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ANALYTIC SEMIGROUPS GENERATED BY AN OPERATOR MATRIX IN $L^2(\Omega) \times L^2(\Omega)$

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Abstract. This article concerns the generation of analytic semigroups by an operator matrix in the space $L^2(\Omega) \times L^2(\Omega)$. We assume that one of the diagonal elements is strongly elliptic and the other is weakly elliptic, while the sum of the non-diagonal elements is weakly elliptic.

1. Introduction

The theory of semigroups of linear operators has applications in many branches of analysis as evolution equations: parabolic and hyperbolic equations and systems with various boundary conditions, harmonic analysis and ergodic theory. In the theory of evolution equations, it is usually shown that a given differential operator A is the infinitesimal generator of a strongly continuous semigroup in a certain concrete Banach space of functions X. This provides us with the existence and uniqueness of a solution of the initial value problem

$$\begin{gathered}\frac{\partial u(x,t)}{\partial t} + Au(x,t) = 0 \\ u(x,0) = u_0(x)\end{gathered}$$

in the sense of the Banach space X.

This article concerns the generation of analytic semigroups by an operator matrix in the space $L^2(\Omega) \times L^2(\Omega)$, where Ω is a bounded open set in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. Passo and Mottoni [4] proved that the operator matrix

$$\mathcal{M} = \begin{pmatrix} a_{11}\Delta & a_{12}\Delta \\ a_{21}\Delta & a_{22}\Delta \end{pmatrix} \tag{1.1}$$

with domain $D(\mathcal{M}) = (H^2(\Omega) \cap H_0^1(\Omega))^2$ generates an analytic semigroup on $L^2(\Omega) \times L^2(\Omega)$ provided $a_{11}, a_{22} \geq 0$, $a_{11} + a_{22} > 0$, $a_{11}a_{22} > a_{12}a_{21}$.

Also, de Oliveira [3] proved that the operator matrix

$$\mathcal{M} = \begin{pmatrix} a_{11}\Delta & \dots & a_{1n}\Delta \\ \vdots & \vdots & \vdots \\ a_{n1}\Delta & \dots & a_{nn}\Delta \end{pmatrix} \tag{1.2}$$

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with domain $D(\mathcal{M}) = (H^2(\Omega) \cap H_0^1(\Omega))^n$ generates an analytic semigroup on $(L^2(\Omega))^n$ provided that all eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

have positive real part.

In this paper we consider the linear operator

$$A(x,D) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) \end{pmatrix} \tag{1.3}$$

where every element A_{hl} is a symmetric second order differential operator given by

$$A_{hl}(x,D)u = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{jk}(x)\frac{\partial u}{\partial x_k}\Big) \tag{1.4}$$

and one of the diagonal operators A_{11} or A_{22} is strongly elliptic and the other diagonal operator is weakly elliptic and the sum of the non-diagonal operators $A_{12}(x,D) + A_{21}(x,D)$ is also weakly elliptic. Under these assumptions we show that this operator matrix generates an analytic semigroup on $L^2(\Omega) \times L^2(\Omega)$.

2. Preliminaries

Let us consider the differential operator

$$A(x,D) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) \end{pmatrix} \tag{2.1}$$

where

$$A_{hl}(x,D)u = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{jk}^{hl}(x)\frac{\partial u}{\partial x_k}\Big) \quad x \in \overline{\Omega},\ h,l = 1,2 \tag{2.2}$$

under the following assumptions:

(H1) The operators A_{hl} $(h,l = 1,2)$ are symmetric; i.e.,

$$a_{kj}^{hl}(x) = a_{jk}^{hl}(x), \quad x \in \overline{\Omega},\ \text{for all } j,\ k = 1,\dots N \tag{2.3}$$

(H2) The operators A_{hl} $(h = 1,2)$ are regular; i.e.,

$$a_{jk}^{hl}(x) \in C^1(\overline{\Omega};\mathbb{R}), \quad h,l = 1,2 \text{ and } j,k = 1,\dots N \tag{2.4}$$

(H3) One of the diagonal operator A_{11} or A_{22} is strongly elliptic; i.e., there is a constant $\mu > 0$ such that for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$,

$$\sum_{j,k=1}^{N} a_{jk}^{mm}(x)\xi_j\xi_k \ge \mu \sum_{j=1}^{N} \xi_j^2 = \mu|\xi|^2,\ m = 1 \text{ or } m = 2 \tag{2.5}$$

(H4) The other diagonal operator A_{ll} ($l = 2$ if $m = 1$ and $l = 1$ if $m = 2$) is weakly elliptic; i.e., for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$

$$\sum_{j,k=1}^{N} a_{jk}^{ll}(x)\xi_j\xi_k \ge 0, \tag{2.6}$$

(H5) The sum non-diagonal operators $A_{12} + A_{21}$ is weakly elliptic; i.e., for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$

$$\sum_{j,k=1}^{N} (a_{jk}^{12} + a_{jk}^{21})(x)\xi_j\xi_k \geq 0. \tag{2.7}$$

We give now some definitions which will be used in the sequel. We define the operator A with domain

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^2, \tag{2.8}$$

as

$$Au \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} u = A(x, D)u \equiv \begin{pmatrix} A_{11}(x, D)u_1 + A_{12}(x, D)u_2 \\ A_{21}(x, D)u_1 + A_{22}(x, D)u_2 \end{pmatrix} \tag{2.9}$$

where $u = \operatorname{col}(u_1, u_2)$. The following results are well known; see, for instance [5, page 213].

