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OPTIMAL CONTROL OF A MODIFIED SWIFT-HOHENBERG EQUATION

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Abstract. In this article, we present the optimal control for the modified Swift-Hohenberg equation, under certain boundary conditions, and show the existence of an optimal solution.

1. Introduction

This article concerns the 1-D modified Swift-Hohenberg equation that was proposed by Doelman et al [2]:

$$u_t + ku_{xxxx} + 2u_{xx} + au + b|u_x|^2 + u^3 = 0, \quad x \in \Omega,\ t \in (0,T). \tag{1.1}$$

On the basis of physical considerations, (1.1) is supplemented with the boundary value condition

$$u(x,t) = u_{xx}(x,t) = 0 \quad \text{for } x \in \partial\Omega, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where Ω is an open connected bounded domain in $\mathbb{R}$, k, a and b are arbitrary constants. $u_0(x)$ is a given function from a suitable phase space.

The Swift-Hohenberg equation is one of the universal equations used in the description of pattern formation in spatially extended dissipative systems, (see [15]), which arise in the study of convective hydrodynamics [16], plasma confinement in toroidal devices [5], viscous film flow and bifurcating solutions of the Navier-Stokes [12]. Note that, the usual Swift-Hohenberg equation [16] is recovered for $b = 0$. The additional term $b|\nabla u|^2$, reminiscent of the Kuramoto-Sivashinsky equation, which arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition, (see [4, 9, 14]), breaks the symmetry $u \to -u$.

During the past years, many authors have paid much attention to the Swift-Hohenberg equation (see, e.g. [6, 8, 16]). However, only a few people dovoted to the modified Swift-Hohenberg equation. It were A. Doelman et al.[2] who first studied the modified Swift-Hohenberg equation for a pattern formation system with two unbounded spatial directions that is near the onset to instability. M. Polat[9]

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also considered the modified Swift-Hohenberg equation. In his paper, the existence of a global attractor is proved for the modified Swift-Hohenberg equation as (1.1)-(1.3). Recently, L. Song et al.[15] studied the long time behavior for modified Swift-Hohenberg equation in H^k $(k \geq 0)$ space. By using an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of global attractor, they proved that problem (1.1)-(1.3) possesses a global attractor in Sobolev space H^k for all $k \geq 0$, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

The optimal control plays an important role in modern control theories, and has a wider application in modern engineering. Two methods are used for studying control problems in PDE: one is using a low model method, and then changing to an ODE model [3]; the other is using a quasi-optimal control method [1]. No matter which one is chosen, it is necessary to prove the existence of optimal solution and establish the optimality system. Many papers have already been published to study the control problems of nonlinear parabolic equations. For example, Yong and Zheng[19], Tian et al.[17, 18], Ryu and Yagi [10, 11], Zhao and Liu[20] and so on.

This article concerns the distributed optimal control problem

$$\text{minimize} \quad J(u,w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|_{L^2(Q_0)}^2, \tag{1.4}$$

subject to

$$\begin{gathered} \frac{\partial u}{\partial t} + ku_{xxxx} + 2u_{xx} + au + b|u_x|^2 + u^3 = Bw, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = u_{xx}(x,t) = 0, \quad x \in \partial\Omega, \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{gathered} \tag{1.5}$$

The control target is to match the given desired state z_d in the L^2-sense by adjusting the body force w in a control volume $Q_0 \subseteq Q = (0,1) \times (0,T)$ in the L^2-sense.

Assume that $V = \{u \in H^2(0,1) \big| u(0,t) = u(1,t) = 0\}$, $U = H_0^1(0,1)$ and $H = L^2(0,1)$. Assume further that V', U' and H' are dual spaces of V, U and H. Then, we obtain

$$V \hookrightarrow U \hookrightarrow H = H' \hookrightarrow U' \hookrightarrow V'.$$

Each embedding being dense. The extension operator $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$ which is called the controller is introduced as

$$Bw = \begin{cases} w, & q \in Q_0, \\ 0, & w \in Q \setminus Q_0. \end{cases}$$

We supply H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and define a space $W(0,T;V)$ as

$$W(0,T;V) = \{y : y \in L^2(0,T;V),\ y_t \in L^2(0,T;V')\},$$

which is a Hilbert space endowed with common inner product.

This paper is organized as follows. In the next section, we prove the existence and uniqueness of weak solution to the equation in a special space. We also discuss the relation among the norms of weak solution, initial value and control item; In section 3, we consider the optimal control problem and prove the existence of optimal solution; Finally in Section 4, conclusions are obtained.

