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EXACT BEHAVIOR OF SINGULAR SOLUTIONS TO PROTTER'S PROBLEM WITH LOWER ORDER TERMS

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Abstract. For the (2+1)-D wave equation Protter formulated (1952) some boundary value problems which are three-dimensional analogues of the Darboux problems on the plane. Protter studied these problems in a 3-D domain, bounded by two characteristic cones and by a planar region. Now it is well known that, for an infinite number of smooth functions in the right-hand side, these problems do not have classical solutions, because of the strong power-type singularity which appears in the generalized solution. In the present paper we consider the wave equation involving lower order terms and obtain new a priori estimates describing the exact behavior of singular solutions of the third boundary value problem. According to the new estimates their singularity is of the same order as in case of the wave equation without lower order terms.

1. Introduction

We denote points in $\mathbb{R}^3$ by $(x,t) = (x_1, x_2, t)$ and consider the wave equation involving lower order terms

$$Lu \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu = f \tag{1.1}$$

in a simply connected region

$$\Omega_0 := \{(x,t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}.$$

The region $\Omega_0 \subset \mathbb{R}^3$ is bounded by the disk

$$\Sigma_0 := \{(x,t) : t = 0, x_1^2 + x_2^2 < 1\}$$

with center at the origin $O(0,0,0)$ and the characteristic surfaces of (1.1):

$$\Sigma_1 := \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$

$$\Sigma_{2,0} := \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}.$$

In this work we will study the problem

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Problem P_α. Find solutions to (1.1) in Ω_0 that satisfy the conditions

$$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0 \setminus O} = 0, \tag{1.2}$$

where $\alpha \in C^1(\bar{\Sigma}_0)$. The adjoint problem to $\mathbf{P}_\alpha$ is as follows.

Problem P^*_α. Find a solution of the adjoint equation

$$L^* u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} - (b_1 u)_{x_1} - (b_2 u)_{x_2} - (bu)_t + cu = g \quad \text{in } \Omega_0$$

with the boundary conditions:

$$u|_{\Sigma_{2,0}} = 0, \quad [u_t + (\alpha + b)u]|_{\Sigma_0} = 0.$$

The following problems were introduced by Protter [31].

Protter's Problems. Find a solution of the wave equation

$$\Box u \equiv \Delta_x u - u_{tt} \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \quad \text{in } \Omega_0 \tag{1.3}$$

with one of the following boundary conditions

$$\begin{aligned} &P1: \quad u|_{\Sigma_0 \cup \Sigma_1} = 0, && P1^*: \quad u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ &P2: \quad u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, && P2^*: \quad u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{aligned}$$

Protter [31] formulated and investigated both Problems $P1$ and $P1^*$ in Ω_0 as multi-dimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems on $\mathbb{R}^2$ are well posed, which is not true for the Protter's problems in $\mathbb{R}^3$ or $\mathbb{R}^4$. The uniqueness of a classical solution of Problem $P1$ in the $(3+1)-D$ case was proved by Garabedian [11]. For recent results concerning the Protter's problems with lower order terms (1.1) – (1.2) see Hristov, Popivanov, Schneider [15] and references therein, also see Grammatikopoulos et al [12]. For further publications in this area see Aldashev [1] – [2], Edmunds and Popivanov [10], Choi and Park [8], Cher [18], Popivanov and Popov [28] – [30]. Let us mention some special orthogonality conditions on f, found in Popivanov and Popov [28] – [30], which in the case of the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$ control the order of singularity of the generalized solutions of Problems $P1$ and $P2$. Unfortunately, we do not know of any such conditions in the more general case of equation (1.1).

On the other hand, Bazarbekov and Bazarbekov [5] gives in $\mathbb{R}^4$ another analogue of the classical Darboux problem in the four-dimensional domain corresponding to Ω_0. Some different statements of Darboux type problems in $\mathbb{R}^3$ or some connected with them Protter problems for mixed type equations (also studied in Protter [32]) can be found in Aldashev [3], Aziz and Schneider [4], Bitsadze [6], Kharibegashvili [17], Popivanov and Schneider [26]. Protter problems for mixed type equations in $\mathbb{R}^3$ involving lower order terms are considered in Rassias [33]–[34] and Hristov et al [16], where uniqueness theorems are proved under some conditions on the coefficients of the equation. In Lupo and Payne [20]–[21] and Lupo et al [22] one finds results for mixed type equations including some special nonlinearity with supercritical exponent term in various situations, namely for the Frankl' and Guderley-Morawetz problem in $\mathbb{R}^2$ and for the Protter problem in $\mathbb{R}^{N+1}$ with $N \geq 2$. The existence of bounded or unbounded solutions for the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$, as well as for the Euler-Poisson-Darboux equation has been studied in Cher [18], Choi [7], Choi and Park [8], Grammatikopoulos et al [13], Popivanov and Popov [30].

According to the ill-possedness of Protter's Problems $P1$ and $P2$, it is interesting to find some of their regularizations. A nonstandard, nonlocal regularization of

Problem P1, can be found in Edmunds and Popivanov [10]. In the present paper we are looking for some other kind of regularization and formulate the following problem.

Open Question 1. Is it possible to find conditions for the coefficients b_1, b_2, b, c and α, under which for all smooth functions f Problem P_α has only regular solutions?

Remark. If the answer to the above question is positive, then, using an operator L_k with lower order perturbations in the wave equation (1.3), we can find possible regularization for Problem $P2$. Solving the equation $L_k u_k = f$, with $L_k \to \Box$ (i.e. $b_{1k}, b_{2k}, b_{3k}, c_k \to 0$) and $\alpha_k \to 0$, we can find an approximated sequence u_k. Due to the fact that in this case the cones Σ_1 and $\Sigma_{2,0}$ are again characteristics for L_k, this process, with respect to our boundary value problem, looks to be natural.

For Problem (1.1), (1.2), i.e. P_α and $\alpha(x) \neq 0$, there are only few publications and we refer the reader to [15] and [12]. In the case of the equation (1.1), which involves either lower order terms or some other type of perturbation, Problem P_α in Ω_0 with $\alpha(x) \equiv 0$ has been studied by Aldashev [1]–[2].

Next, we formulate the following well known result Kwang-Chang [35], Popivanov and Schneider [25], presented here in the terms of the polar coordinates $x_1 = \varrho \cos\varphi$, $x_2 = \varrho \sin\varphi$.

Theorem 1.1. *For all $n \in \mathbb{N}$, $n \geq 4$; a_n, b_n arbitrary constants, the functions*

$$v_n(\varrho,\varphi,t) = t\varrho^{-n}\left(\varrho^2 - t^2\right)^{n-\frac{3}{2}}\left(a_n \cos n\varphi + b_n \sin n\varphi\right) \tag{1.4}$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

$$w_n(\varrho,\varphi,t) = \varrho^{-n}\left(\varrho^2 - t^2\right)^{n-\frac{1}{2}}\left(a_n \cos n\varphi + b_n \sin n\varphi\right) \tag{1.5}$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability (see Bitsadze [6]) of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all smooth functions (1.4) (respectively, (1.5)). The reason of this fact has been found by Popivanov and Schneider in [25], where they announced for Problems $P1$ and $P2$ that there exist singular solutions for the wave equation (1.3) with power type isolated singularities even for very smooth functions f. Using Theorem 1.1, Popivanov and Schneider [27] proved the existence of generalized solutions of Problems $P1$ and $P2$, which have at least power type singularities at the vertex O of the cone $\Sigma_{2,0}$. Considering Problems $P1$ and $P2$, Popivanov and Schneider [25] announced the existence of singular solutions for both wave and degenerate hyperbolic equations (see Popivanov and Schneider [26]). The first a priori estimates for singular solutions of Protter's Problems $P1$ and $P2$, concerning the wave equation in $\mathbb{R}^3$, were obtained in [27]. On the other hand, for the case of the wave equation in $\mathbb{R}^{m+1}$, Aldashev [1] announced that there exist solutions of Problem $P1$ (respectively, $P2$) in the domain Ω_ε, which blow up on the cone $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when $\varepsilon \to 0$ and the cone $\Sigma_{2,\varepsilon} := \{\varrho = t + \varepsilon\}$ approximates $\Sigma_{2,0}$. It is obvious that for $m = 2$ this result can be compared with the estimate (1.6) of Theorem 1.3 below. For the homogeneous Problem P^*_α (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$), even for the wave equation, we do not know of nontrivial solutions analogous to (1.4) and (1.5). Anyway, in Grammatikopoulos et al [12] under appropriate conditions for the coefficients of the general equation (1.1), we derive results which ensure the existence of many singular solutions of Problem P_α. Here we refer also to Khe Kan Cher [18], who gives some nontrivial solutions for the homogeneous Problems $P1^*$ and $P2^*$, but in the case of Euler-Poisson-Darboux equation. These results are closely connected to those of Theorem 1.1.

