Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 164, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INFINITELY MANY SOLUTIONS FOR A FOURTH-ORDER BOUNDARY-VALUE PROBLEM

SEYYED MOHSEN KHALKHALI, SHAPOUR HEIDARKHANI, ABDOLRAHMAN RAZANI

ABSTRACT. In this article we consider the existence of infinitely many solutions to the fourth-order boundary-value problem

$$u^{iv} + \alpha u'' + \beta(x)u = \lambda f(x,u) + h(u), \quad x \in]0,1[$$
$$u(0) = u(1) = 0,$$
$$u''(0) = u''(1) = 0\,.$$

Our approach is based on variational methods and critical point theory.

1. INTRODUCTION

The deformations of an elastic beam in equilibrium, whose two ends are simply supported, can be described by the nonlinear fourth-order boundary-value problem

$$u^{iv} = g(x,u,u',u''), \quad x \in]0,1[$$
$$u(0) = u(1) = 0,$$
$$u''(0) = u''(1) = 0,$$

where $g : [0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous [20, 21]. The importance of existence and multiplicity of solutions of this problem for physicists puts it and its variants at the center of attention of many works in mathematics. The fourth-order boundary-value problem

$$u^{iv} + \alpha u'' + \beta u = \lambda f(x,u), \quad x \in]0,1[$$
$$u(0) = u(1) = 0,$$
$$u''(0) = u''(1) = 0$$

where α, β are some real constants, is the subject of many recent researches by different approaches (See [32, 33, 35, 4, 6, 19]). In [32, 35] the authors by means of a version of Mountain-Pass Theorem of Rabinowitz [34, Theorem 9.12] obtain their results and in [33] by decomposition of operators shown by Chen, and in [6] by means of a Variational theorems of Ricceri and Bonanno, and in [19] by means of Morse Theory.

2000 *Mathematics Subject Classification.* 34B15.
Key words and phrases. Fourth-order equation; infinitely many solutions; critical point theory; variational methods.

Submitted June 28, 2012 Published September 22, 2012.
A. Razani was supported by grant 91470122 from IPM.

In this work, by employing Ricceri's Variational Principle [28, Theorem 2.5] and applying the similar methods used in [6], albeit with different calculations that it seems practically has significant difference with respect to [6], we ensure the existence of infinitely many solutions for

$$\begin{gathered} u^{iv} + \alpha u'' + \beta(x)u = \lambda f(x,u) + h(u), \quad x \in]0,1[\\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{gathered} \tag{1.1}$$

where α is a real constant, $\beta(x)$ is a continuous function on $[0,1]$ and λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^2-Carathéodory function and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function with the Lipschitz constant $L \geqslant 0$; i.e.,

$$|h(t_1) - h(t_2)| \leqslant L|t_1 - t_2| \tag{1.2}$$

for all $t_1, t_2 \in \mathbb{R}$, satisfying $h(0) = 0$.

To be precise, using Ricceri's Variational Principle [28] (see Theorem 1.2), under some appropriate hypotheses on the behavior of the potential of f, under some conditions on the potentials of h, at infinity, we establish the existence of a precise interval of parameters Λ such that, for each $\lambda \in \Lambda$, the problem (1.1) admits a sequence of weak solutions which are unbounded in the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$; see Theorem 3.1. Further, replacing the conditions at infinity of the potentials of f and h, by a similar one at zero, the same results hold and, in addition, the sequence of weak solutions uniformly converges to zero; see Theorem 3.5.

Existence of infinitely many solutions for boundary value problems using Ricceri's Variational Principle [28] and its variants has been widely investigated (see [8, 26]). We refer the reader to the papers [15, 17, 18, 22, 23, 29, 6], and [9]-[13]. We refer the reader also to[1, 3, 4, 5, 14, 25, 31] and their references, in which fourth-order boundary value problems have been studied.

Recall that a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be an L^2-*Carathéodory function*, if

- (C1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;
- (C2) the function $t \to f(x,t)$ is continuous for almost every $x \in [0,1]$;
- (C3) for every $\rho > 0$ there exists a function $\ell_\rho \in L^2([0,1])$ such that

$$\sup_{|t| \leqslant \rho} |f(x,t)| \leqslant \ell_\rho(x) \quad \text{for a.e. } x \in [0,1].$$

A special case of our main result is the following theorem.

Theorem 1.1. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function and denote by $F(x,\xi)$ its antiderivative with respect to its second argument at any $x \in [0,1]$ such that $F(x,0) = 0$. Assume that $\ell_\xi \in L^2([0,1])$ satisfies condition* (C3) *for every $\xi > 0$. Suppose $\pi^4 > |\alpha|\pi^2 + \|\beta\|_\infty + L$ and*

$$\liminf_{\xi \to +\infty} \frac{\|\ell_\xi\|_2}{\xi} = 0 \quad \text{and} \quad \limsup_{\xi \to +\infty} \frac{\int_a^b F(x,\xi)dx}{\xi^2} = +\infty$$

for some $[a,b] \subset]0,1[$ then, the problem

$$\begin{gathered} u^{iv} + \alpha u'' + \beta(x)u = f(x,u) + h(u), \quad x \in]0,1[\\ u(0) = u(1) = 0, \end{gathered}$$

$$u''(0) = u''(1) = 0$$

admits a sequence of pairwise distinct classical solutions.

Our main tool to investigate the existence of infinitely many solutions for the problem (1.1) is a smooth version of [8, Theorem 2.1] which is a more precise version of Ricceri's Variational Principle [28, Theorem 2.5], which we now recall.