2. Existence and uniqueness of weak solutions

In this section, we prove the existence and uniqueness of weak solution for problem (1.5), where $x \in (0,1)$, $t \in [0,T]$, $Bw \in L^2(0,T;H)$ and a control $w \in L^2(Q_0)$. Now, we give the definition of the weak solution in the space $W(0,T;V)$.

Definition 2.1. For all $\eta \in V$, a function $u(x,t) \in W(0,T;V)$ is called a weak solution to problem (1.5), if

$$(\frac{\partial u}{\partial t}, \eta) + k(u_{xx}, \eta_{xx}) - 2(u_x, \eta_x) + a(u,\eta) + b(|u_x|^2, \eta) + (u^3, \eta) = (Bw, \eta). \quad (2.1)$$

We shall give a theorem on the existence and uniqueness of weak solution to problem (1.5).

Theorem 2.2. *Suppose that k is sufficiently large, $u_0 \in V$, $Bw \in L^2(0,T;H)$, then (1.5) admits a unique weak solution $u(x,t) \in W(0,T;V)$.*

Proof. Galerkin's method is applied for this proof. Denote $\mathbb{A} = (-\partial_x^2)^2$ as a differential operator, let $\{\psi_i\}_{i=1}^\infty$ denote the eigenfunctions of the operator $\mathbb{A} = (-\partial_x^2)^2$. For $n \in N$, define the discrete ansatz space by

$$V_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_n\} \subset V.$$

Let $u_n = \sum_{i=1}^n u_i^n(t)\psi_i(x)$ require $u_n(0,\cdot) \to u_0$ in H to hold true.

By analyzing the limiting behavior of sequences of smooth function $\{u_n\}$, we can prove the existence of a weak solution to the modified Swift-Hohenberg equation.

Performing the Galerkin metod for (1.5), we obtain

$$\begin{gathered}(x\frac{\partial u_n}{\partial t}, \eta) + k(u_{n,xx}, \eta_{xx}) - 2(u_{n,x}, \eta_x) + a(u_n, \eta) + b(|u_{n,x}|^2, \eta) + (u_n^3, \eta) \\ = (Bw, \eta), \quad \forall \eta \in V,\ (x,t) \in Q, \\ (u_n(x,0), \eta) = (u_0(x), \eta), \quad \forall \eta \in V,\ x \in (0,1)\end{gathered} \quad (2.2)$$

Then the equation of problem (2.2) is an ordinary differential equation and according to ODE theory, there exists a unique solution in the interval $[0, t_n)$. what we should do is to show that the solution is uniformly bounded when $t_n \to T$. We need also to show that the times t_n there are not decaying to 0 as $n \to \infty$.

Then, we shall prove the existence of solution in the following steps.

Step 1, multiplying the equation in (2.2) by u_n, integrating with respect to x on $(0,1)$, we deduce that

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|u_n\|^2 + k\|u_{n,xx}\|^2 + \|u_n\|_4^4 \\ \le |a|\|u_n\|^2 + 2\|u_{nx}\|^2 + |b|(|u_{nx}|^2, u_n) + (Bw, u_n).\end{gathered} \quad (2.3)$$

By Nirenberg's inequality,

$$\|u_{nx}\|_{8/3} \le c_0\|u_{nxx}\|^{1/2}\|u_n\|_4^{1/2}.$$

Then

$$|b|(|u_{nx}|^2, u_n) \le |b|\|u_{nx}\|_{8/3}^2\|u\|_4 \le c_0^2|b|\|u_{nxx}\|\|u_n\|_4^2 \le \|u_n\|_4^4 + \frac{c_0^4 b^2}{4}\|u_{nxx}\|^2.$$

On the other hand, we have

$$2\|u_{nx}\|^2 = -2(u_n, u_{nxx}) \le \|u_n\|^2 + \|u_{nxx}\|^2,$$

$$(Bw, u_n) \le \|Bw\|\|u_n\| \le \frac{1}{2}\|Bw\|^2 + \frac{1}{2}\|u_n\|^2.$$

Summing up, we have

$$\frac{d}{dt}\|u_n\|^2 + (2k - \frac{c_0^4 b^2}{2} - 2)\|u_{nxx}\|^2 \le (2|a|+3)\|u_n\|^2 + \|Bw\|^2,$$

where k satisfies $2k - \frac{c_0^4 b^2}{2} - 2 > 0$. Since $Bw \in L^2(0,T;H)$ is the control item, we can assume $\|Bw\| \le M$, where M is a positive constant. Then

$$\frac{d}{dt}\|u_n\|^2 + (2k - \frac{c_0^4 b^2}{2} - 2)\|u_{nxx}\|^2 \le (2|a|+3)\|u_n\|^2 + M^2. \tag{2.4}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned} \|u_n\|^2 &\le e^{(2|a|+3)t}\|u_{n,0}\|^2 + \frac{M^2}{2|a|+3} \\ &\le e^{(2|a|+3)T}\|u_{n,0}\|^2 + \frac{M^2}{2|a|+3} = c_1^2, \quad t \in [0,T]. \end{aligned} \tag{2.5}$$