To formulate known results for Problem P_α we first recall the definition of generalized solutions.

Definition 1.2 ([12]). A function $u = u(x_1, x_2, t)$ is called a *generalized solution* of problem P_α in Ω_0, if

(1) $u \in C^1(\bar{\Omega}_0 \backslash O)$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\backslash O} = 0$, $u\big|_{\Sigma_1} = 0$,

(2) the equality

$$\begin{aligned}&\int_{\Omega_0} [u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} + (b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu - f)v] dx_1 dx_2 dt\\&= \int_{\Sigma_0} \alpha(x)(uv)(x,0) dx_1 dx_2\end{aligned}$$

holds for all v from

$$V_0 := \{v \in C^1(\bar{\Omega}_0) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighborhood of } \Sigma_{2,0}\}.$$

The Definition 1.2 assures that generalized solutions of Problem P_α may have singularities on the cone $\Sigma_{2,0}$.

In [12] is proved the following existence theorem for solutions of Problem P_α which have singularities on $\Sigma_{2,0}$.

In next Theorem we denote $a_1 := b_1 \cos\varphi + b_2 \sin\varphi$, $a_2 := \varrho^{-1}(b_2\cos\varphi - b_1 \sin\varphi)$ and we assume that a_1, a_2, b, c are independent on φ, i. e. they are functions of $(|x|, t)$ only and α is function of $(|x|)$.

Theorem 1.3 ([12]). *Let $\alpha \geq 0$; a_1, b, $c \in C^1(\bar{\Omega}_0\backslash O)$, $a_2 \equiv 0$ and*

$$a_1(|x|, t) \geq |b|(|x|, t), \quad a_1(|x|, t) \geq 2|x|c(|x|, t), \quad (x, t) \in \Omega_0.$$

Then for each function

$$f_n(x,t) = |x|^{-n}(|x|^2 - t^2)^{n-1/2} \cos n(\arctan \frac{x_2}{x_1}) \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0),$$

$n \in \mathbb{N}$, $n \geq 4$ the corresponding generalized solution u_n of the problem P_α belongs to $C^2(\bar{\Omega}_0\backslash O)$ and satisfies the estimate

$$|u_n(x,t)|_{t=|x|} \geq c_0|x|^{-n}|\cos n(\arctan\frac{x_2}{x_1})|, \quad 0 < |x| < 1/2, \tag{1.6}$$

where $c_0 = const > 0$.

In the same paper one can find a proof of the uniqueness of the treated problem. Note that the generalized solutions in this theorem have singularities at the vertex O of the cone $\Sigma_{2,0}$ and that these singularities do not propagate in the direction of the bicharacteristics on the characteristic cone $\Sigma_{2,0}$. For results concerning the propagation of singularities for solutions of second order operators see Hörmander [14, Chapter 24.5].

On the other hand, Hristov, Popivanov and Schneider in [15] (see Theorem 4.4 there in) obtained some upper bounds for all the solutions of this problem, considering the case that the coefficients b_1, b_2, b, c and α are smooth functions in $\bar{\Omega}_0$ (the coefficients of the equation (1.1) in polar coordinates, like it is in Theorem 1.3, do not depend on φ) and also assuming the function $f \in C(\bar\Omega_0)$ to be of the form

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi, n \in \mathbb{N}. \tag{1.7}$$

These upper bounds can be written of the form:

$$|u(x,t)| \le C_0 \max_{\bar\Omega_0}\{|f_n^{(1)}| + |f_n^{(2)}|\}|x|^{-n-\psi(K)}, \tag{1.8}$$

where C_0 is a positive constant,

$$K := \max\big\{\sup_{\bar\Omega_0}|b_1|, \sup_{\bar\Omega_0}|b_2|, \sup_{\bar\Omega_0}|b|, \sup_{\bar\Omega_0}|c|, \sup_{0\le|x|\le 1}|\alpha(|x|)|\big\}$$

and $\psi(K)$ is a positive function which blows up as K blows up.

In the present paper this estimate is improved by the following main result

Theorem 1.4. *Let the right-hand side function f in the equation* (1.1) *is of the form* (1.7)*, $b_1, b_2, b, c \in C(\bar\Omega_0)$, $\alpha \in C^1([0,1])$, $f_n^{(i)} \in C(\bar\Omega_0)$, $i = 1,2$ and a_1, a_2, b, c are functions of $(|x|,t)$, $\alpha = \alpha(|x|)$, where $a_1 := b_1\cos(\arctan\frac{x_2}{x_1}) + b_2\sin(\arctan\frac{x_2}{x_1})$, $a_2 := |x|^{-1}(b_2\cos(\arctan\frac{x_2}{x_1}) - b_1\sin(\arctan\frac{x_2}{x_1}))$. Then for the generalized solution $u(x,t)$ of Problem $\mathbf{P}_\alpha$ the following estimate*

$$|u(x,t)| \le C_\sigma \max_{\bar\Omega_0}\{|f_n^{(1)}| + |f_n^{(2)}|\}|x|^{-n-\sigma} \tag{1.9}$$

holds, where σ is an arbitrary positive number and C_σ is a positive constant depending on σ, n and all coefficients of (1.1).

Remark 1.5. A new point here, as distinct from (1.8), is the fact that the order of singularity does not depend on the lower order terms of (1.1) and on the boundary coefficient α.

Comparing this estimate with the lower bound of the singular solutions found in Theorem 1.3, we see that we have obtained their exact asymptotic behavior.

First, in this work we follow the exposition of Hristov et al [15] until Theorem 4.4. This takes the next three sections.

In Section 2 Problem P_α is reduced to a two-dimensional problem in the following steps. First, we transform equation (1.1) in polar coordinates, i.e.

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1u_\varrho + a_2u_\varphi + bu_t + cu = f, \tag{1.10}$$

($a_1 := b_1 cos\varphi + b_2 sin\varphi$, $a_2 := \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi)$), considering, as noted before, a polar symmetry of a_1, a_2, b, c and α, and a special form of the right-hand side (1.7). Next, we ask for generalized solution of the form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi. \tag{1.11}$$

Thus separating the variables we succeed in reducing the problem to a two-dimensional one for functions $\{u_n^{(1)}(\varrho,t), u_n^{(2)}(\varrho,t)\}$, called Problem $P_{\alpha,1}$. Finally, using characteristic coordinates $\xi = 1-\varrho-t$, $\eta = 1-\varrho+t$ and new functions

$$u_n^{(i)}(\xi,\eta) := z_n^{(i)}(\varrho,t) := \varrho^{\frac12}u_n^{(i)}(\varrho,t), i = 1,2, \tag{1.12}$$

we obtain a system for $\{u_n^{(1)}(\xi,\eta), u_n^{(2)}(\xi,\eta)\}$, called Problem $P_{\alpha,2}$.

In Section 3 an equivalent integral equation system of Problem $P_{\alpha,2}$ is constructed.

In Section 4 are presented some results from [15] which we use in the next section. Also, here is formulated the main result of [15], Theorem 4.4, which ensures the existence of a generalized solution of the two-dimensional Problem $P_{\alpha,2}$ and gives upper bounds of possible singularity. Using this theorem, after the inverse transformation to Problem P_α, one comes to (1.8).

In Section 5 we prove Theorem 1.4, the main result of this work.

The next Section 6 is dedicated to the singular solutions. Modifying a little the proof of Theorem 1.3, we deduce the following result.

Theorem 1.6. *Let $\alpha \geq 0$; b_1, b_2, b, $c \in C^1(\bar{\Omega}_0 \backslash O)$ and*

$$b_1 = a_1(|x|, t)\cos(\arctan x_2/x_1), \quad b_2 = a_1(|x|, t)\sin(\arctan x_2/x_1)$$

with some function $a_1(|x|, t)$ for which $a_1 \geq |b|, a_1 \geq 2|x|c$. Then for each function of the form

$$f(x,t) = f_n(|x|, t)\cos n(\arctan x_2/x_1) \quad \text{or}$$
$$f(x,t) = f_n(|x|, t)\sin n(\arctan x_2/x_1), \quad n \in \mathbb{N}$$

in the right-hand side of the equation, satisfying the following conditions:

$$f_n \in C(\bar{\Omega}_0), \quad f_n \not\equiv 0 \quad \text{in } \Omega_0, \quad \text{either } f_n \geq 0 \text{ or } f_n \leq 0 \quad \text{in } \Omega_0,$$

the corresponding generalized solution u_n of the problem P_α satisfies the estimate

$$|u_n(x,t)| \geq C_0|x|^{-n}|\cos n(\arctan \frac{x_2}{x_1})|, \quad C_0 = const > 0 \tag{1.13}$$

in some neighborhood of $O(0,0,0)$.

The difference between this theorem and Theorem 1.3 is that we have the same result for a wider class of right-hand side functions and, as well, in (1.13) we estimate $|u_n(x,t)|$, while in (1.6) is estimated the restriction $|u_n(x,t)|_{t=|x|}$.

