\beta_k|} \tag{3.1}
$$

where β_k is given by (2.7).

Proof. It is well known [7, Thm. 1] that if $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k}$ be an entire function of order $\rho(f)$ then

$$
\rho(f) = \limsup_{\|k\|\to\infty} \frac{\|2k\| \log \|k\|}{-\log|a_k|}. \tag{3.2}
$$

Hence for the function $u(z,0) = \sum_{\|k\|=0}^{\infty} \beta_k z_1^{2k_1}\dots z_n^{2k_n}$, we have

$$
\begin{aligned}
\frac{1}{\rho(u)} &= \liminf_{\|k\|\to\infty} \frac{-\log|\beta_k|}{\|2k\|\log\|2k\|} \\
&= \liminf_{\|k\|\to\infty} \frac{\log|a_k|^{-1} - \log\big[\big\{\Gamma\big(\frac{(a+1)}{2n}\big)\big\}^n \frac{2^n K^{n-m}}{\pi^{m/2}} \big\{\prod_{j=1}^{n} k_j^{p_j}\big\}\big]}{\|2k\|\log\|2k\|} \\
&= \liminf_{\|k\|\to\infty} \frac{\log|a_k|^{-1}}{\|2k\|\log\|2k\|} - \frac{\log\Big[\big\{\Gamma\big(\frac{a+1}{2n}\big)\big\}^n \frac{2^n K^{n-m}}{\pi^{m/2}} \big\{\prod_{j=1}^{n} k_j^{p_j}\big\}\Big]^{1/\|2k\|}}{\log\|2k\|} \\
&= \liminf_{\|k\|\to\infty} \frac{\log|a_k|^{-1}}{\|2k\|\log\|2k\|}.
\end{aligned}
$$

☐

Now using (3.2) for data function $\phi(z)$, we get the required results.

Theorem 3.2. *Let $u(z,t)$ be an entire function converges to solution of* (1.1) *corresponding to given data function $\phi(z)$ in* (1.1) *having type $T(u)$. Then*

$$(e\rho(u)T(u))^{1/\rho(u)} = \limsup_{\|k\|\to\infty} \left\{ \|2k\|^{1/\rho(u)} \left[|\beta_k| d_k(G)\right]^{1/\|2k\|} \right\}, (0 < \rho(u) < \infty).$$

Proof. For an entire function $f(z) = \sum_{\|k\|=0}^{\infty} a_k z^{2k}$, Gol'dberg [7, Thm. 1] obtained type in terms of the coefficients of its Taylor series expansion as

$$(e\rho(f)T(f))^{1/\rho(f)} = \limsup_{\|k\|\to\infty} \left\{ \|2k\|^{1/\rho(f)} [|a_k| d_k(G)]^{1/\|2k\|}, (0 < \rho(f) < \infty) \right\}. \quad (3.3)$$

It can be seen that

$$[|\beta_k| d_k(G)]^{1/\|2k\|} \to [|a_k| d_k(G)]^{1/\|2k\|} \quad \texttt{as} \quad \|k\| \to \infty. \quad (3.4)$$

Hence the result follows by using (3.3) for data function $\phi(z)$ and taking into account the equation (3.4). □

In analogy with the definitions of order $\rho(u)$ and type $T(u)$, we define lower order $\lambda(u)$ and lower type $t(u)$ as

$$\lambda(u) = \liminf_{R\to\infty} \frac{\log\log M_{u,S}(R)}{\log R}$$
$$t(u) = \liminf_{R\to\infty} \frac{\log M_{u,S}(R)}{R^{\rho(u)}}, 0 < \rho(u) < \infty.$$

Theorem 3.3. *Let $u(z,t)$ be an entire function converges to the problem* (1.1) *corresponding to data function $\phi(z)$ in* (1.1) *having lower order $\lambda(u)$. Then*

$$\lambda(u) \geq \liminf_{\|k\|\to\infty} \frac{\|2k\| \log\|2k\|}{-\log|\beta_k|}. \quad (3.5)$$

Also if $|\beta_k|/|\beta_{k'}|$, where $\|k'\| = \|k\| + 1$, is a non-decreasing function of k, then equality holds in (3.5).

Proof. For entire function $f(z) = \sum_{\|k=0}^{\infty} a_k z_1^{2k_1} \dots z_n^{2k_n}$, if $|a_k|/|a_{k'}|$ forms a non-decreasing function of k then we have [11, Thm. 1]

$$\lambda(f) = \liminf_{\|k\|\to\infty} \frac{\|2k\| \log\|2k\|}{\log|a_k|^{-1}}. \quad (3.6)$$

Let $|\beta_k|/|\beta_{k'}|$ forms a non-decreasing function of k for $k > k_0$. Applying Lemma 2.3 and (3.6) to $u(z,0) = \sum_{\|k\|=0}^{\infty} \beta_k z_1^{2k_1} \dots z_n^{2k_n}$, we obtain

$$\frac{1}{\lambda(u)} = \limsup_{\|k\|\to\infty} \frac{\log|a_k|^{-1} - \log\left[C\prod_{j=1}^{n} k_j^{p_j}\right]}{\|2k\|\log\|2k\|} = \limsup_{\|k\|\to\infty} \frac{\log|a_k|^{-1}}{\|2k\|\log\|2k\|}$$

Then $\lambda(u) = \lambda(\phi)$. □

In a similar manner we can prove the following theorem.

Theorem 3.4. *Let $u(z,t)$ be an entire function converging to a solution of* (1.1) *corresponding to data function $\phi(z)$ in* (1.1) *having lower type $t(u)$. Then*

$$t(u) \geq \liminf_{\|k\|\to\infty} \frac{\|2k\|}{e\rho(u)} |\beta_k|^{\rho(u)/\|2k\|}. \quad (3.7)$$

Also, if $|\beta_k|/|\beta_{k'}|$, where $\|k'\| = \|k\| + 1$, is a non-decreasing function of $k > k_0$, then equality holds in (3.7).

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