Theorem 1.2. *Let X be a reflexive real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$ put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$ the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X;*

(b) *if $\gamma < +\infty$ then for every $\lambda \in]0, \frac{1}{\gamma}[$ either I_λ has a global minimum or there is a sequence $\{u_n\}$ of critical points (local minimum) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty;$$

(c) *if $\delta < +\infty$ then for every $\lambda \in]0, \frac{1}{\delta}[$ either there is a global minimum of Φ which is a local minimum of I_λ or there is a sequence of pairwise distinct critical points (local minimum) of I_λ which weakly converges to a global minimum of Φ.*

2. Preliminaries and basic lemmas

Hereafter, let $X = W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ with its usual norm inherited from $W^{2,2}([0,1])$ and $\|\cdot\|_2$ denotes the usual norm of $L^2([0,1])$; i.e.,

$$\|u\|_2 = \Big(\int_0^1 |u(x)|^2 dx \Big)^{1/2}.$$

Since $\beta(x)$ in (1.1), by assumption, is continuous on $[0,1]$, there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$\beta_1 \leqslant \beta(x) \leqslant \beta_2$$

for every $x \in [0,1]$. Therefore,

$$\begin{aligned} \int_0^1 |u''(x)|^2 - \alpha|u'(x)|^2 + \beta_1 |u(x)|^2 dx &\leqslant \int_0^1 |u''(x)|^2 - \alpha|u'(x)|^2 + \beta(x)|u(x)|^2 dx \\ &\leqslant \int_0^1 |u''(x)|^2 - \alpha|u'(x)|^2 + \beta_2 |u(x)|^2 dx. \end{aligned}$$

We need the following Poincaré type inequality.

Lemma 2.1 ([27, Lemma 2.3]). *For every $u \in X$*

$$\|u\|_2 \leqslant \frac{1}{\pi^2}\|u''\|_2. \tag{2.1}$$

From which we have as a consequence

$$\|u'\|_2 \leqslant \frac{1}{\pi}\|u''\|_2. \tag{2.2}$$

Now put

- $\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1$ when $\beta_2 \leqslant 0$ and $\alpha \geqslant 0$;
- $\sigma_1 := 1 + \frac{\beta_1}{\pi^4}$, $\sigma_2 := 1 - \frac{\alpha}{\pi^2}$ when $\beta_2 \leqslant 0$ and $\alpha < 0$;
- $\sigma_1 := 1 - \frac{\alpha}{\pi^2}$ and $\sigma_2 := 1 + \frac{\beta_2}{\pi^4}$ when $\beta_1 \geqslant 0$ and $\alpha \geqslant 0$;
- $\sigma_1 := 1$ and $\sigma_2 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_2}{\pi^4}$ when $\beta_1 \geqslant 0$ and $\alpha < 0$;
- $\sigma_1 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_1}{\pi^4}$ and $\sigma_2 := 1 + \frac{\beta_2}{\pi^4}$ when $\beta_1 < 0 < \beta_2$ and $\alpha \geqslant 0$;
- $\sigma_1 := 1 + \frac{\beta_1}{\pi^4}$ and $\sigma_2 := 1 - \frac{\alpha}{\pi^2} + \frac{\beta_2}{\pi^4}$ when $\beta_1 < 0 < \beta_2$ and $\alpha < 0$.

In each of these cases, if $\sigma_1 > 0$ and

$$\theta_i := \sqrt{\sigma_i} \quad (i = 1, 2) \tag{2.3}$$

then by (2.1) and (2.2)

$$\theta_1\|u''\|_2 \leqslant \|u\| \leqslant \theta_2\|u''\|_2 \tag{2.4}$$

where

$$\|u\| = \Big(\int_0^1 \big(|u''(x)|^2 - \alpha|u'(x)|^2 + \beta(x)|u(x)|^2\big)dx\Big)^{1/2}$$

and so, $\|\cdot\|$ defines a norm on X equivalent to usual norm of X inherited from $W^{2,2}([0,1])$.

In the remainder, we suppose θ_1 defined by (2.3) satisfies $\theta_1 > 0$ and therefore (2.4) holds. The following result is useful for proving our main result.

Proposition 2.2. *For every $u \in X$.*

$$\|u\|_\infty \leqslant \frac{1}{2\pi\theta_1}\|u\|$$

Proof. Similar to the proof of [4, Proposition 2.1], considering (2.2) and (2.4) and using well-known inequality $\|u\|_\infty \leqslant \frac{1}{2}\|u'\|_2$ yields the conclusion. □

A function $u : [0,1] \to \mathbb{R}$ is said a *generalized solution to the problem* (1.1), if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = 0$, $u''(0) = u''(1) = 0$ and $u^{iv} + \alpha u'' + \beta u = \lambda f(x,u) + h(u)$. If f is continuous in $[0,1] \times \mathbb{R}$, then each generalized solution of the problem (1.1) is a classical one. Standard methods (see [4, Proposition 2.2]) show that a weak solution to (1.1) is a generalized one when f is an L^2-Carathéodory function.

We define

$$F(x,\xi) = \int_0^\xi f(x,t)dt \quad \text{and} \quad H(\xi) = \int_0^\xi h(x)dx \tag{2.5}$$

for every $x \in [0,1]$ and $\xi \in \mathbb{R}$.

Lemma 2.3. *Suppose $h : \mathbb{R} \to \mathbb{R}$ satisfies* (1.2) *and $H(\xi)$ defined by* (2.5) *for every $\xi \in \mathbb{R}$. Then the functional $\Theta : X \to \mathbb{R}$ defined by*

$$\Theta(u) := \int_0^1 H(u(x))dx \tag{2.6}$$

is a Gâteaux differentiable sequentially weakly continuous functional on X with compact derivative

$$\Theta'(u)[v] = \int_0^1 h(u(x))v(x)dx$$

for every $v \in X$.