In the case of wave equation without lower order terms and $\alpha \equiv 0$, Theorem 1.6 is in correspondence with the results deduced so far. Actually, in [9] one can find an asymptotic expansion of the generalized solution at the origin. According to this work, the order of singularity of the solution is less than n only if some orthogonality conditions are fulfilled, namely if the function f_n is orthogonal to some solutions of the adjoint homogeneous problem $P2^*$. If f_n does not change its sign, a necessary orthogonality condition is not fulfilled.

In the case of wave equation with lower order terms, we do not know such orthogonality conditions "controlling" the order of singularity of the corresponding solution.

Open Question 2. Can one find some orthogonality conditions in the case of the equation (1.1), under which we have a lower order of singularity?

2. Preliminaries

As we noted in the previous section, we consider (1.1) in polar coordinates (see (1.10)) in case that the right-hand side of the equation is of the form (1.7) and we ask for the generalized solution to be of the form (1.11). Here we assume that all coefficients of (1.10) depend only on ϱ and t, and we set $\alpha(x) \equiv \alpha(\varrho) \in C^1[0,1]$.

Thus from (1.1) we obtain the system

$$\begin{aligned} &\frac{1}{\varrho}(\varrho u^{(1)}_{n,\varrho})_\varrho - u^{(1)}_{n,tt} + a_1 u^{(1)}_{n,\varrho} + b u^{(1)}_{n,t} + (c - \frac{n^2}{\varrho^2}) u^{(1)}_n + n a_2 u^{(2)}_n = f^{(1)}_n, \\ &\frac{1}{\varrho}(\varrho u^{(2)}_{n,\varrho})_\varrho - u^{(2)}_{n,tt} + a_1 u^{(2)}_{n,\varrho} + b u^{(2)}_{n,t} + (c - \frac{n^2}{\varrho^2}) u^{(2)}_n - n a_2 u^{(1)}_n = f^{(2)}_n. \end{aligned} \tag{2.1}$$

To deal with singularities on $t = \varrho$, especially at $(0,0)$, we consider (2.1) in the domain

$$G_\varepsilon = \{(\varrho, t) : t > 0, \varepsilon + t < \varrho < 1 - t\}, \varepsilon > 0$$

which is bounded by the disc $S_0 = \{(\varrho, t) : t = 0, 0 < \varrho < 1\}$, and

$$S_1 = \{(\varrho, t) : \varrho = 1 - t\}, \quad S_{2,\varepsilon} = \{(\varrho, t) : \varrho = t + \varepsilon\}$$

and treat the following problem (omitted the index n):

Problem $P_{\alpha,1}$. Find solutions $u = (u^{(1)}, u^{(2)})$ of system (2.1) which satisfy

$$u^{(i)}|_{S_1 \cap \partial G_\varepsilon} = 0, \quad [u^{(i)}_t + \alpha(\varrho) u^{(i)}]|_{S_0 \cap \partial G_\varepsilon} = 0, \quad i = 1, 2.$$

Definition 2.1. A function $u = (u^{(1)}, u^{(2)})(\varrho, t)$ is called a *generalized solution* of Problem $P_{\alpha,1}$ in G_ε, $\varepsilon > 0$, if:

(1) $u \in C^1(\bar G_\varepsilon)$, $[u^{(i)}_t + \alpha(\varrho) u^{(i)}]\big|_{S_0 \cap \partial G_\varepsilon} = 0$, $u^{(i)}\big|_{S_1 \cap \partial G_\varepsilon} = 0, i = 1, 2;$

(2) The equalities

$$\begin{aligned} &\int_{G_\varepsilon} \big[u^{(1)}_t v_{1,t} - u^{(1)}_\varrho v_{1,\varrho} + \big(a_1 u^{(1)}_\varrho + b u^{(1)}_t + (c - \frac{n^2}{\varrho^2}) u^{(1)} + n a_2 u^{(2)} - f^{(1)}\big) v_1\big] \varrho d\varrho\, dt \\ &= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u^{(1)} v_1 \varrho\, d\varrho, \\ &\int_{G_\varepsilon} [u^{(2)}_t v_{2,t} - u^{(2)}_\varrho v_{2,\varrho} + \big(a_1 u^{(2)}_\rho + b u^{(2)}_t + (c - \frac{n^2}{\varrho^2}) u^{(2)} - n a_2 u^{(1)} - f^{(2)}\big) v_2\big] \varrho d\varrho\, dt \\ &= \int_{S_0 \cap \partial G_\varepsilon} \alpha(\varrho) u^{(2)} v_2 \varrho\, d\varrho \end{aligned}$$

hold for all

$$v_1, v_2 \in V^{(1)}_\varepsilon = \{v \in C^1(\bar G_\varepsilon) : [v_t + (\alpha + b) v]\big|_{S_0 \cap \partial G_\varepsilon} = 0, v\big|_{S_{2,\varepsilon} \cap \partial G_\varepsilon} = 0\}.$$

Introducing a new function

$$z^{(i)}(\varrho, t) = \varrho^{\frac12} u^{(i)}(\varrho, t) = z^{(i)}(\varrho(\xi, \eta), t(\xi, \eta)) =: U^{(i)}(\xi, \eta), \ i = 1, 2, \tag{2.2}$$

in characteristic coordinates

$$\xi = 1 - \varrho - t, \quad \eta = 1 - \varrho + t \tag{2.3}$$

we obtain the system

$$\begin{aligned} &U^{(1)}_{\xi\eta} - A_1 U^{(1)}_\xi - B_1 U^{(1)}_\eta - C_1 U^{(1)} - D_1 U^{(2)} = F^1(\xi, \eta) \quad \text{in } D_\varepsilon, \\ &U^{(2)}_{\xi\eta} - A_2 U^{(2)}_\xi - B_2 U^{(2)}_\eta - C_2 U^{(2)} - D_2 U^{(1)} = F^2(\xi, \eta) \quad \text{in } D_\varepsilon, \end{aligned} \tag{2.4}$$

where $D_\varepsilon = \{(\xi, \eta) : 0 < \xi < \eta < 1 - \varepsilon\}$ and

$$F^{(i)}(\xi, \eta) = \frac{1}{4\sqrt2}(2 - \xi - \eta)^{\frac12} f^{(i)}(\varrho(\xi, \eta), t(\xi, \eta)), i = 1, 2, \tag{2.5}$$

$$
\begin{aligned}
&A_1 = A_2 = \frac{1}{4}(a_1 + b), \quad B_1 = B_2 = \frac{1}{4}(a_1 - b),\\
&D_2 = -D_1 = \frac{1}{4}na_2, \quad C_1 = C_2 = \frac{1}{4}\Big\{\frac{4n^2-1}{(2-\xi-\eta)^2} + \frac{a_1}{2-\xi-\eta} - c\Big\}.
\end{aligned}
\tag{2.6}
$$

Note, that Problem $P_{\alpha,1}$ is reduced to the Darboux-Goursat problem for the system (2.4) in D_ε. Note also, that if we consider this problem in D_0 , then the coefficients $C_i, D_i (i = 1, 2)$ are singular at the point $(1, 1)$.

To investigate the smoothness or the singularities of solutions at the original problem P_α on $\Sigma_{2,0}$, we are looking for classical solutions for the system (2.4) not only in the domain D_ε, but also in the domain

$$
D_\varepsilon^{(1)} := \{(\xi, \eta) : 0 < \xi < \eta < 1, 0 < \xi < 1 - \varepsilon\}, \quad \varepsilon > 0,
$$

where $D_\varepsilon \subset D_\varepsilon^{(1)}$. Thus we come to the following question.

Problem $P_{\alpha,2}$. Find solutions $(U^{(1)}, U^{(2)})(\xi, \eta)$ of system (2.4) in $D_\varepsilon^{(1)}$, which satisfy the boundary conditions

$$
U^{(i)}(0, \eta) = 0, (U_\eta^{(i)} - U_\xi^{(i)})(\xi, \xi) + \alpha(1 - \xi)U^{(i)}(\xi, \xi) = 0, \tag{2.7}
$$

$i = 1, 2$, $\xi \in (0, 1 - \varepsilon)$, $\eta \in (0, 1)$.

