Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 152, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE POSITIVE SOLUTIONS FOR SECOND-ORDER THREE-POINT BOUNDARY-VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITIES

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Abstract. In this article, we study the second-order three-point boundary-value problem

$$u''(t) + a(t)u'(t) + f(t, u) = 0, \quad 0 \le t \le 1,$$
$$u'(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $0 < \alpha$, $\eta < 1$, $a \in C([0, 1], (-\infty, 0))$ and f is allowed to change sign. We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem.

1. Introduction

The study of multi-point boundary-value problems for linear second-order ordinary differential equations was initiated by Kiguradze and Lomtatidze [11], Lomtatidze [15], Il'in and Moviseev [9, 10], Agarwal and Kiguradze [1], Lomtatidze and Malaguti [16]. Motivated by the study of [9, 10], Gupta [6] studied certain three-point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors. We refer the reader to Gupta [7], Li, Liu and Jia [13], Liu [14], Ma[17, 18] for some references along this line. Using results from fixed point theory, such as the fixed-point theorems by Banach, Krasnosel'skii, Leggett-Williams etc., in studying second-order dynamic systems is a standard and useful tool (see, e.g. [2, 3, 4]).

Recently, Ma [17] studied the three-point boundary-value problem (BVP)

$$u''(t) + a(t)f(u) = 0, \quad 0 \le t \le 1,$$
$$u(0) = 0, \quad u(1) = \alpha u(\eta),$$

where $0 < \eta < 1$, α is a positive constant, $a \in C[0, 1]$, $f \in C([0, +\infty), [0, +\infty))$ and there exists $x_0 \in (0, 1)$ such that $a(x_0) > 0$. Author got the existence and multiplicity of positive solutions theorems under the condition that f is either superlinear or sublinear by using Krasnoselskii's fixed point theorem.

2000 *Mathematics Subject Classification.* 34B15, 34B25.
Key words and phrases. Multiple positive solutions; sign changing; fixed-point theorem.

Submitted March 14, 2012. Published September 7, 2012.

In 2001, Ma[18] considered m-point boundary-value problem

$$
\begin{gathered}
u''(t)+h(t)f(u)=0, \quad 0\le t\le 1,\\
u(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\beta_i u(\xi_i),
\end{gathered}
$$

where $\beta_i>0$ $(i=1,2,\dots,m-2)$, $0<\xi_1<\xi_2<\dots<\xi_{m-2}<1$, $h\in C([0,1],[0,+\infty))$ and $f\in C([0,+\infty),[0,+\infty))$. Author established the existence of positive solutions under the condition that f is either superlinear or sublinear.

In [13], the authors studied the three-point boundary-value problem

$$
\begin{gathered}
u''(t)+a(t)u'(t)+\lambda f(t,u)=0, \quad 0\le t\le 1,\\
u'(0)=0, \quad u(1)=\alpha u(\eta),
\end{gathered}
$$

where $0<\eta<1$, α is a positive constant, $a\in C([0,1],(-\infty,0))$, $f\in C([0,1]\times\mathbb{R}^+,\mathbb{R})$ and there exists $M>0$ such that $f(t,u)\ge -M$ for $(t,u)\in[0,1]\times\mathbb{R}^+$. They obtained the existence of one positive solution by using Krasnoselskii's fixed point theorem.

Motivated by the results mentioned above, in this paper, we study the existence of positive solutions of three-point boundary-value problem

$$
\begin{gathered}
u''(t)+a(t)u'(t)+f(t,u)=0, \quad 0\le t\le 1,\\
u'(0)=0, \quad u(1)=\alpha u(\eta),
\end{gathered}
\tag{1.1}
$$

where $0<\alpha$, $\eta<1$, $a\in C([0,1],(-\infty,0))$ and f is allowed to change sign. We show that there exist two positive solutions by using Leggett-Williams fixed-point theorem. Our ideas are similar those used in [13], but a little different. By applying Leggett-Williams fixed-point theorem, we get the new results, which are different from the previous results and the conditions are easy to be checked. In particular, we do not need that f be either superlinear or sublinear which was required in [7, 13, 17, 18].

In the rest of the paper, we make the following assumptions

- (H1) $0<\alpha$, $\eta<1$;
- (H2) $a\in C([0,1],(-\infty,0))$;
- (H3) $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}$ is continuous and there exists $M>0$ such that $f(t,u)\ge -M$ for $(t,u)\in[0,1]\times\mathbb{R}^+$.