Integrating (2.4) with on $[0,T]$,

$$\begin{aligned} &\int_0^T \|u_{n,xx}\|^2 dt \\ &\le \frac{2}{4k - c_0^4 b^2 - 4}\Big((2|a|+3)\int_0^T \|u_n\|^2 dt + M^2 T + \|u_{n,0}\|^2\Big) \\ &\le \frac{2}{4k - c_0^4 b^2 - 4}\big((2|a|+3)c_1^2 T + M^2 T + \|u_{n,0}\|^2\big) = c_2^2. \end{aligned} \tag{2.6}$$

Multiplying the equation in (2.2) by u_{nxx}, integrating with respect to x on $(0,1)$, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_{n,x}\|^2 + k\|u_{n,xxx}\|^2 \\ &= 2\|u_{nxx}\|^2 - a\|u_{nx}\|^2 + ((u_n)^3, u_{nxx}) + b(|u_{nx}|^2, u_{nxx}) - (Bw, u_{n,xx}). \end{aligned} \tag{2.7}$$

Noticing that

$$2\|u_{nxx}\|^2 = -2(u_{nx}, u_{nxxx}) \le \frac{k}{12}\|u_{nxxx}\|^2 + \frac{12}{k}\|u_{nx}\|^2,$$

and

$$\begin{aligned} -(Bw, u_{nxx}) &\le \|Bw\|\|u_{nxx}\| \le \frac{M^2}{2} + \frac{1}{2}\|u_{nxx}\|^2 \\ &\le \frac{M^2}{2} + \frac{1}{2}(\frac{k}{6}\|u_{nxxx}\|^2 + \frac{3}{2k}\|u_{nx}\|^2). \end{aligned}$$

By Nirenberg's inequality,

$$\|u_n\|_6 \le c_0\|u_{nxxx}\|^{1/9}\|u_n\|^{8/9}, \quad \|u_{nx}\|_4 \le c_0\|u_{nxxx}\|^{5/12}\|u_n\|^{7/12}.$$

Hence

$$\begin{aligned} ((u_n)^3, u_{nxx}) &\le 2\|u_{nxx}\|^2 + \frac{1}{8}\|u_n\|_6^6 \\ &\le \frac{k}{12}\|u_{nxxx}\|^2 + \frac{12}{k}\|u_{nx}\|^2 + \frac{k}{12}\|u_{nxxx}\|^2 + c(c_1) \\ &= \frac{12}{k}\|u_{nx}\|^2 + \frac{k}{6}\|u_{nxxx}\|^2 + c(c_1), \end{aligned}$$

and

$$\begin{aligned}|b|((u_{nx})^2, u_{nxx}) &= |b|\int_0^1 (u_{nx})^2 u_{nxx}dx \le \frac{|b|^2}{8}\|u_{nx}\|_4^4 + 2\|u_{nxx}\|^2\\ &\le \frac{k}{12}\|u_{nxxx}\|^2 + c(c_1) + \frac{k}{12}\|u_{nxxx}\|^2 + \frac{12}{k}\|u_{nx}\|^2\\ &= \frac{k}{6}\|u_{nxxx}\|^2 + c(c_1) + \frac{12}{k}\|u_{nx}\|^2.\end{aligned}$$

Summing up,

$$\frac{d}{dt}\|u_{nx}\|^2 + k\|u_{nxxx}\|^2 \le (\frac{72}{k} + 2|a| + \frac{3}{2k})\|u_{nx}\|^2 + 2c(c_1) + M^2.$$

Using Gronwall's inequality, we deduce that

$$\begin{aligned}\|u_{n,x}\|^2 &\le e^{(\frac{72}{k}+2|a|+\frac{3}{2k})t}\|u_{n,x}(0)\|^2 + \frac{2k(2c(c_1)+M^2)}{144+4k|a|+3}\\ &\le e^{(\frac{72}{k}+2|a|+\frac{3}{2k})T}\|u_{n,x}(0)\|^2 + \frac{2k(2c(c_1)+M^2)}{144+4k|a|+3} = c_3^2, \quad t\in[0,T].\end{aligned} \tag{2.8}$$