3. A system of integral equations for problem $P_{\alpha,2}$

We consider a point $(\xi_0, \eta_0) \in D_\varepsilon^{(1)}$ and rectangle R, triangle T defined by

$$
\begin{gathered}
R := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi_0 < \eta < \eta_0\},\\
T := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi < \eta < \xi_0\}.
\end{gathered}
$$

By use of Green's theorem in

$$
\begin{aligned}
I_R^{(i)} &:= \iint_R U_{\xi\eta}^{(i)}(\xi, \eta)\, d\xi d\eta = \int_0^{\xi_0} \Big(\int_{\xi_0}^{\eta_0} U_{\xi\eta}^{(i)}(\xi, \eta)\, d\eta\Big)\, d\xi,\\
I_T^{(i)} &:= \iint_T U_{\xi\eta}^{(i)}(\xi, \eta)\, d\xi d\eta = \int_0^{\xi_0} \Big(\int_{\xi}^{\xi_0} U_{\xi\eta}^{(i)}(\xi, \eta)\, d\eta\Big)\, d\xi,
\end{aligned}
\tag{3.1}
$$

$i = 1, 2$, and the boundary conditions (2.7) we obtain

$$
I_R^{(i)} + 2I_T^{(i)} = U^{(i)}(\xi_0, \eta_0) - \int_0^{\xi_0} \alpha(1 - \xi)U^{(i)}(\xi, \xi)\, d\xi. \tag{3.2}
$$

We set $p^{(i)} := U_\xi^{(i)}$, $q^{(i)} := U_\eta^{(i)}$ and define (see (2.4))

$$
\begin{aligned}
E^{(1)}(\xi, \eta) &:= [F^1 + A_1p^{(1)} + B_1q^{(1)} + C_1U^{(1)} + D_1U^{(2)}](\xi, \eta),\\
E^{(2)}(\xi, \eta) &:= [F^2 + A_2p^{(2)} + B_2q^{(2)} + C_2U^{(2)} + D_2U^{(1)}](\xi, \eta).
\end{aligned}
\tag{3.3}
$$

Using (3.1) - (3.3) and (2.4) we obtain six integral equations ($i = 1, 2$)

$$
\begin{aligned}
U^{(i)}(\xi_0, \eta_0) = &\int_0^{\xi_0} \Big(\int_{\xi_0}^{\eta_0} E^{(i)}(\xi, \eta)\, d\eta\Big)d\xi + 2\int_0^{\xi_0} \Big(\int_0^{\eta} E^{(i)}(\xi, \eta)\, d\xi\Big)d\eta\\
&+ \int_0^{\xi_0} \alpha(1 - \xi)U^{(i)}(\xi, \xi)d\xi,
\end{aligned}
\tag{3.4}
$$

$$
p^{(i)}(\xi_0, \eta_0) = \int_0^{\xi_0} E^{(i)}(\xi, \xi_0)d\xi + \int_{\xi_0}^{\eta_0} E^{(i)}(\xi_0, \eta)d\eta + \alpha(1 - \xi_0)U^{(i)}(\xi_0, \xi_0), \tag{3.5}
$$

$$q^{(i)}(\xi_0,\eta_0)=\int_0^{\xi_0} E^{(i)}(\xi,\eta_0)d\xi \tag{3.6}$$

The system (3.4)–(3.6) is equivalent to the system (2.4) with the boundary conditions (2.7).

Remark 3.1. We recall that in Section 2 the index n in system (2.1) was omitted. We see that in (2.4) the coefficients C_i, D_i $(i=1,2)$ depend on n, where on the right-hand side we have

$$F^{(i)}(\xi,\eta)=\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\frac{1}{2}} f_n^{(i)}(\varrho(\xi,\eta),t(\xi,\eta)).$$

Therefore for fixed $n\in\mathbb{N}$ solutions $(U^{(1)},U^{(2)})$ of the integral equation system (3.4) - (3.6) depend on n and will be later marked by $(U_n^{(1)},U_n^{(2)})$, which gives functions $(u_n^{(1)},u_n^{(2)})$ by relation $\varrho^{\frac{1}{2}}u^{(i)}(\varrho,t)=U_n^{(i)}(\xi,\eta)$ (see (2.2)).

Furthermore we observe that classical solutions $(U_n^{(1)},U_n^{(2)})\in C^1(\bar{D}_\varepsilon^{(1)})$, $U_{n,\xi\eta}^{(i)}\in C(\bar{D}_\varepsilon^{(1)})$ of the integral equation system define functions $(u_n^{(1)},u_n^{(2)})$ which are generalized solutions of Problem $P_{\alpha,1}$ in $\bar{G}_0\backslash(0,0)$.

4. Solutions of the system and first upper estimates

We define in $D_\varepsilon^{(1)}$ functions $(U_m^{(i)},p_m^{(i)},q_m^{(i)})$, $i=1,2$, $m\in\mathbb{N}$, by the formulas

$$\begin{gathered}
U_{m+1}^{(i)}(\xi_0,\eta_0)=\int_0^{\xi_0}\Big(\int_{\xi_0}^{\eta_0}E_m^{(i)}(\xi,\eta)\,d\eta\Big)\,d\xi+2\int_0^{\xi_0}\Big(\int_0^{\eta}E_m^{(i)}(\xi,\eta)\,d\xi\Big)\,d\eta\\
+\int_0^{\xi_0}\alpha(1-\xi)U_m^{(i)}(\xi,\xi)\,d\xi,\quad i=1,2;\quad m=0,1,2\dots\\
p_{m+1}^{(i)}(\xi_0,\eta_0)=\int_0^{\xi_0}E_m^{(i)}(\xi,\xi_0)\,d\xi+\int_{\xi_0}^{\eta_0}E_m^{(i)}(\xi_0,\eta)\,d\eta\\
+\alpha(1-\xi_0)U_m^{(i)}(\xi_0,\xi_0),\quad i=1,2;\quad m=0,1,2\dots\\
q_{m+1}^{(i)}(\xi_0,\eta_0)=\int_0^{\xi_0}E_m^{(i)}(\xi,\eta_0)\,d\xi,\quad i=1,2;\quad m=0,1,2\dots\\
U_0^{(i)}(\xi_0,\eta_0)=0,\quad p_0^{(i)}(\xi_0,\eta_0)=0, q_0^{(i)}(\xi_0,\eta_0)=0,\quad i=1,2,
\end{gathered}\tag{4.1}$$

in $D_\varepsilon^{(1)}$, where

$$\begin{gathered}
E_m^{(1)}(\xi,\eta):=[F^1+A_1p_m^{(1)}+B_1q_m^{(1)}+C_1U_m^{(1)}+D_1U_m^{(2)}](\xi,\eta),\\
E_m^{(2)}(\xi,\eta):=[F^2+A_2p_m^{(2)}+B_2q_m^{(2)}+C_2U_m^{(2)}+D_2U_m^{(1)}](\xi,\eta).
\end{gathered}\tag{4.2}$$

Now we formulate some results from Hristov et al [15] which we use later.

Lemma 4.1 ([15])**.** *Let for $(\xi_0,\eta_0)\in D_\varepsilon^{(1)}=\{(\xi,\eta):0<\xi<\eta<1,0<\xi<1-\varepsilon\},\varepsilon>0$, and $\mu\in\mathbb{R}_+$ define*

$$I_\mu:=\int_0^{\xi_0}\Big(\int_{\xi_0}^{\eta_0}(2-\xi-\eta)^{-\mu-2}d\eta\Big)d\xi+2\int_0^{\xi_0}\Big(\int_{\xi}^{\xi_0}(2-\xi-\eta)^{-\mu-2}d\eta\Big)d\xi.$$

Then

$$I_\mu\le\frac{1}{\mu(\mu+1)}(2-\xi_0-\eta_0)^{-\mu}.$$

As we mentioned in the introduction, we treat in this paper the equation (1.1) in case that its coefficients are continuous in $\bar{\Omega}_0$, so we may set

$$\sup_{\bar{\Omega}_0}\{|b_1|,|b_2|,|b|\}\le K_1,\quad \sup_{\bar{\Omega}_0}|c|\le K_0,\quad \sup_{[0,1]}|\alpha(\varrho)|\le K_\alpha. \tag{4.3}$$

Then, from (2.6) we obtain the following bounds

$$\begin{gathered}
|a_1|\le 2K_1, |a_2|\le \frac{2K_1}{\rho}, |A_1|=|A_2|\le \frac{3K_1}{4},\\
|B_1|=|B_2|\le \frac{3K_1}{4}, |D_1|=|D_2|\le \frac{nK_1}{2\rho}=\frac{nK_1}{2-\xi-\eta},\\
|C_1|=|C_2|\le \frac{\nu(\nu+1)}{(2-\xi-\eta)^2}+\frac{K_1}{2(2-\xi-\eta)}+\frac{K_0}{4},
\end{gathered}$$

where $\nu:=n-\frac{1}{2}$. According to (4.2)

$$E_m^{(i)}(\xi,\eta):=[F^i+A_ip_m^{(i)}+B_iq_m^{(i)}+C_iU_m^{(i)}+D_iU_m^{(\gamma_i)}](\xi,\eta),$$

with $\gamma_1=2,\gamma_2=1$ and thus for $i=1,2$ we have

$$\begin{aligned}
&|(E_m^{(i)}-E_{m-1}^{(i)})(\xi,\eta)|\\
&\le\Big\{\frac{\nu(\nu+1)}{(2-\xi-\eta)^2}+\frac{K_1}{2(2-\xi-\eta)}+\frac{K_0}{4}\Big\}|U_m^{(i)}-U_{m-1}^{(i)}|\\
&\quad+\frac{(\nu+1/2)K_1}{2-\xi-\eta}|U_m^{(\gamma_i)}-U_{m-1}^{(\gamma_i)}|+\frac{3K_1}{4}|p_m^{(i)}-p_{m-1}^{(i)}|+\frac{3K_1}{4}|q_m^{(i)}-q_{m-1}^{(i)}|.
\end{aligned}\tag{4.4}$$