By a positive solution of (1.1), we understand a function u which is positive on $(0,1)$ and satisfies the differential equations as well as the boundary conditions in (1.1).

2. Preliminaries

In this section, we give some definitions and lemmas.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $P\subset E$ is said to be a cone provided that

- (i) $u\in P$ and $a\ge 0$ imply $au\in P$;
- (ii) $u,\ -u\in P$ implies $u=0$;
- (ii) $u,v\in P$ implies $u+v\in P$.

Definition 2.2. Given a cone P in a real Banach space E, an operator $\psi : P \to P$ is said to be increasing on P, provided $\psi(x) \le \psi(y)$, for all $x, y \in P$ with $x \le y$.

A functional $\alpha : P \to [0, \infty)$ is said to be nonnegative continuous concave on P, provided $\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$, for all $x, y \in P$ with $t \in [0, 1]$.

Let $a, b, r > 0$ be constants with P and α as defined above, we note

$$P_r = \{y \in P : \|y\| < r\}, \quad P\{\alpha, a, b\} = \{y \in P : \alpha(y) \ge a,\ \|y\| \le b\}.$$

The main tool of this paper is the following well known Leggett-Williams fixed-point theorem.

Theorem 2.3 ([8, 12]). *Assume E be a real Banach space, $P \subset E$ be a cone. Let $A : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(y) \le \|y\|$, for $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \le c$ such that*

(i) $\{y \in P(\alpha,\ b,\ d) |\ \ \alpha(y) > b\} \neq \emptyset$ and $\alpha(Ay) > b$, for all $y \in P(\alpha,\ b,\ d)$;

(ii) *$\|Ay\| < a$, for all $\|y\| \le a$;*

(iii) *$\alpha(Ay) > b$ for all $y \in P(\alpha, b, c)$ with $\|Ay\| > d$.*

Then A has at least three fixed points y_1, y_2, y_3 satisfying

$$\|y_1\| < a, \quad b < \alpha(y_2), \quad \|y_3\| > a, \quad \alpha(y_3) < b.$$

Lemma 2.4 ([13]). *Assume that* (H1), (H2) *hold. Then for any $y \in C[0, 1]$ the BVP*

$$\begin{gathered} u''(t) + a(t)u'(t) + y(t) = 0, \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered} \tag{2.1}$$

has unique solution

$$\begin{aligned} u(t) = &-\int_0^t \Big(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau\Big)ds + \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau\Big)ds \\ &- \frac{\alpha}{1-\alpha} \int_0^\eta \Big(\frac{1}{p(s)} \int_0^s p(\tau)y(\tau)d\tau\Big)ds, \end{aligned} \tag{2.2}$$

where $p(t) = \exp\big(\int_0^t a(\tau)d\tau\big)$.

Lemma 2.5 ([13]). *Assume that* (H1), (H2) *hold. Let $y \in C[0, 1]$ and $y(t) \ge 0$ for all $t \in [0, 1]$, then the unique solution of* (2.1) *satisfies $u(t) \ge 0$.*

Lemma 2.6 ([13]). *Let* (H1), (H2) *hold. If $y \in C[0, 1]$ and $y(t) \ge 0$ for all $t \in [0, 1]$, the unique solution of* (2.1) *satisfies*

$$\min_{t\in[0,1]} u(t) \ge \gamma\|u\|,$$

where $\gamma = \frac{\alpha(1-\eta)}{1-\alpha\eta}$.

Lemma 2.7 ([13]). *Let ω be the unique solution of the initial-value problem*

$$\begin{gathered} u''(t) + a(t)u'(t) + 1 = 0, \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = \alpha u(\eta). \end{gathered} \tag{2.3}$$

Then $\omega(t) \le \Gamma\gamma$ for $t \in [0, 1]$, where $\gamma = (\alpha(1-\eta))/(1-\alpha\eta)$ and

$$\Gamma = \Big(\int_0^\eta (\frac{1}{p(s)} \int_0^s p(\tau)d\tau)ds + \frac{1}{1-\alpha} \int_\eta^1 (\frac{1}{p(s)} \int_0^s p(\tau)d\tau)ds\Big)\frac{1}{\gamma}.$$

3. Main Results

For convenience, we let $\gamma = \frac{\alpha(1-\eta)}{1-\alpha\eta}$,

$$l = \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau)d\tau\Big)ds, \quad h = \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau)d\tau\Big)ds.$$

Let $E = C[0,1]$, then E is Banach space, with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. We define a cone in E by

$$P = \{u \in E : u \geq 0, \min_{t\in[0,1]} u(t) \geq \gamma\|u\|\}.$$

Our main results are the following theorems.