Proof. If $u_n \rightharpoonup u$ in X then compactness of embedding $X \hookrightarrow C([0,1])$ implies $u_n \to u$ in $C([0,1])$ i.e. $u_n \to u$ uniformly on $[0,1]$ (see Proposition 2.2.4 of [16]). Hence, for some constant $M > 0$ and any $n \in \mathbb{N}$ we have $\|u_n\|_\infty \leqslant M$, and so

$$|H(u_n(x)) - H(u(x))|dx \leqslant L\Big|\int_{u(x)}^{u_n(x)} |t|dt\Big| \leqslant \frac{L}{2}(|u_n(x)|^2 + |u(x)|^2) \leqslant \frac{L}{2}(M^2 + \|u\|_\infty^2)$$

for every $n \in \mathbb{N}$ and $x \in [0,1]$. Furthermore, $H(u_n(x)) \to H(u(x))$ at any $x \in [0,1]$ and therefore, the Lebesgue Convergence Theorem yields

$$\Theta(u_n) = \int_0^1 H(u_n(x))dx \to \int_0^1 H(u(x))dx = \Theta(u).$$

For proving Gâteaux differentiability of Θ suppose $u, v \in X$ and $t \neq 0$ then

$$\begin{aligned}
&\Big|\frac{\Theta(u+tv) - \Theta(u)}{t} - \int_0^1 h(u(x))v(x)dx\Big| \\
&\leqslant \int_0^1 \Big|\frac{H(u+tv) - H(u)}{t} - h(u(x))v(x)\Big|dx \\
&= \int_0^1 |h\big(u(x) + t\zeta(x)v(x)\big) - h(u(x))||v(x)|dx \\
&\leqslant L\|v\|_\infty^2|t|
\end{aligned}$$

in which $0 < \zeta(x) < 1$ for every $x \in [0,1]$. Therefore, $\Theta : X \to \mathbb{R}$ is a Gâteaux differentiable at every $u \in X$ with derivative

$$\Theta'(u)[v] = \int_0^1 h(u(x))v(x)dx$$

for every $v \in X$. Also, since

$$\big(\Theta'(u) - \Theta'(v)\big)[w] \leqslant L\int_0^1 |u(x) - v(x)||w(x)|dx \leqslant \frac{L}{2\pi\theta_1}\|u - v\|_\infty\|w\|$$

for every three elements u, v and w of X, then

$$\|\Theta'(u) - \Theta'(v)\|_{X^*} \leqslant \frac{L}{2\pi\theta_1}\|u - v\|_\infty$$

which implies compactness of $\Theta' : X \to X^*$. □

Lemma 2.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function and $F(x,\xi)$ defined by* (2.5). *Then $\Psi : X \to \mathbb{R}$ defined by*

$$\Psi(u) := \int_0^1 F\big(x, u(x)\big)dx$$

is a Gâteaux differentiable sequentially weakly continuous functional on X.

Proof. If $u_n \rightharpoonup u$ in X, in Lemma 2.3 was proved that $u_n \to u$ uniformly on $[0,1]$ and there exists $M > 0$ such that $\|u_n\|_\infty \leqslant M$ for any $n \in \mathbb{N}$. Since $F(x,\xi)$ is differentiable with respect to ξ for a.e. $x \in [0,1]$ so $F\big(x,u_n(x)\big) \to F\big(x,u_n(x)\big)$ a.e. on $[0,1]$. Moreover, by the assumption (C3) on $f(x,t)$

$$F\big(x,u_n(x)\big) \leqslant M\ell_M(x)$$

and by the Lebesgue Convergence Theorem

$$\Psi(u_n) = \int_0^1 F\big(x,u_n(x)\big)dx \to \int_0^1 F\big(x,u(x)\big)dx = \Psi(u).$$

Therefore Ψ is a sequentially weakly continuous functional on X. For proving the Gâteaux differentiability of Ψ, let $u,v \in X$ with $\|u\| < 2\pi\theta_1 M$ and $\|v\| < 2\pi\theta_1 M$ for some $M > 0$. Then for $t \neq 0$ by the Mean Value Theorem

$$\begin{aligned}
&\Big|\frac{\Psi(u+tv)-\Psi(u)}{t} - \int_0^1 f(x,u(x))v(x)dx\Big| \\
&\leqslant \int_0^1 \big|f(x,u(x)+t\zeta(x)v(x)) - f(x,u(x))\big||v(x)|dx \\
&\leqslant \|v\|_\infty \int_0^1 \big|f(x,u(x)+t\zeta(x)v(x)) - f(x,u(x))\big|dx
\end{aligned}$$

where $0 < \zeta(x) < 1$ for every $x \in [0,1]$ for which $F(x,\xi)$ is differentiable with respect to ξ. Since the assumption $\mathbf{C}_2$ on $f(x,t)$ implies

$$\lim_{t\to 0} f(x,u(x)+t\zeta(x)v(x)) = f(x,u(x)) \quad \text{for a.e. } x \in [0,1]$$

and by Proposition 2.2 we have $\|v\|_\infty \leqslant M$ and $\|u\|_\infty \leqslant M$, then by the assumption (C3) on $f(x,t)$ we have

$$|f(x,u(x)+t\zeta(x)v(x)) - f(x,u(x))| \leqslant \ell_{2M}(x) + \ell_M(x)$$

for any $|t| < 1$. Therefore the Lebesgue Convergence Theorem implies

$$\lim_{t\to 0} \frac{\Psi(u+tv)-\Psi(u)}{t} = \int_0^1 f(x,u(x))v(x)dx.$$

Since for every $v \in X$, some constant $M > 0$ can be found so that both of inequalities $\|u\| < 2\pi\theta_1 M$ and $\|v\| < 2\pi\theta_1 M$ hold, thus Ψ is Gâteaux differentiable at every $u \in X$. □

3. Main Results

Our approach closely depends on the test function $v_0 \in X$ defined by

$$v_0(x) = \begin{cases} \frac{2ax-x^2}{a^2} & \text{if } x \in [0,a[, \\ 1 & \text{if } x \in [a,b], \\ \frac{2bx-x^2-2b+1}{(1-b)^2} & \text{if } x \in]b,1]. \end{cases}$$

Let

$$K(a,b) := \frac{4\pi^2\theta_1^2}{\|v_0\|^2}$$

for every $0 < a \leqslant b < 1$, we get a positive continuous function

$$k(\epsilon) := \min\big\{K(a,b) : a,b \in [\epsilon, 1-\epsilon],\ a \leqslant b\big\} \tag{3.1}$$

which is defined for every $0 < \epsilon < 1/2$.