Lemma 4.2 ([15]). *Let the conditions* (4.3) *be fulfilled and there exists a constant* $A>0$*, such that*

$$\begin{gathered}
|(U_m^{(i)}-U_{m-1}^{(i)})(\xi_0,\eta_0)|\le A(2-\xi_0-\eta_0)^{-\mu},\\
|(p_m^{(i)}-p_{m-1}^{(i)})(\xi_0,\eta_0)|\le \mu A(2-\xi_0-\eta_0)^{-\mu-1},\\
|(q_m^{(i)}-q_{m-1}^{(i)})(\xi_0,\eta_0)|\le \mu A(2-\xi_0-\eta_0)^{-\mu-1},
\end{gathered}$$

where $\mu\in\mathbb{R}_+,\mu>\nu=n-1/2,m\in\mathbb{N}$. *If the parameter* δ_ν *is such, that*

$$(\mu-\nu)(\mu+\nu+1)\ge\delta_\nu\mu(\mu+1)+(3\mu+2\nu+2)K_1+2(\mu+1)K_\alpha+K_0, \tag{4.5}$$

then for $m\in\mathbb{N}$, $i=1,2$ *we have*

$$\begin{gathered}
|(U_{m+1}^{(i)}-U_m^{(i)})(\xi_0,\eta_0)|\le A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu},\\
|(p_{m+1}^{(i)}-p_m^{(i)})(\xi_0,\eta_0)|\le \mu A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu-1},\\
|(q_{m+1}^{(i)}-q_m^{(i)})(\xi_0,\eta_0)|\le \mu A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu-1}.
\end{gathered}$$

Lemma 4.3 ([15]). *Let now* $\nu=n-1/2,n\in\mathbb{N}$ *be fixed. If the parameter* μ *is large enough,* $\mu>\nu$*, then*

$$(\mu-\nu)(\mu+\nu+1)-[(3\mu+2\nu+2)K_1+2(\mu+1)K_\alpha+K_0]>0 \tag{4.6}$$

and we can choose the parameter $\delta_\nu>0$*, such that the condition* (4.5) *to be fulfilled.*

In [15] the integral equation system (3.4)–(3.6) is solved by the successive approximations method and the following important theorem is proved.

Theorem 4.4 ([15]). *Let* $n\in\mathbb{N}$ *be fixed. Assume:*

(i) *$a_1 = b_1 cos\varphi + b_2 sin\varphi, a_2 = \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi), b, c$ are functions of (ϱ, t), $\alpha = \alpha(\rho)$;*
(ii) *$b_1, b_2, b, c \in C(\bar{\Omega}_0)$, $\alpha(\varrho) \in C^1([0,1])$, $f_n^{(i)} \in C(\bar{\Omega}_0)$, $i = 1, 2$;*
(iii) *the parameter $\mu = \mu_n$ is such large, that*

$$(\mu - \nu)(\mu + \nu + 1) > (3\mu + 2\nu + 2)K_1 + 2(\mu + 1)K_\alpha + K_0$$

(see Lemma 4.3).

Then there exists a classical solution $(U_n^{(1)}, U_n^{(2)}) \in C^1(\bar{D}_\varepsilon^{(1)})$, $U_{n,\xi_0\eta_0}^{(i)} \in C(\bar{D}_\varepsilon^{(1)})$ of Problem $P_{\alpha,2}$ and the following estimates hold:

$$\begin{gathered} |U_n^{(i)}(\xi, \eta)| \le A_\mu \delta_\nu^{-1} (2 - \xi - \eta)^{-\mu}, \\ |U_{n,\xi}^{(i)}(\xi, \eta)| \le \mu A_\mu \delta_\nu^{-1} (2 - \xi - \eta)^{-\mu-1}, \\ |U_{n,\eta}^{(i)}(\xi, \eta)| \le \mu A_\mu \delta_\nu^{-1} (2 - \xi - \eta)^{-\mu-1}, \end{gathered} \tag{4.7}$$

where

$$A_\mu := \frac{1}{\mu(\mu+1)} \max_{\bar{G}_0} |\frac{1}{4\sqrt{2}} (2\varrho)^{\mu + \frac{5}{2}} f_n^{(i)}(\varrho, t)|,$$

$$\delta_\nu := \frac{1}{\mu(\mu+1)} \{(\mu - \nu)(\mu + \nu + 1) - [(3\mu + 2\nu + 2)K_1 + 2(\mu + 1)K_\alpha + K_0]\}$$

After the inverse transformation to Problem P_α (using the relation (2.2)), we see that the first estimate of (4.7) is equivalent to (1.8). Next, we aim to refine this result.

5. New (exact) upper estimates

Theorem 5.1. *Let $n \in \mathbb{N}$ be fixed and the conditions (i) and (ii) from Theorem 4.4 be fulfilled. Then for each number $\sigma > 0$ there exists a positive constant C_σ, such that the inequalities*

$$\begin{gathered} |U_n^{(i)}(\xi, \eta)| \le C_\sigma \max_{\bar{D}_0^{(1)}} |F^{(i)}| (2 - \xi - \eta)^{-\nu - \sigma}, \\ |U_{n,\xi}^{(i)}(\xi, \eta)| \le (\nu + \sigma) C_\sigma \max_{\bar{D}_0^{(1)}} |F^{(i)}| (2 - \xi - \eta)^{-\nu - \sigma - 1}, \\ |U_{n,\eta}^{(i)}(\xi, \eta)| \le (\nu + \sigma) C_\sigma \max_{\bar{D}_0^{(1)}} |F^{(i)}| (2 - \xi - \eta)^{-\nu - \sigma - 1} \end{gathered} \tag{5.1}$$

hold in $\bar{D}_\varepsilon^{(1)}$, $i = 1, 2$. $C_\sigma > 0$ depends on the numbers ν, σ, K_1, K_0 and K_α.

Proof. Let us choose and fix some $\mu > \nu$ satisfying Lemma 4.3. Next, we choose and fix an arbitrary positive number σ, such that $\nu + \sigma < \mu$. Further, we choose $\delta_\nu \in (0, 1)$ satisfying the condition (4.5) from Lemma 4.2. From Lemma 4.3 we see that it is possible. Now we introduce the positive number

$$\tau := \max\{(1 - \delta_\nu), \theta\} < 1, \tag{5.2}$$

where

$$\theta := \frac{\nu(\nu+1)}{(\nu + \sigma/2)(\nu + \sigma/2 + 1)}.$$

For shortness in the further calculations, we denote

$$N(K_1, K_0, K_\alpha) := \frac{(5\nu + 3\sigma + 2)K_1 + K_0 + 2(\nu + \sigma + 1/2)K_\alpha + 1}{(\nu + \sigma - 1/2)(\nu + \sigma + 1/2)}.$$

Note that $N(K_1, K_0, K_\alpha) > 0$ and
$$\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)} < \theta.$$
Next, we divide $D_\varepsilon^{(1)}$ by the line
$$2-\xi-\eta = \frac{1}{N(K_1,K_0,K_\alpha)^2}\left(\theta - \frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}\right)^2$$
and obtain two parts:
$$\begin{aligned} D1 := \Big\{&(\xi,\eta) : 0<\xi<\eta<1, \quad 0<\xi<1-\varepsilon,\\ &(2-\xi-\eta)^{1/2} > \frac{1}{N(K_1,K_0,K_\alpha)}\Big(\theta - \frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}\Big)\Big\}, \end{aligned}$$
and
$$\begin{aligned} D2 := \Big\{&(\xi,\eta) : 0<\xi<\eta<1, \quad 0<\xi<1-\varepsilon,\\ &(2-\xi-\eta)^{1/2} \le \frac{1}{N(K_1,K_0,K_\alpha)}\Big(\theta - \frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}\Big)\Big\}. \end{aligned}$$
It is possible that $D1 = \emptyset$ or $D2 = \emptyset$.

Finally, for $\lambda > 0$ we denote
$$A_\lambda := \frac{1}{\lambda(\lambda+1)} \max_{(\xi_0,\eta_0)\in \bar{D}_0^{(1)}} |(2-\xi_0-\eta_0)^{\lambda+2} F^{(i)}(\xi_0,\eta_0)| \tag{5.3}$$
and
$$C_1 = \max\Big\{A_{\nu+\sigma},\, \frac{\mu}{\nu+\sigma} A_\mu \max_{\overline{D1}} (2-\xi_0-\eta_0)^{-\mu+\nu+\sigma}\Big\} \le C_{\mu,\sigma} \max_{\bar{D}_0^{(1)}} |F^{(i)}|, \tag{5.4}$$
where $C_{\mu,\sigma} > 0$ do not depend on $F^{(i)}$. If $D1 = \emptyset$ we set $\max_{\overline{D1}}(\dots) = 1$. Now, we are ready to prove Theorem 5.1 by induction.