Theorem 3.1. *Suppose conditions* (H1)–(H3) *hold and there exist positive constants e, b, c, N with $M\Gamma < e < e + M\Gamma\gamma < b < \gamma^2 c$, $\frac{1}{\gamma} < N < \frac{cl}{bh}$ such that*

(A1) $f(t,u) < \frac{e}{h} - M$ *for* $t \in [0,1]$, $0 \leq u \leq e$;
(A2) $f(t,u) \geq \frac{b}{l}N - M$ *for* $t \in [0,1]$, $b - M\Gamma\gamma \leq u \leq \frac{b}{\gamma^2}$;
(A3) $f(t,u) \leq \frac{c}{h} - M$ *for* $t \in [0,1]$, $0 \leq u \leq c$,

where the number Γ is defined in Lemma 2.7. Then (1.1) *has at least two positive solutions.*

Proof. Let ω be a solution of (2.3) and $z = M\omega$. By Lemma 2.7 we have $z(t) = M\omega(t) \leq M\Gamma\gamma < e\gamma$. It is easy to see that (1.1) has a positive solution u if and only if $u + z = \overline{u}$ is a solution of the boundary-value problem

$$\begin{gathered} u''(t) + a(t)u'(t) = -g(t, u - z), \quad 0 \leq t \leq 1, \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered} \tag{3.1}$$

and $\overline{u} > z$ for $t \in (0,1)$, where $g : [0,1] \times R \to [0, +\infty)$ is defined by

$$g(t,y) = \begin{cases} f(t,y) + M, & (t,y) \in [0,1] \times [0,+\infty), \\ f(t,0) + M, & (t,y) \in [0,1] \times (-\infty, 0). \end{cases}$$

For $v \in P$, define the operator

$$\begin{aligned} Tv(t) = &- \int_0^t \Big(\frac{1}{p(s)} \int_0^s p(\tau)g(\tau, v(\tau) - z(\tau))d\tau\Big)ds \\ &+ \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau)g(\tau, v(\tau) - z(\tau))d\tau\Big)ds \\ &- \frac{\alpha}{1-\alpha} \int_0^\eta \Big(\frac{1}{p(s)} \int_0^s p(\tau)g(\tau, v(\tau) - z(\tau))d\tau\Big)ds, \end{aligned}$$

where p is defined in Lemma 2.4. By Lemmas 2.4, 2.5 and 2.6, we can check $T(P) \subseteq P$. It is easy to check T is completely continuous by Arzela-Ascoli theorem.

In the following, we show that all the conditions of Theorem 2.3 are satisfied. Firstly, we define the nonnegative, continuous concave functional $\alpha : P \to [0, \infty)$ by

$$\alpha(v) = \min_{t\in[0,1]} v(t),$$

Obviously, for every $v \in P$, $\alpha(v) \leq \|v\|$.

We first show that $T(\overline{P}_c) \subseteq \overline{P}_c$. Let $v \in \overline{P}_c$ and $t \in [0,1]$ be arbitrary. When $v(t) \geq z(t)$, we have $0 \leq v(t) - z(t) \leq v(t) \leq c$ and thus $g(t, v(t) - z(t)) = f(t, v(t) - z(t)) + M \geq 0$. By (A3) we have

$$g(t, v(t) - z(t)) \leq \frac{c}{h}.$$

When $v(t) < z(t)$, we have $v(t) - z(t) < 0$ and then $g(t, v(t) - z(t)) = f(t, 0) + M \geq 0$. Again by (A3) we have

$$g(t, v(t) - z(t)) \leq \frac{c}{h}.$$

Therefore, we have proved that, if $v \in \overline{P}_c$, then $g(t, v(t) - z(t)) \leq \frac{c}{h}$ for $t \in [0,1]$. Then,

$$\begin{aligned}
\|Tv\| = Tv(0) &= \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\quad - \frac{\alpha}{1-\alpha} \int_0^\eta \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\leq \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\leq \frac{c}{h} \frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) d\tau\Big) ds = c.
\end{aligned}$$

Thus $Tv \in \overline{P}_c$. Therefore, we have $T(\overline{P}_c) \subseteq \overline{P}_c$. Especially, if $v \in \overline{P}_e$, then assumption (A1) yields $g(t, v(t) - z(t)) \leq \frac{e}{h}$ for $t \in [0,1]$. So, we have $T : \overline{P}_e \to P_e$, i.e., the assumption (ii) of Theorem 2.3 holds.