Theorem 2.1. *The operator A_{hl} with domain*

$$D(A_{hl}) = H^2(\Omega) \cap H_0^1(\Omega) \tag{2.10}$$

and defined by

$$A_{hl}u = A_{hl}(x, D)u \tag{2.11}$$

is closed.

Theorem 2.2. *Let $1 \leq p < \infty$, $L_n^p(\Omega) = \prod_{j=1}^n L^p(\Omega)$, and $(L_n^p(\Omega))'$ the dual space of $L_n^p(\Omega)$. Then, to every $\varphi \in (L_n^p(\Omega))'$ there corresponds unique $v = (v_1, \dots, v_n) \in L_n^q(\Omega)$ such that for every $u = (u_1, \dots, u_n) \in L_n^p(\Omega)$:*

$$\varphi(u) = \sum_{j=1}^{n} \langle u_j, vj \rangle \tag{2.12}$$

Moreover, $\|\varphi; (L_n^p(\Omega))'\| = \|v; L_n^q(\Omega)\|$, where q is the conjugate exponent of p and $\langle u_k, v_k \rangle = \int_\Omega u_k(x)v_k(x)dx$. Therefore, $(L_n^p(\Omega))' \sim L_n^q(\Omega)$.

For a proof of the above theorem, see [1, page 47].

Definition 2.3. Let X be a Banach space and let X^* be its dual. For every $x \in X$, the duality set is defined by

$$J(x) = \{x^* \in X^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.13}$$

3. Main results

Theorem 3.1. *Assume that* (2.1)-(2.11) *hold. Then, the operator A generates a strongly continuous semigroup of contractions on the space $X = L^2(\Omega) \times L^2(\Omega)$ endowed with the norm $\|u\| = (\|u_1\|_2^2 + \|u_2\|_2^2)^{1/2}$, where $u = (u_1, u_2)$ and $\|u_1\|_2^2 = \int_\Omega |u_1(x)|^2 dx$.*

To prove this theorem we will need some lemmas.

Lemma 3.2. *For every $\lambda > 0$ and $u \in D(A)$ we have*

$$\lambda \|u\| \leq \|(\lambda I + A)u\| \tag{3.1}$$

Proof. We denote the pairing between $L_2^2(\Omega)$ and itself by $\langle , \rangle$. If $u = \mathrm{col}(u_1, u_2) \in D(A)\backslash\{0\}$ then the function $u^* = \mathrm{col}(u_1^*, u_2^*)$ is in the duality map $J(u)$ (see Definition 2.3 and Theorem 2.1), where $u_h^* = \overline{u_h}$ for $h = 1, 2$. We have

$$\langle Au, u^* \rangle = \langle A_{11}u_l, u_1^* \rangle + \langle A_{22}u_2, u_2^* \rangle + \langle A_{12}u_2, u_1^* \rangle + \langle A_{21}u_1, u_2^* \rangle \tag{3.2}$$

Integration by parts yields

$$\begin{aligned}\langle A_{hh}u_h, u_h^* \rangle &= -\int_\Omega \sum_{j,k=1}^N \frac{\partial}{\partial x_j}(a_{jk}^{hh}(x)\frac{\partial u_h}{\partial x_k})\overline{u_h}dx \\ &= \int_\Omega \sum_{j,k=1}^N a_{jk}^{hh}(x)\frac{\partial u_h}{\partial x_k}\frac{\partial \overline{u_h}}{\partial x_j}dx \,.\end{aligned}$$

Denoting

$$\frac{\partial u_h}{\partial x_j} = \alpha_{hj} + i\beta_{hj}, \quad h = 1, 2,\ j = 1, \dots, N$$

where $\alpha_{hj}, \beta_{hj} \in \mathbb{R}$, we find that

$$\langle A_{hh}u_h, u_h^* \rangle = \int_\Omega \sum_{j,k=1}^N a_{jk}^{hh}(x)(\alpha_{hk}\alpha_{hj} + \beta_{hk}\beta_{hj})dx, \quad h = 1, 2\,. \tag{3.3}$$

Also, integrating by parts we have

$$\begin{aligned}\langle A_{12}u_2, u_1^* \rangle &= \int_\Omega \sum_{j,k=1}^N a_{jk}^{12}(x)(\alpha_{1j}\alpha_{2k} + \beta_{1j}\beta_{2k})dx \\ &\quad + i\Big(\int_\Omega \sum_{j,k=1}^N a_{jk}^{12}(x)(\alpha_{1j}\beta_{2k} - \alpha_{2k}\beta_{1j})dx\Big)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\langle A_{21}u_1, u_2^* \rangle &= \int_\Omega \sum_{j,k=1}^N a_{jk}^{21}(x)(\alpha_{1k}\alpha_{2j} + \beta_{1k}\beta_{2j})dx \\ &\quad + i\Big(\int_\Omega \sum_{j,k=1}^N a_{jk}^{21}(x)(\alpha_{2j}\beta_{1k} - \alpha_{1k}\beta_{2j})dx\Big)\end{aligned} \tag{3.5}$$