(i) For $m = 0$:
$$\begin{gathered} U_{n,0}^{(i)}(\xi,\eta) = p_{n,0}^{(i)}(\xi,\eta) = q_{n,0}^{(i)}(\xi,\eta) \equiv 0 \text{ in } \bar{D}_\varepsilon^{(1)},\\ E_{n,0}^{(i)}(\xi,\eta) = F_n^{(i)}(\xi,\eta). \end{gathered}$$

(ii) For $m = 1$:
$$\begin{aligned} &(U_{n,1}^{(i)} - U_{n,0}^{(i)})(\xi_0,\eta_0)\\ &= \int_0^{\xi_0} \Big(\int_{\xi_0}^{\eta_0} E_{n,0}^{(i)}(\xi,\eta)\,d\eta\Big)\,d\xi + 2\int_0^{\xi_0} \Big(\int_0^{\eta} E_{n,0}^{(i)}(\xi,\eta)\,d\xi\Big)\,d\eta\\ &= \int_0^{\xi_0} \Big(\int_{\xi_0}^{\eta_0} (2-\xi-\eta)^{-\lambda-2}(2-\xi-\eta)^{\lambda+2} F^{(i)}(\xi,\eta)\,d\eta\Big)\,d\xi\\ &\quad + 2\int_0^{\xi_0} \Big(\int_0^{\eta} (2-\xi-\eta)^{-\lambda-2}(2-\xi-\eta)^{\lambda+2} F^{(i)}(\xi,\eta)\,d\xi\Big)\,d\eta. \end{aligned}$$
Applying Lemma 4.1 and recalling (5.3), we obtain
$$|(U_{n,1}^{(i)} - U_{n,0}^{(i)})(\xi_0,\eta_0)| \le A_\lambda (2-\xi_0-\eta_0)^{-\lambda}. \tag{5.5}$$

Likewise we have

$$(p_{n,1}^{(i)} - p_{n,0}^{(i)})(\xi_0, \eta_0) = \int_0^{\xi_0} F^{(i)}(\xi, \xi_0) d\xi + \int_{\xi_0}^{\eta_0} F^{(i)}(\xi_0, \eta) d\eta$$

and with integration

$$|(p_{n,1}^{(i)} - p_{n,0}^{(i)})(\xi_0, \eta_0)| \leq \lambda A_\lambda (2 - \xi_0 - \eta_0)^{-\lambda-1}, \tag{5.6}$$

respectively

$$|(q_{n,1}^{(i)} - q_{n,0}^{(i)})(\xi_0, \eta_0)| \leq \lambda A_\lambda (2 - \xi_0 - \eta_0)^{-\lambda-1}. \tag{5.7}$$

For $\lambda = \nu + \sigma$ we have

$$\begin{gathered} |(U_{n,1}^{(i)} - U_{n,0}^{(i)})(\xi_0, \eta_0)| \leq A_{\nu+\sigma} (2 - \xi_0 - \eta_0)^{-\nu-\sigma}, \\ |(p_{n,1}^{(i)} - p_{n,0}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) A_{\nu+\sigma} (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1}, \\ |(q_{n,1}^{(i)} - q_{n,0}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) A_{\nu+\sigma} (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1}. \end{gathered} \tag{5.8}$$

(iii) For $m = 2, 3, \dots$ Now with Lemma 4.2, the inequalities (5.5)–(5.7) for $\lambda = \mu$ and induction, there exist sequences $\{U_{n,m}^{(i)}\}$, $\{p_{n,m}^{(i)}\}$ and $\{q_{n,m}^{(i)}\}$, $m \in \mathbb{N}$, of continuous functions and the estimates

$$\begin{gathered} |(U_{n,m+1}^{(i)} - U_{n,m}^{(i)})(\xi_0, \eta_0)| \leq A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\mu}, \\ |(p_{n,m+1}^{(i)} - p_{n,m}^{(i)})(\xi_0, \eta_0)| \leq \mu A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\mu-1}, \\ |(q_{n,m+1}^{(i)} - q_{n,m}^{(i)})(\xi_0, \eta_0)| \leq \mu A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\mu-1} \end{gathered} \tag{5.9}$$

hold for $m = 0, 1, 2, \dots$.

For the points $(\xi_0, \eta_0) \in D1$ from (5.9) we obtain:

$$\begin{aligned} &|(U_{n,m+1}^{(i)} - U_{n,m}^{(i)})(\xi_0, \eta_0)| \\ &\leq A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma} \max_{\bar{D}1} (2 - \xi_0 - \eta_0)^{-\mu+\nu+\sigma}, \\ &|(p_{n,m+1}^{(i)} - p_{n,m}^{(i)})(\xi_0, \eta_0)| \\ &\leq (\nu + \sigma) \frac{\mu}{\nu + \sigma} A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1} \max_{\bar{D}1} (2 - \xi_0 - \eta_0)^{-\mu+\nu+\sigma}, \\ &|(q_{n,m+1}^{(i)} - q_{n,m}^{(i)})(\xi_0, \eta_0)| \\ &\leq (\nu + \sigma) \frac{\mu}{\nu + \sigma} A_\mu (1 - \delta_\nu)^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1} \max_{\bar{D}1} (2 - \xi_0 - \eta_0)^{-\mu+\nu+\sigma}. \end{aligned}$$

Thus using (5.2) and (5.4) in $D1$ for $m \in N$ we obtain

$$\begin{gathered} |(U_{n,m+1}^{(i)} - U_{n,m}^{(i)})(\xi_0, \eta_0)| \leq C_1 \tau^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma}, \\ |(p_{n,m+1}^{(i)} - p_{n,m}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) C_1 \tau^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1}, \\ |(q_{n,m+1}^{(i)} - q_{n,m}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) C_1 \tau^m (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1}. \end{gathered} \tag{5.10}$$

For $(\xi_0, \eta_0) \in D2$ we will show that such estimates hold too.

Our induction hypothesis is that for some $m \in \mathbb{N}$ is true

$$\begin{gathered} |(U_{n,m}^{(i)} - U_{n,m-1}^{(i)})(\xi_0, \eta_0)| \leq C_1 \tau^{m-1} (2 - \xi_0 - \eta_0)^{-\nu-\sigma}, \\ |(p_{n,m}^{(i)} - p_{n,m-1}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) C_1 \tau^{m-1} (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1}, \\ |(q_{n,m}^{(i)} - q_{n,m-1}^{(i)})(\xi_0, \eta_0)| \leq (\nu + \sigma) C_1 \tau^{m-1} (2 - \xi_0 - \eta_0)^{-\nu-\sigma-1} \end{gathered} \tag{5.11}$$

in $D_\varepsilon^{(1)}$, which for $m=1$ is fulfilled according to (5.8) and $C_1 \geq A_{\nu+\sigma}$. Now, we are trying to approve (5.10) in $D_\varepsilon^{(1)}$, which is already known in $D1$.

By setting the inequalities (5.11) in (4.4) we derive

$$
\begin{aligned}
&|(E_m^{(i)} - E_{m-1}^{(i)})|(\xi,\eta)\\
&\leq \Big\{\frac{\nu(\nu+1)}{(2-\xi-\eta)^2} + \frac{K_1}{2(2-\xi-\eta)} + \frac{K_0}{4}\Big\} C_1\tau^{m-1}(2-\xi-\eta)^{-\nu-\sigma}\\
&\quad + \frac{(\nu+1/2)K_1}{2-\xi-\eta} C_1\tau^{m-1}(2-\xi-\eta)^{-\nu-\sigma}\\
&\quad + \frac{3K_1}{2}(\nu+\sigma)C_1\tau^{m-1}(2-\xi-\eta)^{-\nu-\sigma-1}\\
&\leq C_1\tau^{m-1}\Big\{\nu(\nu+1)(2-\xi-\eta)^{-\nu-\sigma-2} + (5\nu+3\sigma+2)K_1(2-\xi-\eta)^{-\nu-\sigma-3/2}\\
&\quad + K_0(2-\xi-\eta)^{-\nu-\sigma-3/2}\Big\}
\end{aligned}
$$

everywhere in $D_\varepsilon^{(1)}$. Now we are ready to apply Lemma 4.1 for all the terms in the brackets, since they are of power less than -2. We substitute the last inequality in the formulas (4.1) and with integration and Lemma 4.1 we obtain:

$$
\begin{aligned}
&|(U_{n,m+1}^{(i)} - U_{n,m}^{(i)})(\xi_0,\eta_0)|\\
&\leq C_1\tau^{m-1}\Big\{\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}(2-\xi_0-\eta_0)^{-\nu-\sigma}\\
&\quad + \frac{(5\nu+3\sigma+2)K_1+K_0}{(\nu+\sigma-1/2)(\nu+\sigma+1/2)}(2-\xi_0-\eta_0)^{-\nu-\sigma+1/2}\Big\}\\
&\quad + K_\alpha\int_0^{\xi_0} |(U_{n,m}^{(i)} - U_{n,m-1}^{(i)})(\xi,\xi)|d\xi\\
&\leq C_1\tau^{m-1}(2-\xi_0-\eta_0)^{-\nu-\sigma}\Big\{\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}\\
&\quad + (2-\xi_0-\eta_0)^{1/2}N(K_1,K_0,K_\alpha)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&|(p_{n,m+1}^{(i)} - p_{n,m}^{(i)})(\xi_0,\eta_0)|\\
&\leq (\nu+\sigma)C_1\tau^{m-1}\Big\{\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}(2-\xi_0-\eta_0)^{-\nu-\sigma-1}\\
&\quad + \frac{(5\nu+3\sigma+2)K_1+K_0}{(\nu+\sigma-1/2)(\nu+\sigma+1/2)}(2-\xi_0-\eta_0)^{-\nu-\sigma-1/2}\Big\}\\
&\quad + 2K_\alpha C_1\tau^{m-1}(2-2\xi_0)^{-\nu-\sigma-1/2}\\
&\leq (\nu+\sigma)C_1\tau^{m-1}(2-\xi_0-\eta_0)^{-\nu-\sigma-1}\Big\{\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}\\
&\quad + (2-\xi_0-\eta_0)^{1/2}N(K_1,K_0,K_\alpha)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&|(q_{n,m+1}^{(i)} - q_{n,m}^{(i)})(\xi_0,\eta_0)|\\
&\leq (\nu+\sigma)C_1\tau^{m-1}(2-\xi_0-\eta_0)^{-\nu-\sigma-1}\Big\{\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)}
\end{aligned}
$$

$$+ (2-\xi_0-\eta_0)^{1/2} N(K_1, K_0, K_\alpha)\Big\}$$

in $D_\varepsilon^{(1)}$. Since

$$\frac{\nu(\nu+1)}{(\nu+\sigma)(\nu+\sigma+1)} + (2-\xi_0-\eta_0)^{1/2} N(K_1, K_0, K_\alpha) \le \theta \le \tau \quad \text{in } D2$$

by definition, for the points $(\xi_0, \eta_0) \in D2$ from the last three inequalities we obtain (5.10). Then by induction we conclude that the estimates (5.10) hold in $D_\varepsilon^{(1)}$ for $m = 2, 3, \dots$.

The functions $\{U_{n,m}^{(i)}, p_{n,m}^{(i)}, q_{n,m}^{(i)}\}_{m=0}^{\infty}$ belong to $C(\bar{D}_\varepsilon^{(1)})$ and we have uniform convergence to some functions $\{U_n^{(i)}, p_n^{(i)}, q_n^{(i)}\} \in C(\bar{D}_\varepsilon^{(1)})$, as $m \to \infty$ and

$$\begin{aligned}
|U_n^{(i)}(\xi_0,\eta_0)| &= \Big|\sum_{m=0}^{\infty} (U_{n,m+1}^{(i)} - U_{n,m}^{(i)})(\xi_0,\eta_0)\Big| \\
&\le C_1(1-\tau)^{-1}(2-\xi_0-\eta_0)^{-\nu-\sigma}, \\
|U_{n,\xi_0}^{(i)}(\xi_0,\eta_0)| &= \Big|\sum_{m=0}^{\infty} (p_{n,m+1}^{(i)} - p_{n,m}^{(i)})(\xi_0,\eta_0)\Big| \\
&\le (\nu+\sigma) C_1(1-\tau)^{-1}(2-\xi_0-\eta_0)^{-\nu-\sigma-1}, \\
|U_{n,\eta_0}^{(i)}(\xi_0,\eta_0)| &= \Big|\sum_{m=0}^{\infty} (q_{n,m+1}^{(i)} - q_{n,m}^{(i)})(\xi_0,\eta_0)\Big| \\
&\le (\nu+\sigma) C_1(1-\tau)^{-1}(2-\xi_0-\eta_0)^{-\nu-\sigma-1}.
\end{aligned}$$

In view of (5.4), these estimates coincide with (5.1) with $C_\sigma = C_{\mu,\sigma}(1-\tau)^{-1}$. □

Proof of Theorem 1.4. First, we note that the conditions (i) and (ii) of Theorem 4.4 are fulfilled, hence we can apply Theorem 5.1. Using the relations (2.2) and (2.3), we make the inverse transformation from Problem $P_{\alpha,2}$ to Problem P_α and we see that the generalized solution $u(x,t)$ belongs to $C^1(\bar{\Omega}_0 \backslash O)$ and the estimates

$$\begin{gathered}
|u(x,t)| \le C_{n,\sigma} \max_{\bar{\Omega}_0} \{|f_n^{(1)}| + |f_n^{(2)}|\} |x|^{-n-\sigma}, \\
\sum_{|\beta|=1} |D^\beta u(x,t)| \le n C_{n,\sigma} \max_{\bar{\Omega}_0} \{|f_n^{(1)}| + |f_n^{(2)}|\} |x|^{-n-\sigma-1}
\end{gathered}$$

hold, where $C_{n,\sigma} > 0$ depends on n, σ and all coefficients of (1.1). □

It is easy to generalize this result in the following way.

Theorem 5.2. *Let the right-hand function $f(\varrho,\varphi,t)$ of (1.10) be a trigonometric polynomial*

$$f = \sum_{n=0}^{l} f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi, \quad l \in \mathbb{N}. \tag{5.12}$$

If conditions (i) and (ii) of Theorem 4.4 are fulfilled, then there exists one and only one generalized solution $u(x,t) \in C^1(\bar{\Omega}_0 \backslash O)$ of Problem P_α and the a priori estimates

$$|u(x,t)| \le C_{l,\sigma} \max_{\bar{\Omega}_0} \{|f_l^{(1)}| + |f_l^{(2)}|\} |x|^{-l-\sigma} + O(|x|^{-l-\sigma+1}),$$

$$\sum_{|\beta|=1} |D^\beta u(x,t)| \le C_{l,\sigma} \max_{\bar\Omega_0} \{|f_l^{(1)}| + |f_l^{(2)}|\}|x|^{-l-\sigma-1} + O(|x|^{-l-\sigma})$$

hold.

6. On the singularity of solutions of problem $P_{\alpha,2}$

In this section we derive some sufficient conditions on the coefficients and the right-hand side of (1.1) for the existence of singular solutions of the problem we treat. We follow Grammatikopoulos et al [12] (see Theorem 1.3) and making some modifications we extend this result.

First, we represent an important lemma

Lemma 6.1 ([12])**.** *Consider Problem $P_{\alpha,2}$. Let $F^i, A_i, B_i, C_i, D_i \in C(\bar D_\varepsilon^{(1)})$, $i = 1,2$,*

$$A_i \ge 0, \quad B_i \ge 0, \quad C_i \ge 0, \quad D_i \ge 0, \quad \alpha(1-\xi) \ge 0 \quad \text{in } \bar D_\varepsilon^{(1)}, \quad i = 1,2 \tag{6.1}$$

and

$$F^{(i)} \ge 0 \quad \text{in } \bar D_\varepsilon^{(1)}, \quad i = 1,2. \tag{6.2}$$

Then for the solution $(U^{(1)}, U^{(2)})$ of Problem $P_{\alpha,2}$ we have

$$U^{(i)}(\xi,\eta) \ge 0, \quad U_\eta^{(i)}(\xi,\eta) \ge 0, \quad U_\xi^{(i)}(\xi,\eta) \ge 0 \quad \text{for } (\xi,\eta) \in \bar D_\varepsilon^{(1)},\ i = 1,2. \tag{6.3}$$

Note that in view of $D_1 = -D_2$ (see (2.6)) for (6.1) to be fulfilled is necessary $D_1 = D_2 \equiv 0$, so in this case we may consider the system (2.4) as two independent single equations

$$U_{\xi\eta} - AU_\xi - BU_\eta - CU = F(\xi,\eta) \tag{6.4}$$

with boundary conditions

$$U(0,\eta) = 0, \quad (U_\eta - U_\xi)(\xi,\xi) + \alpha(1-\xi)U(\xi,\xi) = 0. \tag{6.5}$$

Next, we formulate the main result in this section.

Theorem 6.2. *Consider the problem* (6.4)*,* (6.5)*. Let for the coefficients we assume $A, B, C \in C(\bar D_\varepsilon^{(1)}), \alpha(1-\xi) \in C^1([0, 1-\varepsilon])$ and*

$$A \ge 0, \quad B \ge 0, \quad C \ge \frac{4n^2-1}{4(2-\xi-\eta)^2}, \quad \alpha(1-\xi) \ge 0 \text{ in } \bar D_\varepsilon^{(1)}. \tag{6.6}$$

Additionally, let $F(\xi,\eta) \in C(\bar D_\varepsilon^{(1)})$ does not change its sign (that means either $F \ge 0$ or $F \le 0$) and $F \not\equiv 0$ in $D_0^{(1)}$.