Then, by (2.5), (2.6) and (2.8), we obtain

$$\int_0^T \|u_n(x,t)\|_{H^2}^2 dt \le c.$$

Using Sobolev's embedding theorem, we also have

$$\|u_n\|_\infty \le c_4. \tag{2.9}$$

Step 2, we prove a uniform $L^2(0,T;V')$ bound on a sequence $\{u_{n,t}\}$. In order to obtain the result, we first establish the H^2-norm estimate for problem (2.2).

Multiplying the equation in (2.2) by u_{nxxxx}, integrating with respect to x on $(0,1)$, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_{n,xx}\|^2 + k\|u_{n,xxxx}\|^2\\ &= 2\|u_{nxxx}\|^2 - a\|u_{nxx}\|^2 - ((u_n)^3, u_{nxxxx}) - b(|u_{nx}|^2, u_{nxxxx}) + (Bw, u_{nxxxx}).\end{aligned} \tag{2.10}$$

By Nierenberg's inequality,

$$\|u_{nx}\|_4 \le c_0\|u_{nxxxx}\|^{1/12}\|u_{nx}\|^{11/12}.$$

Therefore,

$$b((u_{nx})^2, u_{nxxxx}) \le \frac{k}{10}\|u_{nxxxx}\|^2 + \frac{5|b|^2}{2k}\|u_{nx}\|_4^4 \le \frac{k}{5}\|u_{nxxxx}\|^2 + c(c_2).$$

On the other hand, we have

$$\begin{gathered}((u_n)^3, u_{nxxxx}) \le \sup_{x\in[0,1]}|u_n|^3\cdot\|u_{nxxxx}\|_1 \le \frac{k}{10}\|u_{nxxxx}\|^2 + c(c_4),\\ 2\|u_{nxxx}\|^2 = -2(u_{nxx}, u_{nxxxx}) \le \frac{k}{10}\|u_{nxxxx}\|^2 + \frac{10}{k}\|u_{nxx}\|^2,\\ (Bw, u_{nxxxx}) \le \|Bw\|\|u_{nxxxx}\| \le \frac{k}{10}\|u_{nxxxx}\|^2 + \frac{5M^2}{2k}.\end{gathered}$$

Summing up,

$$\frac{d}{dt}\|u_{nxx}\|^2 + k\|u_{nxxxx}\|^2 \le (\frac{20}{k} + 2|a|)\|u_{nxx}\|^2 + \frac{5M^2}{k} + 2c(c_2) + 2c(c_4).$$

Using Gronwall's inequality, we derive that

$$\begin{aligned} \|u_{nxx}\|^2 &\le e^{(\frac{20}{k}+2|a|)t}\|u_{nxx}(0)\|^2 + \frac{5M^2 + 2k(c(c_2) + c(c_4))}{20 + 2k|a|} \\ &\le e^{(\frac{20}{k}+2|a|)T}\|u_{nxx}(0)\|^2 + \frac{5M^2 + 2k(c(c_2) + c(c_4))}{20 + 2k|a|} \\ &= c_5^2, \quad \forall t \in [0, T]. \end{aligned} \tag{2.11}$$

It then follows from (2.5), (2.6) and (2.11) that

$$\|u_{nx}\|_\infty \le c_6. \tag{2.12}$$

Notice that

$$\begin{gathered} (u_{nxxxx}, \eta) = (u_{nxx}, \eta_{xx}) \le \|u_{nxx}\|\|\eta_{xx}\| \le \|u_{nxx}\|\|\eta\|_V, \\ (|u_{nx}|^2, \eta) \le \sup_{x\in[0,1]} |u_{nx}| \cdot (u_{nx}, \eta) \le c_6\|u_{nx}\|\|\eta\| \le c_6\|u_{nx}\|\|\eta\|_V, \\ ((u_n)^3, \eta) \le \sup_{x\in[0,1]} |u_n|^2 \cdot (u_n, \eta) \le c_4^2\|u_n\|\|\eta\| \le c_4^2\|u_n\|\|\eta\|_V, \\ (u_{nxx}, \eta) = (u_n, \eta_{xx}) \le \|u_n\|\|\eta_{xx}\| \le \|u_n\|\|\eta\|_V, \quad (u_n, \eta) \le \|u_n\|\|\eta\| \le \|u_n\|\|\eta\|_V, \end{gathered}$$