To verify condition (i) of Theorem 2.3, let $v(t) = \frac{b}{\gamma^2}$, then $v \in P$, $\alpha(v) = b/\gamma^2 > b$. That is $\{v \in P(\alpha, b, \frac{b}{\gamma^2}) : \alpha(v) > b\} \neq \emptyset$. Moreover, if $v \in P(\alpha, b, \frac{b}{\gamma^2})$, then $\alpha(v) \geq b$, so $b \leq \|v\| \leq \frac{b}{\gamma^2}$. Thus, $0 < b - M\Gamma\gamma \leq v(t) - z(t) \leq v(t) \leq \frac{b}{\gamma^2}$, $t \in [0,1]$. From assumption (A2) we obtain $g(t, v(t) - z(t)) \geq \frac{b}{l} N$ for $t \in [0,1]$. By the definition of α and above-proved inclusion $T(P) \subseteq P$, we have

$$\begin{aligned}
\alpha(Tv) &= \min_{t \in [0,1]} Tv(t) \geq \gamma \|Tv\| = \gamma Tv(0) \\
&= \gamma \Big(\frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\quad - \frac{\alpha}{1-\alpha} \int_0^\eta \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds\Big) \\
&\geq \gamma \Big(\frac{1}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\quad - \frac{\alpha}{1-\alpha} \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds\Big) \\
&= \gamma \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) g(\tau, v(\tau) - z(\tau)) d\tau\Big) ds \\
&\geq \gamma \frac{b}{l} N \int_0^1 \Big(\frac{1}{p(s)} \int_0^s p(\tau) d\tau\Big) ds \\
&= \gamma N b > b.
\end{aligned}$$

Therefore, condition (i) of Theorem 2.3 is satisfied with $d = b/\gamma^2$.

Finally, we address condition (iii) of Theorem 2.3. For this we choose $v \in P(\alpha, b, c)$ with $\|Tv\| > b/\gamma^2$. Then from above-proved inclusion $T(P) \subseteq P$, we have

$$\alpha(Tv) = \min_{t\in[0,1]} Tv(t) \geq \gamma\|Tv\| \geq \frac{b}{\gamma} > b.$$

Hence, condition (iii) of Theorem 2.3 holds with $\|Tv\| > b/\gamma^2$.

To sum up, all the hypotheses of Theorem 2.3 are satisfied. Hence T has at least three positive fixed points v_1, v_2 and v_3 such that

$$\|v_1\| < e, \quad b < \alpha(v_2), \quad \|v_3\| > e, \quad \alpha(v_3) < b.$$

Further, $u_i = v_i - z$ $(i = 1, 2, 3)$ are solutions of (3.1). Moreover,

$$v_2(t) \geq \gamma\|v_2\| \geq \gamma\alpha(v_2) > \gamma b > \gamma M\Gamma \geq z(t), \quad t \in [0,1],$$
$$v_3(t) \geq \gamma\|v_3\| > \gamma e > \gamma M\Gamma \geq z(t), \quad t \in [0,1].$$

So $u_2 = v_2 - z$, $u_3 = v_3 - z$ are two positive solutions of (1.1). This completes the proof. □

Theorem 3.2. *Suppose* (H1)–(H3) *hold, and there exist positive constants* a_i, b_i, N *with* $M\Gamma < a_i < a_i + M\Gamma\gamma < b_i < \gamma^2 a_{i+1}$, $\frac{1}{\gamma} < N < \frac{a_{i+1}l}{b_i h}$, $(i = 1, 2, \dots, n-1)$ *such that*

(A4) $f(t, u) < \frac{a_i}{h} - M$ *for* $t \in [0,1]$, $0 \leq u \leq a_i$ $(i = 1, 2, \dots, n)$*;*