Theorem 3.1. *Suppose that $L < \pi^4\theta_1^2$. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function and $F(x,\xi)$ defined by (2.5). Assume that $\ell_\xi \in L^2([0,1])$ satisfies (C3) condition on $f(x,t)$ for every $\xi > 0$. Furthermore, suppose that there exist an interval $[a,b] \subset [\epsilon, 1-\epsilon]$ for some $0 < \epsilon < \frac{1}{2}$ for which $k(\epsilon)$ defined by (3.1) and two positive constants T and p and a function $q \in L^2([0,1])$ such that*

(i) *$f(x,t) \geqslant q(x) - p|t|$ for every $(x,t) \in \big([0,a] \cup [b,1]\big) \times \{t \in \mathbb{R} \mid t \geqslant T\}$;*

(ii) $\liminf_{\xi\to\infty} \frac{\|\ell_\xi\|_2}{(\pi^4\theta_1^2 - L)\xi} < \frac{\pi k(\epsilon)}{2(\pi^4\theta_1^2+L)} \limsup_{\xi\to\infty} \frac{\int_a^b F(x,\xi)\,dx}{\xi^2}$.

Then, for every

$$\lambda \in \Lambda := \Big] \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} \frac{1}{\limsup_{\xi\to\infty} \frac{\int_a^b F(x,\xi)\,dx}{\xi^2}},\ \limsup_{\xi\to\infty} \frac{(\pi^4\theta_1^2 - L)\xi}{\pi\|\ell_\xi\|_2} \Big[$$

problem (1.1) has an unbounded sequence of generalized solutions in X.

Proof. Put

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_0^1 H\big(u(x)\big)dx = \frac{1}{2}\|u\|^2 - \Theta(u), \quad \Psi(u) = \int_0^1 F\big(x, u(x)\big)dx$$

for every $u \in X$. Since (1.2) holds for every $t_1, t_2 \in \mathbb{R}$ and $h(0) = 0$, we have $|h(t)| \leqslant L|t|$ for every $t \in \mathbb{R}$, and so using (2.1), (2.4) and Lemma 2.3 we obtain

$$\Phi(u) \geqslant \frac{1}{2}\|u\|^2 - \Big|\int_0^1 H\big(u(x)\big)dx\Big| \geqslant \frac{1}{2}\|u\|^2 - \frac{L}{2}\int_0^1 |u(x)|^2 dx \geqslant \Big(\frac{1}{2} - \frac{L}{2\pi^4\theta_1^2}\Big)\|u\|^2, \tag{3.2}$$

and similarly

$$\Phi(u) \leqslant \frac{1}{2}\|u\|^2 + \Big|\int_0^1 H\big(u(x)\big)dx\Big| \leqslant \Big(\frac{1}{2} + \frac{L}{2\pi^4\theta_1^2}\Big)\|u\|^2. \tag{3.3}$$

Also, since $\Phi + \Theta$ is a continuous functional on X and Θ, by Lemma 2.3, is a Gâteaux differentiable weakly continuous and therefore continuous functional on X then Φ is a continuous functional on X and by a routine argument can be proved that Φ is a Gâteaux differentiable functional with the differential

$$\Phi'(u)[v] = \int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta(x)u(x)v(x)]dx - \int_0^1 h(u(x))v(x)dx$$

and it is sequentially weakly lower semicontinuous since Θ is sequentially weakly continuous, and if $u_n \rightharpoonup u$ in X then

$$\liminf_{n\to\infty} \Phi(u_n) = \liminf_{n\to\infty} \frac{1}{2}\|u_n\|^2 - \lim_{n\to\infty} \Theta(u_n) \geqslant \frac{1}{2}\|u\|^2 - \Theta(u) = \Phi(u).$$

It is easy to see that the critical points of the functional $I_\lambda = \Phi - \lambda\Psi$ and the weak solutions (and therefore generalized solutions) of the problem (1.1) are the same and by Theorem 1.2 we prove our result.

Assume that $\{\xi_n\}_{n=1}^\infty$ is a sequence of positive numbers such that $\xi_n \to \infty$ and

$$\lim_{n\to\infty} \frac{\|\ell_{\xi_n}\|_2}{(\pi^4\theta_1^2 - L)\xi_n} = \liminf_{\xi\to\infty} \frac{\|\ell_\xi\|_2}{(\pi^4\theta_1^2 - L)\xi}$$

and let $r_n = \frac{2(\pi^4\theta_1^2 - L)}{\pi^2}\xi_n^2$ then by (3.2) for any $v \in X$ such that $\Phi(v) < r_n$ we have

$$\|v\| \leqslant \pi^2\theta_1\sqrt{\frac{2\Phi(v)}{\pi^4\theta_1^2 - L}} < \pi^2\theta_1\sqrt{\frac{2r_n}{\pi^4\theta_1^2 - L}} = 2\pi\theta_1\xi_n \tag{3.4}$$

and by Proposition 2.2,

$$\|v\|_\infty < \xi_n. \tag{3.5}$$

On the other hand, by condition (C3) on $f(x,t)$ and (3.5)

$$|F(x,v(x))| \leqslant \Big| \int_0^{v(x)} \ell_{\xi_n}(x)dt \Big| = |v(x)|\ell_{\xi_n}(x)$$

and so by the Hölder inequality and Lemma 2.1 and (2.4)

$$|\Psi(v)| \leqslant \int_0^1 |v(x)||\ell_{\xi_n}(x)|dx \leqslant \frac{1}{\pi^2\theta_1}\|v\|\|\ell_{\xi_n}\|_2. \tag{3.6}$$