Therefore,

$$\begin{aligned} \|u_{nt}\|_{V'} &\le k\|u_{nxx}\| + 2\|u_n\| + |a|\|u_n\| + |b|c_6\|u_{nx}\| + c_4^2\|u_n\| + \|Bw\| \\ &\le (kc_5 + 2c_1 + |a|c_1 + |b|c_6c_3 + c_4^2c_1 + M). \end{aligned}$$

Hence,

$$\|u_{n,t}\|_{L^2(0,T;V)} \le (kc_5 + 2c_1 + |a|c_1 + |b|c_6c_3 + c_4^2c_1 + M)T = c_7. \tag{2.13}$$

Collecting the previous results, we obtain:

(1) For every $t \in [0, T]$, the sequence $\{u_n\}_{n\in N}$ is bounded in $L^2(0, T; V)$, which is independent of the dimension of ansatz space n.

(2) For every $t \in [0, T]$, the sequence $\{u_{n,t}\}_{n\in N}$ is bounded in $L^2(0, T; V')$, which is independent of the dimension of ansatz space n.

By the above discussion, we obtain $u(x, t) \in W(0, T; V)$. It is easy to check that $W(0, T; V)$ is compactly embedded into $C(0, T; H)$ which denote the space of continuous functions. We concludes convergence of a subsequences, again denoted by $\{u_n\}$ weak into $W(0, T; V)$, weak-star in $L^\infty(0, T; H)$ and strong in $L^2(0, T; H)$ to functions $u(x, t) \in W(0, T; V)$.

Since the proof of uniqueness is easy, we omit it. Then, Theorem 2.2 is proved. □

Now, we shall discuss the relation among the norm of the weak solution, the initial value, and the control item.

Theorem 2.3. *Suppose that k is sufficiently large, $u_0 \in V$, $Bw \in L^2(0, T; H)$, then there exists positive constants C_1 and C_2 such that*

$$\|u\|^2_{W(0,T;V)} \le C_1(\|u_0\|_V^2 + \|w\|^2_{L^2(Q_0)}) + C_2, \tag{2.14}$$

Proof. Clearly, (2.14) implies

$$\|u\|^2_{L^2(0,T;V)} + \|u_t\|^2_{L^2(0,T;V')} \le C_1(\|u_0\|^2_V + \|Bw\|^2_{L^2(H)}) + C_2. \tag{2.15}$$

Passing to the limit in (2.3), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + k\|u_{xx}\|^2 + \|u\|_4^4 \le |a|\|u\|^2 + 2\|u_x\|^2 + |b|(|u_x|^2, u) + (Bw, u). \tag{2.16}$$

Using the same method as in the proof of the above theorem, we derive that

$$\frac{d}{dt}\|u\|^2 + (2k - \frac{c_0^4 b^2}{2} - 2)\|u_{xx}\|^2 \le (2|a|+3)\|u\|^2 + \|Bw\|^2. \tag{2.17}$$

Then, by Gronwall's inequality,

$$\begin{aligned}\|u\|^2 &\le e^{(2|a|+3)t}\|u_0\|^2 + \frac{1}{2|a|+3}\|Bw\|^2 \\ &\le c_8\|u_0\|^2 + c_9\|Bw\|^2,\ \forall t \in [0,T].\end{aligned} \tag{2.18}$$

Therefore,

$$\|u\|^2_{L^2(0,T;H)} \le c_8 T\|u_0\|^2 + c_9\|Bw\|^2_{L^2(0,T;H)}. \tag{2.19}$$

Integrating (2.17) with respect to t on $[0,T]$, we obtain

$$\begin{aligned}&\|u(T)\|^2 - \|u_0\|^2 + (2k - \frac{c_0^4 b^2}{2} - 2)\|u_{xx}\|^2_{L^2(H)} \\ &\le \|Bw\|^2_{L^2(H)} + (2|a|+3)\|u\|^2_{L^2(H)}.\end{aligned}$$

By (2.19) and the above inequality,

$$\begin{aligned}&\|u_{xx}\|^2_{L^2(H)} \\ &\le \frac{2}{4k - c_0^4 b^2 - 4}\Big(\|Bw\|^2_{L^2(H)} + (2|a|+3)(c_8 T\|u_0\|^2 + c_9\|Bw\|^2_{L^2(H)}) + \|u_0\|^2\Big) \\ &\le c_{10}\|Bw\|^2_{L^2(H)} + c_{11}\|u_0\|^2.\end{aligned} \tag{2.20}$$