(A5) $f(t, u) \geq \frac{b_i}{l} N - M$ *for* $t \in [0,1]$, $b_i - M\Gamma\gamma \leq u \leq \frac{b_i}{\gamma^2}$ $(i = 1, 2, \dots, n-1)$*.*

Then, (1.1) *has at least* $2(n-1)$ *positive solutions.*

Proof. When $n = 2$, the assumptions of Theorem 3.1 hold (with $c = a_2$), so we can get at least two positive solutions u_2 and u_3 such that $a_1 < u_2 \leq a_2$ and $u_3 \leq a_2$. Following the identical fashion, by the induction method we immediately complete the proof. □

Our results are different from those in [13], in particular, the following condition that was used in [13], is not needed in this article

$$\lim_{u\to\infty} \frac{f(t,u)}{u} = +\infty \quad \text{uniformly on } [0,1].$$

Acknowledgments. The first author is supported by grants ZR2012AQ024 from the Natural Science Foundation of Shandong Province of China, and J10LA62 from the University Science and Technology Foundation of Shandong Provincial Education Department. The second author is supported by grant 10871116 from the National Natural Science Foundation of China, ZR2010AM005 from the Natural Science Foundation of Shandong Province of China, and 200804460001 from the Doctoral Program Foundation of Education Ministry of China.

References

[1] R. P. Agarwal, I. Kiguradze; *On multi-point boundary-value problems for linear ordinary differential equations with singularities,* J. Math. Anal. Appl. **297** (2004), 131-151.

[2] D. R. Anderson; *Solutions to second-order three-point problems on time scales,* J. Differ. Equations Appl. **8** (2002), 673-688.

[3] D. R. Anderson, C. Zhai; *Positive solutions to semi-positone second-order three-point problems on time scales,* Appl. Math. Comput. **215** (2000), 3713-3720.

[4] M. Bohner and S. Stević; *Linear perturbations of a nonoscillatory second-order dynamic equation,* J. Differ. Equations Appl. **15** (11-12) (2009), 1211-1221.

[5] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cone,* Academic Press, San Diego, 1988.
[6] C. P. Gupta; *Solvability of a three-point nonlinear boundary-value problem for a second order ordinary differential equation,* J. Math. Anal. Appl. **168** (1992), 540-551.
[7] C. P. Gupta; *A generalited multi-point boundary-value problem for second order ordinary differential equations,* Appl. Math. Comput. **89** (1998), 133-146.
[8] J. Henderson, H. B. Thompson; *Multiple symmetric positive solutions for second order boundary-value problems,* Proc. Amer. Math. Soc. **128** (2000), 2373-2379.
[9] V. A. Il'in, E. I. Moviseev; *Nonlocal boundary-value problem of the second kind for a Sturm-Liouville operator,* J. Differential Equations. **23** (1987), 979-987.
[10] V. A. Il'in, E. I. Moviseev; *Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects,* J. Differential Equations. **23** (1987), 803-810.
[11] I. T. Kiguradze, A. G. Lomtatidze; *On certain boundary-value problems for second order linear ordinary differential equations with singularities,* J. Math. Anal. Appl. **101** (1984), No. 2, 325-347.
[12] R. W. Leggett, R. L. Williams; *Multiple positive fixed points of nonlinear operators on ordered Banach spaces,* Indiana. Univ. Math. J. **28** (1979), 673-688.
[13] G. Li, X. Liu, M. Jia; *Positive solutions to a type of nonlinear three-point boundary-value problem with sign changing nonlinearities,* Comput. Math. Appl. **571** (2009), 348-355.
[14] B. Liu; *Positive solutions of a nonlinear three-point boundary-value problem,* Comput. Math. Appl. **44** (2002), 201-211.
[15] A. G. Lomtatidze; *Positive solutions of boundary-value problems for second-order ordinary differential equations with singular points,* Differ. Equ. **23** (1987), No. 10, 1146-1152.
[16] A. Lomtatidze, L. Malaguti; *On a nonlocal boundary-value problem for second order nonlinear singular differential equations,* Georgian Math. J. **7** (2000), No. 1, 133-154.
[17] R. Ma; *Positive solutions of a nonlinear three-point boundary-value problem,* Electron. J. Differential Equations. **1999**, 34 (1999), 1-8.
[18] R. Ma; *Positive solutions for a nonlinear m-point boundary-value problems,* Comput. Math. Appl. **42** (2001), 755-765.

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