Therefore, since $L < \pi^4\theta_1^2$, by (3.2)

$$\sup_{v\in\Phi^{-1}(]-\infty,r_n[)} \Psi(v) = \sup_{v\in\Phi^{-1}([0,r_n[)} \Psi(v) \leqslant \frac{2\xi_n}{\pi}\|\ell_{\xi_n}\|_2 \tag{3.7}$$

and then by (3.6) and (3.7)

$$\begin{aligned} \varphi(r_n) &\leqslant \inf_{u\in\Phi^{-1}([0,r_n[)} \frac{\sup_{v\in\Phi^{-1}([0,r_n[)}\Psi(v) - \Psi(u)}{r_n - \Phi(u)} \\ &\leqslant \inf_{u\in\Phi^{-1}([0,r_n[)} \frac{\|\ell_{\xi_n}\|_2}{\pi^2\theta_1} \frac{2\pi\theta_1\xi_n + \|u\|}{r_n - \Phi(u)} \\ &\leqslant \frac{\pi\|\ell_{\xi_n}\|_2}{(\pi^4\theta_1^2 - L)\xi_n} \end{aligned}$$

and hence

$$\gamma \leqslant \liminf_{\xi\to\infty} \frac{\pi\|\ell_\xi\|_2}{(\pi^4\theta_1^2 - L)\xi} < +\infty. \tag{3.8}$$

Then (3.8) in conjunction with the assumption (ii) imply

$$\Lambda \subset]0, \frac{1}{\gamma}[$$

and by (3.2) the functional Φ is coercive, since $L < \pi^4\theta_1^2$. Therefore part *b)* of Theorem 1.2 implies either the functional $I_\lambda = \Phi - \lambda\Psi$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of problem (1.1) such that $\lim_{n\to\infty}\|u_n\| = \infty$ for every $\lambda \in \Lambda$.

Now we prove unboundedness of I_λ from below under condition (ii) and thus the existence of infinitely many solutions of problem (1.1) is proved. If $\lambda \in \Lambda$, then

$$\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} < \lambda \limsup_{\xi\to\infty} \frac{\int_a^b F(x,\xi)dx}{\xi^2}$$

and there exist a constant c and a sequence of reals $\{\eta_n\}$ so that, $\eta_n \geqslant n$ and

$$\lim_{n\to\infty} \frac{\int_a^b F(x,\eta_n)dx}{\eta_n^2} = \limsup_{\xi\to\infty} \frac{\int_a^b F(x,\xi)dx}{\xi^2}$$

and in addition

$$\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} < c < \lambda \lim_{n\to\infty} \frac{\int_a^b F(x,\eta_n)dx}{\eta_n^2}. \tag{3.9}$$

Let $\{v_n\}$ be a sequence in X which is defined by

$$v_n(x) = v_0(x)\eta_n = \begin{cases} \eta_n \frac{2ax-x^2}{a^2} & \text{if } x \in [0,a[, \\ \eta_n & \text{if } x \in [a,b], \\ \eta_n \frac{2bx-x^2-2b+1}{(1-b)^2} & \text{if } x \in]b,1], \end{cases}$$

then from (3.1) and (3.3) we observe that

$$\Phi(v_n) \leqslant \frac{\pi^4\theta_1^2 + L}{2\pi^4\theta_1^2}\|v_n\|^2 \leqslant \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)}\eta_n^2. \tag{3.10}$$

On the other hand, by assumption (i),

$$F(x,u(x)) \geqslant -|q(x)||u(x)| - \frac{p|u(x)|^2}{2} - |u(x)|\ell_T(x) \quad (x \in [0,a[\cup]b,1])$$

and then the Hölder inequality, Lemma 2.1 and (2.4) imply

$$\int_0^a F(x,u(x))dx + \int_b^1 F(x,u(x))dx \geqslant -\frac{2\|\ell_T\|_2 + 2\|q\|_2 + p}{2\pi^2\theta_1}\|u\|. \tag{3.11}$$

So by (3.9), (3.10) and (3.11), there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$

$$\begin{aligned} I_\lambda(v_n) = \Phi(v_n) - \lambda\Psi(v_n) &\leqslant \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)}\eta_n^2 - \lambda\int_0^1 F(x,v_n(x))dx \\ &\leqslant \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)}\eta_n^2 - \lambda\int_a^b F(x,\eta_n)dx + \frac{p + 2\|q\|_2 + 2\|\ell_T\|_2}{2\pi^2\theta_1}\|v_n\| \\ &\leqslant \Big(\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} - c\Big)\eta_n^2 + \frac{p + 2\|q\|_2 + 2\|\ell_T\|_2}{\pi\sqrt{k(\epsilon)}}\eta_n. \end{aligned}$$

Since $\lim_{n\to\infty} \eta_n = \infty$ then (3.9) implies the functional I_λ is unbounded from below and the proof is completed. □

Remark 3.2. If in Theorem 3.1 instead of *i)* we assume $F(x,t) \geqslant 0$ for every $(x,t) \in [0,a[\cup]b,1] \times \mathbb{R}$ then the assumption *ii)* can be replaced by a more general one like

(ii') There exist two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers such that $|a_n| \leqslant b_n\sqrt{k(\epsilon)\frac{\pi^4\theta_1^2 - L}{\pi^4\theta_1^2 + L}}$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = +\infty$ and

$$\liminf_{n\to+\infty} \frac{\|\ell_{b_n}\|_2(b_n\sqrt{k(\epsilon)} + a_n)}{\frac{\pi^4\theta_1^2 - L}{\pi^4\theta_1^2 + L}b_n^2 k(\epsilon) - a_n^2} < \frac{\pi\sqrt{k(\epsilon)}}{2}\limsup_{\xi\to+\infty}\frac{\int_a^b F(x,\xi)dx}{\xi^2}.$$