Then for $\eta \in (0,1]$ and $\varepsilon \in (0, \varepsilon_F)$, where $\varepsilon_F \in (0,1)$ is a number depending on F, holds

$$|U(1-\varepsilon,\eta)| \ge C_0 \varepsilon^{-(n-\frac12)}, \quad C_0 = const > 0. \tag{6.7}$$

Proof. We will consider the case $F \ge 0$. The case $F \le 0$ is obviously analogous. In [12] was shown the existence of classical solution $U(\xi,\eta)$ of the problem we treat.

We introduce a function

$$W(\xi,\eta) := \frac{(1-\xi)^{n-1/2}(1-\eta)^{n-1/2}}{(2-\xi-\eta)^{n-1/2}}.$$

We see that $W(\xi,\eta) > 0$ in $D_\varepsilon^{(1)}$. Next, since $F \not\equiv 0$ in $D_0^{(1)}$ and it is continuous in each $D_\varepsilon^{(1)}$, we conclude that there exists an open ball in $D_0^{(1)}$ where $F > 0$.

Therefore, if we consider ε small enough (smaller than some ε_F), we have the inequality

$$\int_{D_\varepsilon} (FW)(\xi,\eta)d\xi d\eta \geq K, \quad K = \text{const} > 0. \tag{6.8}$$

Recall that $D_\varepsilon = \{(\xi,\eta) : 0 < \xi < \eta < 1-\varepsilon\}$, $D_\varepsilon \subset D_\varepsilon^{(1)}$.

Using (6.4) we transform (6.8) in the following way:

$$\begin{aligned}
0 < K &\leq \int_{D_\varepsilon} (FW)(\xi,\eta)d\xi d\eta \\
&= \int_{D_\varepsilon} (U_{\xi\eta}W)(\xi,\eta)d\xi d\eta - \int_{D_\varepsilon} [(AU_\xi + BU_\eta)W](\xi,\eta)d\xi d\eta \\
&\quad - \int_{D_\varepsilon} (CUW)(\xi,\eta)d\xi d\eta := I_1 - I_2 - I_3.
\end{aligned}$$

Since Lemma 6.1 is fulfilled (consequently, $U \geq 0, U_\xi \geq 0, U_\eta \geq 0$) and $W \geq 0$, we see that $I_2 \geq 0$ and we may neglect this term:

$$0 < K \leq I_1 - I_3. \tag{6.9}$$

Taking into account the first boundary condition from (6.5) and integrating by parts we compute:

$$\begin{aligned}
I_1 &= \int_{D_\varepsilon} (U_{\xi\eta}W)(\xi,\eta)d\xi d\eta \\
&= \int_{D_\varepsilon} (UW_{\xi\eta})(\xi,\eta)d\xi d\eta - \int_0^{1-\varepsilon} (U_\xi W + UW_\eta)(\xi,\xi)d\xi \\
&\quad + \int_0^{1-\varepsilon} (U_\xi W)(\xi, 1-\varepsilon)d\xi := I_{D_\varepsilon} - I_{\partial 1} + I_{\partial 2}.
\end{aligned}$$

Next, we calculate

$$W_{\xi\eta}(\xi,\eta) = \frac{4n^2-1}{4(2-\xi-\eta)^2} W(\xi,\eta).$$

From here and (6.6) it follows that

$$I_{D_\varepsilon} - I_3 = \int_{D_\varepsilon} \Big(\frac{4n^2-1}{4(2-\xi-\eta)^2} - C\Big)(UW)(\xi,\eta)d\xi d\eta \leq 0.$$

Using this conclusion, from (6.9) we derive

$$0 < K \leq I_1 - I_3 = I_{D_\varepsilon} - I_{\partial 1} + I_{\partial 2} - I_3 \leq -I_{\partial 1} + I_{\partial 2}.$$

A calculation shows that

$$W_\eta(\xi,\xi) = \frac{1}{2}[W(\xi,\xi)]_\xi. \tag{6.10}$$

On the other hand, using the second boundary condition from (6.5) we compute

$$U_\xi(\xi,\xi) = \frac{1}{2}[U(\xi,\xi)]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi). \tag{6.11}$$

Then substituting (6.10) and (6.11) in the expression for $I_{\partial 1}$ gives

$$\begin{aligned}
I_{\partial 1} &= \int_0^{1-\varepsilon} (U_\xi W + UW_\eta)(\xi,\xi)d\xi \\
&= \int_0^{1-\varepsilon} \Big\{\frac{1}{2}[U(\xi,\xi)]_\xi W(\xi,\xi) + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi)W(\xi,\xi)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}U(\xi,\xi)[W(\xi,\xi)]_\xi\Big\}d\xi\\
&=\frac{1}{2}\int_0^{1-\varepsilon}[UW(\xi,\xi)]_\xi d\xi+\frac{1}{2}\int_0^{1-\varepsilon}\alpha(1-\xi)(UW)(\xi,\xi)d\xi\\
&=\frac{1}{2}(UW)(1-\varepsilon,1-\varepsilon)+\frac{1}{2}\int_0^{1-\varepsilon}\alpha(1-\xi)(UW)(\xi,\xi)d\xi\ge 0,
\end{aligned}
$$

where in the last inequality we use the sign of α from (6.6). Thus (6.9) becomes

$$0<K\le I_{\partial 2}. \tag{6.12}$$

It is easy to check that $W_\xi\le 0$ in $\bar{D}_\varepsilon$ and we can estimate $I_{\partial 2}$,

$$
\begin{aligned}
I_{\partial 2}&=\int_0^{1-\varepsilon}(U_\xi W)(\xi,1-\varepsilon)d\xi\\
&=-\int_0^{1-\varepsilon}(UW_\xi)(\xi,1-\varepsilon)d\xi+(UW)(1-\varepsilon,1-\varepsilon)\\
&\le U(1-\varepsilon,1-\varepsilon)\int_0^{1-\varepsilon}|W_\xi(\xi,1-\varepsilon)|d\xi\\
&=U(1-\varepsilon,1-\varepsilon)W(0,1-\varepsilon)\\
&=U(1-\varepsilon,1-\varepsilon)(1+\varepsilon)^{-(n-\frac{1}{2})}\varepsilon^{n-\frac{1}{2}}.
\end{aligned}
$$

We set this estimate in (6.12) and conclude that

$$U(1-\varepsilon,1-\varepsilon)\ge(1+\varepsilon)^{n-\frac{1}{2}}K\varepsilon^{-(n-\frac{1}{2})},\quad \varepsilon\in(0,\varepsilon_F).$$

Recalling once again that Lemma 6.1 implies $U_\eta\ge 0$ in $\bar{D}_\varepsilon^{(1)}$ we see that $U(1-\varepsilon,\eta)\ge U(1-\varepsilon,1-\varepsilon)$ in $\bar{D}_\varepsilon^{(1)}$ for $\eta\in(0,1]$.

From this fact and the last estimate immediately follows the assertion of this theorem. $\square$

Proof of Theorem 1.6. First, we transform Problem P_α to Problem $P_{\alpha,1}$ and in view of the relations

$$a_1=b_1\cos\varphi+b_2\sin\varphi,\quad a_2=\varrho^{-1}(b_2\cos\varphi-b_1\sin\varphi)$$

we see that the following relations hold

$$a_2\equiv 0,\quad a_1\ge|b|,\quad a_1\ge 2\varrho c,\quad \alpha(\varrho)\ge 0\quad\text{in}\quad G_\varepsilon$$

for the coefficients of system (2.1). Next, we reduce Problem $P_{\alpha,1}$ to Problem $P_{\alpha,2}$ and recalling the relations

$$
\begin{gathered}
A_1=A_2=\frac{1}{4}(a_1+b),\quad B_1=B_2=\frac{1}{4}(a_1-b),\\
D_2=-D_1=\frac{1}{4}na_2,\quad C_1=C_2=\frac{1}{4}\Big\{\frac{4n^2-1}{(2-\xi-\eta)^2}+\frac{a_1}{2-\xi-\eta}-c\Big\},
\end{gathered}
$$

we see that $D_1=D_2\equiv 0$ and the inequalities (6.6) hold. Furthermore, it is easy to check that the remaining conditions of Theorem 6.2 are also fulfilled, hence for $\eta\in(0,1]$ and $\varepsilon\in(0,\varepsilon_F)$ the estimate (6.7) holds. From this, making the inverse transformation from Problem $P_{\alpha,2}$ to Problem P_α, we obtain the estimate (1.13) in some neighborhood of $O(0,0,0)$. $\square$

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