Passing to the limit in (2.7), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + k\|u_{xxx}\|^2 \\ &= 2\|u_{xx}\|^2 - a\|u_x\|^2 + ((u)^3, u_{xx}) + b(|u_x|^2, u_{xx}) - (Bw, u_{xx}).\end{aligned}$$

Using the same method as in the proof of the above theorem, we derive that

$$\frac{d}{dt}\|u_x\|^2 + k\|u_{xxx}\|^2 \le 2c(c_1) + \|Bw\|^2 + (\frac{72}{k} + 2|a| + \frac{3}{2k})\|u_x\|^2.$$

By Gronwall's inequality,

$$\begin{aligned}\|u_x\|^2 &\le e^{(\frac{72}{k}+2|a|+\frac{3}{2k})t}\|u_{x0}\|^2 + \frac{4kc(c_1)}{144+4k|a|+3} + \frac{2k}{144+4k|a|+3}\|Bw\|^2 \\ &\le c_{12}\|u_{x0}\|^2 + c_{13}\|Bw\|^2 + c_{14}.\end{aligned} \tag{2.21}$$

Therefore,

$$\|u\|_\infty \le c, \quad \|u_x\|^2_{L^2(H)} \le c_{12}T\|u_{x0}\|^2 + c_{13}\|Bw\|^2_{L^2(H)} + c_{14}T. \tag{2.22}$$

Adding (2.19), (2.20) and (2.22) gives

$$\|u\|^2_{L^2(0,T;V)} \le c_{15}(\|Bw\|^2_{L^2(0,T;H)} + \|u_0\|^2_U) + c_{16}. \tag{2.23}$$

On the other hand, passing to the limit in (2.10), a simple calculation shows that

$$\frac{d}{dt}\|u_{xx}\|^2 + k\|u_{xxxx}\|^2 \le (\frac{20}{k} + 2|a|)\|u_{xx}\|^2 + \frac{5}{k}\|Bw\|^2 + 2c(c_2) + 2c(c_4).$$

Using Gronwall's inequality,

$$\begin{aligned} \|u_{xx}\|^2 &\le e^{(\frac{20}{k}+2|a|)t}\|u_{xx0}\|^2 + \frac{5\|Bw\|^2}{20+2k|a|} + \frac{kc(c_2)+kc(c_4)}{10+|k|a|} \\ &\le c_{17}(\|Bw\|^2 + \|u_{xx0}\|^2) + c_{18}. \end{aligned} \tag{2.24}$$

It then follows from (2.18), (2.21) and (2.24) that

$$\|u_x(x,t)\| \le c.$$

On the other hand, we have

$$\begin{gathered} (u_{xxxx}, \eta) = (u_{xx}, \eta_{xx}) \le \|u_{xx}\|\|\eta_{xx}\| \le \|u_{xx}\|\|\eta\|_V, \\ (|u_x|^2, \eta) \le \sup_{x\in[0,1]} |u_x| \cdot (u_x, \eta) \le c\|u_x\|\|\eta\| \le c\|u_x\|\|\eta\|_V, \\ ((u)^3, \eta) \le \sup_{x\in[0,1]} |u|^2 \cdot (u, \eta) \le c^2\|u\|\|\eta\| \le c^2\|u\|\|\eta\|_V. \\ (u_{xx}, \eta) = (u, \eta_{xx}) \le \|u\|\|\eta_{xx}\| \le \|u\|\|\eta\|_V, \quad (u, \eta) \le \|u\|\|\eta\| \le \|u\|\|\eta\|_V, \end{gathered}$$

Therefore,

$$\begin{aligned} &\|u_t\|_{V'} \\ &\le k\|u_{xx}\| + 2\|u\| + |a|\|u\| + |b|c\|u_x\| + c^2\|u\| + \|Bw\| \\ &\le k(c_{17}(\|Bw\|^2 + \|u_{xx0}\|^2) + c_{18})^{1/2} + (2+|a|+c^2)(c_8\|u_0\|^2 + c_9\|Bw\|^2)^{1/2} \\ &\quad + |b|c(c_{12}\|u_{x0}\|^2 + c_{13}\|Bw\|^2 + c_{14})^{1/2} + \|Bw\| \end{aligned}$$

Hence,

$$\|u_{n,t}\|^2_{L^2(0,T;V)} \le c_{19}(\|u_0\|_V^2 + \|Bw\|^2) + c_{20}. \tag{2.25}$$

By (2.23), (2.25) and the definition of extension operator B, we obtain (2.15). Then, Theorem 2.3 is proved. □

3. Optimal control problem

In this section, we consider the optimal control problem associated with the fourth-order parabolic equation and prove the existence of optimal solution basing on J. L. Lions' theory (see [7]).