Obviously, from (ii') we obtain (ii), by choosing $a_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume *ii')* instead of (ii) and set $r_n = \frac{2(\pi^4\theta_1^2 - L)b_n^2}{\pi^2}$ for all $n \in \mathbb{N}$, by the same arguing as inside in Theorem 3.1, we have

$$\begin{aligned} \varphi(r_n) &= \inf\Big\{\frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n[)}\Psi(v) - \Psi(u)}{r_n - \Phi(u)} : u \in \Phi^{-1}(]-\infty,r_n[)\Big\} \\ &\leqslant \frac{\sup_{v\in\Phi^{-1}([0,r_n[)}\Psi(v) - \Psi(v_n)}{r_n - \Phi(v_n)} \\ &\leqslant \frac{\|\ell_{b_n}\|_2}{\pi^2\theta_1}\,\frac{2\pi\theta_1 b_n + \|v_n\|}{\frac{2(\pi^4\theta_1^2 - L)}{\pi^2}b_n^2 - \frac{\pi^4\theta_1^2 + L}{2\pi^4\theta_1^2}\|v_n\|^2} \end{aligned}$$

$$\leqslant \frac{\pi\|\ell_{b_n}\|_2}{\pi^4\theta_1^2+L}\frac{b_n+\frac{a_n}{\sqrt{k(\epsilon)}}}{\frac{\pi^4\theta_1^2-L}{\pi^4\theta_1^2+L}b_n^2-\frac{a_n^2}{k(\epsilon)}}$$

where

$$v_n(x)=\begin{cases} a_n\frac{2ax-x^2}{a^2} & \text{if } x\in[0,a[,\\ a_n & \text{if } x\in[a,b],\\ a_n\frac{2bx-x^2-2b+1}{(1-b)^2} & \text{if } x\in]b,1],\end{cases}$$

Therefore $\gamma<\infty$. Similarly, the second part of the proof of Theorem 3.1 can be improved so that the conclusion of the theorem can be obtained for the interval

$$\Lambda'=\Big]\frac{2(\pi^4\theta_1^2+L)}{\pi^2k(\epsilon)\limsup_{\xi\to+\infty}\frac{\int_a^b F(x,\xi)dx}{\xi^2}},\frac{\pi^4\theta_1^2+L}{\pi\sqrt{k(\epsilon)}}\limsup_{n\to+\infty}\frac{\frac{\pi^4\theta_1^2-L}{\pi^4\theta_1^2+L}b_n^2k(\epsilon)-a_n^2}{\|\ell_{b_n}\|_2\big(b_n\sqrt{k(\epsilon)}+a_n\big)}\Big[$$

instead of Λ.

Now we point out a simple consequence of Theorem 3.1.

Corollary 3.3. *Suppose that $L<\pi^4\theta_1^2$. Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be an L^2-Carathéodory function. Assume that $\ell_\xi\in L^2([0,1])$ satisfies (C3) condition on $f(x,t)$ for every $\xi>0$ and there exists an interval $[a,b]\subset[\epsilon,1-\epsilon]$ for some $0<\epsilon<\frac12$ such that assumption (i) in Theorem 3.1 holds. Furthermore, suppose that*

(iii) $\liminf_{\xi\to\infty}\frac{\|\ell_\xi\|_2}{\xi}<\frac{(\pi^4\theta_1^2-L)}{\pi}$;

(iv) $\limsup_{\xi\to\infty}\frac{\int_a^b F(x,\xi)\,dx}{\xi^2}>\frac{2(\pi^4\theta_1^2+L)}{\pi^2k(\epsilon)}$,

then the problem

$$\begin{gathered} u^{iv}+\alpha u''+\beta(x)u=h(u)+f(x,u),\quad x\in(0,1)\\ u(0)=u(1)=0,\\ u''(0)=u''(1)=0\end{gathered}\tag{3.12}$$

has an unbounded sequence of generalized solutions in X.

Note that Theorem 1.1 is an immediate consequence of Corollary 3.3. Now we present the following example to illustrate our results.

Example 3.4. Let $r>0$ be a real number and $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequences of reals that recursively defined by

$$t_1=r,\ s_1=2r$$

and for $n\geqslant1$ by

$$t_{2n}=(2^{2n+1}-1)t_{2n-1},\quad t_{2n+1}=(2-\frac{1}{2^{2n+1}})t_{2n},$$

$$s_{2n}=\frac{t_{2n}}{2^n}=(2-\frac{1}{2^{2n}})s_{2n-1},\quad s_{2n+1}=2^{n+1}t_{2n+1}=(2^{2n+2}-1)s_{2n}.$$

If $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be the function defined as

$$f(x,t)=\begin{cases}2g(x)t & (x,t)\in[0,1]\times[0,t_1],\\ g(x)\Big(s_{n-1}+\frac{s_n-s_{n-1}}{t_n-t_{n-1}}(t-t_{n-1})\Big) & (x,t)\in[0,1]\times[t_{n-1},t_n]\\ & \text{for some } n>1\end{cases}$$

where $g : [0,1] \to \mathbb{R}$ is a positive continuous function with $0 < m \leqslant g(x) \leqslant M$. Then $f(x,t)$ is an L^2-Carathéodory function and since $f(x,t)$ is strictly increasing with respect to t argument at every $x \in [0,1]$, the function $\ell_\xi(x) := f(x,\xi)$ satisfies in (C3) condition on $f(x,t)$; i.e.,

$$\sup_{|t|\leqslant\xi} |f(x,t)| \leqslant \ell_\xi(x) \quad \forall x \in [0,1].$$

Now we have

$$\begin{aligned}
\liminf_{\xi\to+\infty} \frac{\|\ell_\xi\|_2}{\xi} &\leqslant \lim_{n\to\infty} \frac{Ms_{2n}}{t_{2n}} = 0,\\
\limsup_{\xi\to+\infty} \frac{\int_a^b F(x,\xi)dx}{\xi^2} &\geqslant \lim_{n\to\infty} \frac{m(b-a)(t_{2n+1}-t_{2n})(s_{2n+1}+s_{2n})}{2t_{2n+1}^2}\\
&\geqslant \lim_{n\to\infty} \frac{m(b-a)2^{3n+2}(2^{2n+1}-1)}{(2^{2n+2}-1)^2} = +\infty
\end{aligned}$$

for every $[a,b] \subset [\epsilon, 1-\epsilon]$ with $a < b$ and any $0 < \epsilon < \frac{1}{2}$. Hence by Corollary 3.3, for every $\lambda \in]0,+\infty[$, the boundary value problem

$$\begin{gathered}
u^{iv} + \alpha u'' + \beta(x)u = \lambda f(x,u) + u^+,\\
u(0) = u(1) = 0,\\
u''(0) = u''(1) = 0
\end{gathered}$$

where $\alpha < \pi^2 - \frac{1}{\pi^2}$ is a real constant and $\beta(x)$ is any non-negative continuous function on $[0,1]$ and $u^+ = \max\{u,0\}$, has an unbounded sequence of generalized solutions in X (for instance, $\alpha = 9$ and $\beta(x) = \sin \pi x$).

Arguing as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 1.2 instead of (b), the following result holds.

Theorem 3.5. *Suppose that $L < \pi^4\theta_1^2$. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^2-Carathéodory function. Assume that $\ell_\xi \in L^2([0,1])$ satisfies in* (C3) *condition on $f(x,t)$ for every $\xi > 0$ and there exists an interval $[a,b] \subset [\epsilon, 1-\epsilon]$ for some $0 < \epsilon < \frac{1}{2}$ such that*

(i) *$F(x,t) \geqslant 0$ for every $(x,t) \in [0,a[\cup]b,1] \times \mathbb{R}$;*
(ii)

$$\liminf_{\xi\to0^+} \frac{\|\ell_\xi\|_2}{(\pi^4\theta_1^2 - L)\xi} < \frac{\pi k(\epsilon)}{2(\pi^4\theta_1^2 + L)} \limsup_{\xi\to0^+} \frac{\int_a^b F(x,\xi)\,dx}{\xi^2}.$$

Then, for every

$$\lambda \in \Lambda := \Big] \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} \frac{1}{\limsup_{\xi\to0^+} \frac{\int_a^b F(x,\xi)\,dx}{\xi^2}}, \ \limsup_{\xi\to0^+} \frac{(\pi^4\theta_1^2 - L)\xi}{\pi\|\ell_\xi\|_2} \Big[$$

Problem (1.1) *has a sequence of non-zero generalized solutions in X that converges weakly to* 0.

Proof. Since $\inf_X \Phi = \min_X \Phi = 0$ as a consequence of (3.2) and the assumption $L < \pi^4\theta_1^2$. Exactly as in the proof of Theorem 3.1 it can be shown that

$$\delta = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r) \leqslant \frac{\pi}{\pi^4\theta_1^2 - L} \liminf_{\xi\to0^+} \frac{\|\ell_\xi\|_2}{\xi} < +\infty$$

and therefore

$$\Lambda \subset]0, \frac{1}{\delta}[.$$

If $\lambda \in \Lambda$ then

$$\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} < \lambda \limsup_{\xi \to 0^+} \frac{\int_a^b F(x,\xi)dx}{\xi^2}$$

and there exist a constant c and a sequence of reals $\{\zeta_n\}$ so that, $\zeta_n \leqslant \frac{1}{n}$ and

$$\lim_{n\to\infty} \frac{\int_a^b F(x,\zeta_n)dx}{\zeta_n^2} = \limsup_{\xi \to 0^+} \frac{\int_a^b F(x,\xi)dx}{\xi^2}$$

and in addition

$$\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} < c < \lambda \lim_{n\to\infty} \frac{\int_a^b F(x,\eta_n)dx}{\eta_n^2}. \tag{3.13}$$

Let $\{w_n\}$ be a sequence in X defined by

$$w_n(x) = \begin{cases} \zeta_n \frac{2ax - x^2}{a^2} & \text{if } x \in [0, a[, \\ \zeta_n & \text{if } x \in [a, b], \\ \zeta_n \frac{2bx - x^2 - 2b + 1}{(1-b)^2} & \text{if } x \in]b, 1], \end{cases} \tag{3.14}$$

then $\{w_n\}$ converges strongly to 0 in X and by (3.3)

$$\Phi(w_n) \leqslant \frac{\pi^4\theta_1^2 + L}{2\pi^4\theta_1^2}\|w_n\|^2 \leqslant \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)}\zeta_n^2$$

hence, by (i) and the similar arguments as in the proof of Theorem 3.1, there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$

$$\begin{aligned} I_\lambda(w_n) &\leqslant \frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)}\zeta_n^2 - \lambda \int_a^b F(x,\zeta_n)dx \\ &\leqslant \Big(\frac{2(\pi^4\theta_1^2 + L)}{\pi^2 k(\epsilon)} - c\Big)\zeta_n^2. \end{aligned}$$

Since $I_\lambda(0) = 0$ therefore (3.13) implies 0 is not a local minimum of I_λ and then according to (c) of Theorem 1.2 there exists a sequence $\{u_n\}$ of local minimums of I_λ that weakly converges to 0. □

Remark 3.6. Since the embedding $X \hookrightarrow C([0,1])$ is compact, by [16, Proposition 2.2.4], every weakly convergent sequence in X converges strongly in $C([0,1])$; i.e., converges uniformly on $[0,1]$. Therefore the generalized solutions of the problem (1.1) established in Theorem 3.5 converges uniformly to zero on $[0,1]$.

References

[1] G. A. Afrouzi, S. Heidarkhani, D. O'Regan; *Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem,* Taiwanese J. Math. 15 (2011) 201-210.

[2] A. Ambrosetti, G. Prodi; *A primer of nonlinear analysis,* Cambridge: Cambridge University Press, 1993.

[3] Z. Bai; *Positive solutions of some nonlocal fourth-order boundary value problem,* Appl. Math. Comput. 215 (2010) 4191-4197.

[4] G. Bonanno, B. Di Bella; *A boundary value problem for fourth-order elastic beam equations,* J. Math. Anal. Appl. 343 (2008) 1166-1176.