In the following, we suppose $L^2(Q_0)$ is a Hilbert space of control variables, we also suppose $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$ is the controller and a control $w \in L^2(Q_0)$, consider the following control system

$$\begin{gathered} \frac{\partial u}{\partial t} + ku_{xxxx} + 2u_{xx} + au + b|u_x|^2 + u^3 = Bw, \quad (x,t) \in (0,1)\times(0,T), \\ u(x,t) = u_{xx}(x,t) = 0, \quad x = 0,1, \\ u(x,0) = u_0(x), \quad x \in (0,1). \end{gathered} \tag{3.1}$$

Here, it is assumed that $u_0 \in V$. By Theorem 2.2, we can define the solution map $w \to u(w)$ of $L^2(Q_0)$ into $W(0,T;V)$. The solution u is called the state of the control system (3.1). The observation of the state is assumed to be given by Cu. Here $C \in \mathcal{L}(W(0,T;V), S)$ is an operator, which is called the observer, S is a real

Hilbert space of observations. The cost function associated with the control system (3.1) is given by

$$J(u,w)=\frac{1}{2}\|Cu-z_d\|_S^2+\frac{\delta}{2}\|w\|^2_{L^2(Q_0)}, \tag{3.2}$$

where $z_d\in S$ is a desired state and $\delta>0$ is fixed. An optimal control problem about problem (3.1) is

$$\text{minimize}\quad J(u,w). \tag{3.3}$$

Let $X=W(0,T;V)\times L^2(Q_0)$ and $Y=L^2(0,T;V)\times H$. We define an operator $e=e(e_1,e_2):X\to Y$, where

$$e_1=G=(\Delta^2)^{-1}(\frac{\partial u}{\partial t}+ku_{xxxx}+2u_{xx}+au+b|u_x|^2+u^3-Bw),$$
$$e_2=u(x,0)-u_0.$$

Here Δ^2 is an operator from V to V'. Then, we write (3.3) in the form

$$\text{minimize}\quad J(u,w)\quad \text{subject to } e(u,w)=0.$$

Theorem 3.1. *Suppose that k is sufficiently large, $u_0\in V$, $Bw\in L^2(0,T;H)$, then there exists an optimal control solution (u^*,w^*) to problem* (3.1).

Proof. Suppose (u,w) satisfy $e(u,w)=0$. In view of (3.2), we deduce that

$$J(u,w)\ge\frac{\delta}{2}\|w\|^2_{L^2(Q_0)}.$$

By Theorem 2.3, we obtain $\|u\|_{W(0,T;V)}\to\infty$ yields $\|w\|_{L^2(Q_0)}\to\infty$. Therefore,

$$J(u,w)\to\infty,\quad \text{when}\quad \|(u,w)\|_X\to\infty. \tag{3.4}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since, for all $(u,w)\in X$, $J(u,w)\ge 0$, there exists $\lambda\ge 0$ defined by

$$\lambda=\inf\{J(u,w)|(u,w)\in X,\ e(u,w)=0\},$$

which imlies the existence of a minimizing sequence $\{(u^n,w^n)\}_{n\in N}$ in X such that

$$\lambda=\lim_{n\to\infty}J(u^n,w^n)\quad\text{and}\quad e(u^n,w^n)=0,\quad \forall n\in\mathbb{N}.$$

From (3.4), there exists an element $(u^*,w^*)\in X$ such that when $n\to\infty$,

$$u^n\to u^*,\quad \text{weakly},\quad u\in W(0,T;V), \tag{3.5}$$
$$w^n\to w^*,\quad \text{weakly},\quad w\in L^2(Q_0). \tag{3.6}$$

Using (3.5), we obtain

$$\lim_{n\to\infty}\int_0^T(u_t^n(x,t)-u_t^*,\psi(t))_{V',V}dt=0,\quad \forall\psi\in L^2(0,T;V),$$
$$\lim_{n\to\infty}\int_0^T(u^n(x,t)-u^*,\psi(t))_{V',V}dt=0,\quad \forall\psi\in L^2(0,T;V),$$
$$\lim_{n\to\infty}\int_0^T(u_{xx}^n(x,t)-u_{xx}^*,\psi(t))_{V',V}dt=0,\quad \forall\psi\in L^2(0,T;V),$$

Since $W(0,T;V)$ is compactly embedded into $L^2(0,T;L^\infty)$, we have $u^n\to u^*$ strongly in $L^2(0,T;L^\infty)$. On the other hand, we know that $u_n\in L^\infty(0,T;V)$ and $u_{n,t}\in L^2(0,T;V^*)$. Hence by [13, Lemma 4] we have $u^n\to u^*$ strongly in $C(0,T;L^\infty)$, $u^n_x\to u^*_x$ strongly in $C(0,T;H)$, as $n\to\infty$.