[5] G. Bonanno, B. Di Bella; *A fourth-order boundary value problem for a SturmLiouville type equation,* Appl. Math. Comput. 217 (2010) 3635-3640.

[6] G. Bonanno, B. Di Bella; *Infinitely many solutions for a fourth-order elastic beam equation,* Nonlinear Differential Equations and Applications NoDEA, 18 (2011) 357-368.
[7] G. Bonanno, G. Molica Bisci; *A remark on perturbed elliptic Neumann problems,* Studia Univ. "Babeş-Bolyai", Mathematica, Volume **LV**, Number 4, December 2010.
[8] G. Bonanno, G. Molica Bisci; *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities,* Bound. Value Probl. 2009 (2009) 1-20.
[9] G. Bonanno, G. Molica Bisci; *Infinitely many solutions for a Dirichlet problem involving the p-Laplacian,* Proceedings of the Royal Society of Edinburgh 140A (2010) 737-752.
[10] G. Bonanno, G. Molica Bisci, D. O'Regan; *Infinitely many weak solutions for a class of quasilinear elliptic systems,* Math. Comput. Modelling 52 (2010) 152-160.
[11] G. Bonanno, G. Molica Bisci, V. Rădulescu; *Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces,* Monatsh. Math., doi:10.1007/s00605-010-0280-2, in press.
[12] G. Bonanno, G. Molica Bisci, V. Rădulescu; *Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz-Sobolev spaces,* C. R. Acad. Sci. Paris, Ser. I 349 (2011) 263-268.
[13] P. Candito; *Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities,* Proc. Edin. Math. Soc. 45 (2002) 397-409.
[14] G. Chai; *Existence of positive solutions for fourth-order boundary value problem with variable parameters,* Nonlinear Anal. 66 (2007) 870-880.
[15] G. Dai; *Infinitely many solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian,* Nonlinear Anal. 70 (2009) 2297-2305.
[16] P. Drábek, J. Milota; *Methods of Nonlinear Analysis: Applications to Differential Equations*, Birkhäzur, Basel-Boston-Berlin, 2007.
[17] X. Fan, C. Ji; *Existence of infinitely many solutions for a Neumann problem involving the p(x)-Laplacian,* J. Math. Anal. Appl. 334 (2007) 248 - 260.
[18] J. R. Graef, S. Heidarkhani, L. Kong; *Infinitely many solutions for systems of multi-point boundary value problems using variational methods,* Topol. Methods in Nonlinear Anal., to appear.
[19] M. R. Grossinho, L. Sanchez, S. A. Tersian; *On the solvability of a boundary value problem for a fourth-order ordinary differential equation,* Appl. Math. Lett. 18 (2005) 439 - 444.
[20] C. P. Gupta; *Existence and uniqueness theorems for a bending of an elastic beam equation,* Appl. Anal. 26 (1988) 289-304.
[21] C. P. Gupta; *Existence and uniqueness results for the bending of an elastic beam equation at resonance,* J. Math. Anal. Appl. 135 (1988) 208-225.
[22] S. Heidarkhani, J. Henderson; *Infinitely many solutions for a class of nonlocal elliptic systems of $(p_1, ..., p_n)$-Kirchhoff type,* Electronic Journal of Differential Equations, Vol. 2012 (2012) No. 69, pp. 1-15.
[23] A. Kristály; *Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$,* J. Differential Equations 220 (2006) 511-530.
[24] A. C. Lazer, P. J. McKenna; *Large amplitude periodic oscillations in suspension bridges: Some new connections with nonlinear analysis,* SIAM Rev. 32 (1990) 537-578.
[25] C. Li; *The existence of infinitely many solutions of a class of nonlinear elliptic equations with a Neumann boundary conditions for both resonance and oscillation problems,* Nonlinear Anal. 54 (2003) 431-443.
[26] S. Marano, D. Motreanu; *Infinitely many critical points of non-differentiable functions and applications to the Neumann-type problem involving the p-Laplacian,* J. Differential Equations 182 (2002) 108-120.
[27] L. A. Peletier, W. C. Troy, R. C. A. M. Van der Vorst; *Stationary solutions of a fourth-order nonlinear diffusion equation*, Differ. Equ. 31 (1995) 301-314.
[28] B. Ricceri; *A general variational principle and some of its applications,* J. Comput. Appl. Math. 113 (2000) 401-410.
[29] B. Ricceri; *Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian,* Bull. London Math. Soc. 33(3) (2001) 331-340.
[30] B. Ricceri; *Sublevel sets and global minima of coercive functionals and local minima of their perturbations,* J. Nonlinear Conex Anal. 5(2004) 157-168.

[31] V. Shanthi, N. Ramanujam; *A numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations,* Appl. Math. Comput. 129 (2002) 269-294.
[32] X. L. Liu, W. T. Li, *Existence and multiplicity of solutions for fourth-order boundary value problems with parameters,* J. Math. Anal. Appl. 327 (2007) 362-375.
[33] X. L. Liu, W. T. Li; *Existence and multiplicity of solutions for fourth-order boundary value problems with three parameters,* Math. Comp. Model. 46 (2007) 525-534.
[34] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMS Regional Conference, 1984.
[35] S. Tersian, J. Chaparova; *Periodic and Homoclinic Solutions of Extended Fisher-Kolmogorov Equations,* J. Math. Anal. Appl. 260 (2001) 490-506.
[36] E. Zeidler; *Nonlinear functional analysis and its applications*, Vol. II/B. Berlin-Heidelberg-New York 1985.

Seyyed Mohsen Khalkhali
Department of Mathematics, Science and Research branch, Islamic Azad University, Tehran, Iran
E-mail address: sm.khalkhali@srbiau.ac.ir

Shapour Heidarkhani
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: s.heidarkhani@razi.ac.ir

Abdolrahman Razani
Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box 19395-5746, Tehran, Iran
E-mail address: razani@ikiu.ac.ir