As the sequence $\{u^n\}_{n\in\mathbb{N}}$ converges weakly, then $\|u^n\|_{W(0,T;V)}$ is bounded. And $\|u^n\|_{L^2(0,T;L^\infty)}$ is also bounded based on the embedding theorem.

Because $u_x^n \to u_x^*$ in $L^2(0,T;L^\infty)$ as $n\to\infty$, we know that $\|u_x^*\|_{L^2(0,T;L^\infty)}$ is bounded too.

By (3.5), we deduce that

$$
\begin{aligned}
\Big|\int_0^T\int_0^1 \big((u_x^n)^2-(u_x^*)^2\big)\eta\,dx\,dt\Big| &= \Big|\int_0^T\int_0^1 (u_x^n+u_x^*)(u_x^n-u_x^*)\eta\,dx\,dt\Big| \\
&\le \Big|\int_0^T \|u_x^n+u_x^*\|_{L^\infty}\|u_x^n-u_x^*\|_H\|\eta\|_H dt\Big| \\
&\le \|u_x^n+u_x^*\|_{L^2(L^\infty)}\|u_x^n-u_x^*\|_{C(H)}\|\eta\|_{L^2(H)} \\
&\to 0, \quad n\to\infty,\ \forall \eta\in L^2(0,T;H).
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_0^T\int_0^1 \big((u^n)^3-(u^*)^3\big)\,\eta\,dx\,dt\Big| \\
&\le \Big|\int_0^T\int_0^1 ((u^n)^2+u_nu^*+(u^*)^2)(u^n-u^*)\eta\,dx\,dt\Big| \\
&\le \Big|\int_0^T \|(u^n)^2+u_nu^*+(u^*)^2\|_{L^\infty}\|u^n-u^*\|_H\|\eta\|_H dt\Big| \\
&\le \|(u^n)^2+u_nu^*+(u^*)^2\|_{L^2(L^\infty)}\|u^n-u^*\|_{C(H)}\|\eta\|_{L^2(H)} \\
&\to 0, \quad n\to\infty,\ \forall\eta\in L^2(0,T;H).
\end{aligned}
$$

Using (3.6) again,

$$
\Big|\int_0^T\int_0^1 (Bw-Bw^*)\eta\,dx\,dt\Big| \to 0, \quad \text{as } n\to\infty,\ \forall\eta\in L^2(0,T;H).
$$

In view of the above discussions,

$$
e_1(u^*,w^*)=0, \quad \forall n\in N.
$$

Noticing that $u^*\in W(0,T;V)$, we derive that $u^*(0)\in H$. Since $u^n\to u^*$ weakly in $W(0,T;V)$, we can infer that $u^n(0)\to u^*(0)$ weakly as $n\to\infty$. Thus,

$$
(u^n(0)-u^*(0),\eta)\to 0, \quad \text{as } n\to\infty,\ \forall\eta\in H,
$$

which means $e_2(u^*,w^*)=0$. Therefore, we obtain

$$
e(u^*,w^*)=0, \quad \text{in } Y.
$$

So, there exists an optimal solution (u^*,w^*) to problem (3.1). Then, Theorem 3.1 is proved. □

4. Conclusions

The modified Swift-Hohenberg equation is an important mathematical physical model. Because of the complexity of nonlinear parts of the equation, there has been no research on the optimal control and boundary control of this equation. In this paper, we study the distributed optimal control problem for problem (1.1)-(1.3) using a series of mathematical estimates. Our research is motivated by the study of the optimal control problem for the viscous Degasperis-Procesi equation, viscous Camassa-Holm equation [17, 18], and the existence theory of optimal control of distributed parameter systems. We also prove the existence of an optimal solution to problem (1.1)-(1.3). In order to realize optimal solutions of optimal control problems in practice one must be able to recompute the optimal solutions in the presence of disturbances in real time unless one gives up optimality. We will use mathematical theory and related numerical methods to solve that problem numerically, which is our intention in the future.

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