Then substituting (3.3)–(3.5) into (3.2) yields

$$\begin{aligned}\langle Au, u^* \rangle &= \sum_{h=1}^2 \int_\Omega \sum_{j,k=1}^N a_{jk}^{hh}(x)(\alpha_{hk}\alpha_{hj} + \beta_{hk}\beta_{hj})dx \\ &\quad + \int_\Omega \sum_{j,k=1}^N (a_{jk}^{12} + a_{jk}^{21})(x)(\alpha_{1j}\alpha_{2k} + \beta_{1j}\beta_{2k})dx \\ &\quad + i\Big\{\int_\Omega \sum_{j,k=1}^N (a_{jk}^{12} - a_{jk}^{21})(x)(\alpha_{1j}\beta_{2k} - \alpha_{2j}\beta_{1k})dx\Big\}\end{aligned} \tag{3.6}$$

Set

$$|\alpha_h|^2 = \sum_{j=1}^N \int_\Omega \alpha_{hj}^2 dx, \quad |\beta_h|^2 = \sum_{j=1}^N \int_\Omega \beta_{hj}^2 dx, \quad h = 1,\ 2 \tag{3.7}$$

Then from (3.6)–(3.7) and using (H3)-(H5), we have that the real part of $\langle Au, u^*\rangle$ satisfies

$$\operatorname{Re}\langle Au, u^*\rangle \geq 2\mu(\sum_{h=1}^{2} |\alpha_h|^2 + \sum_{h=1}^{2} |\beta_h|^2) \geq 0 \tag{3.8}$$

From (3.8), the linear operator $-A$ is dissipative. It follows that for every $\lambda > 0$ and $u \in D(A)$ we have $\lambda\|u\| \leq \|(\lambda I + A)u\|$ (see [5, page 14]. □

Lemma 3.3. *The operator A is closed.*

Proof. The adjoint operator of A is

$$A^* = \begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix} \tag{3.9}$$

where A_{hl}^* is the adjoint operator of A_{hl}, for $h, l = 1, 2$. As the domain $D(A^*) = D(A)$ is dense in $L_2^2(\Omega)$, then the operator $(A^*)^*$ is closed (see [2, page 28]). Also, as $L^2(\Omega)$ is reflexive, then $L^2(\Omega) \times L^2(\Omega)$ is reflexive (see [1, page 8]); whence $(A^*)^* = A$ [2, page 46]. We finally conclude that A is closed. □

Lemma 3.4. *for every $\lambda > 0$, the operator $\lambda I + A$ is bijective.*

Proof. From (3.1) it follows that $\lambda I + A$ is injective. As in lemma 3.2, we can prove that for every every $\lambda > 0$ and $u \in D(A)$,

$$\lambda\|u\| \leq \|(\lambda I + A)^* u\| \tag{3.10}$$

then the operator $((\lambda I + A)^*)^* = \lambda I + A$ is surjective (see [2, page 30]). □

Proof of Theorem 3.1. The domain $D(A)$ of A contains $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ and it is therefore dense in $X \equiv L^2(\Omega) \times L^2(\Omega)$. Also, A is closed and as a consequence of Lemmas 3.2 and 3.4 we have

$$\|(\lambda I + A)^{-1}\| \leq \frac{1}{\lambda}, \quad \text{for all } \lambda > 0 \tag{3.11}$$

The Hille-Yosida theorem [5, page 8] now implies that $-A$ is the infinitesimal generator of a strongly continuous semigroup of contractions on $L^2(\Omega) \times L^2(\Omega)$. □

Theorem 3.5. *The semigroup generated in theorem 3.1 is also analytic.*

Proof. Let X be a Banach space and let X^* be its dual. If $A : X \to X$ is a linear operator in X, the numerical range of A is the set

$$\mathcal{N}(A) = \{\langle x^*, Ax\rangle : x \in D(A),\ x^* \in X^*,\ \langle x^*, x\rangle = \|x\| = \|x^*\| = 1\} \tag{3.12}$$

If we put

$$|a_{jk}^{hl}(x)| \leq M, \quad \text{for all } h, l = 1, 2 \text{ and } j, k = 1, \dots, N \tag{3.13}$$

we get from (3.6) that the imaginary part of $\langle Au, u^*\rangle$

$$|\operatorname{Im}\langle Au, u^*\rangle| \leq M\Big(\sum_{h=1}^{2} |\alpha_h|^2 + \sum_{h=1}^{2} |\beta_h|^2\Big) \tag{3.14}$$

and hence from (3.8) and (3.14), we find that

$$\frac{|\operatorname{Im}\langle Au, u^*\rangle|}{|\operatorname{Re}\langle Au, u^*\rangle|} \leq \frac{M}{2\mu} \tag{3.15}$$

We observe by (3.8) and (3.15) that the numerical range $\mathcal{N}(-A)$ of $-A$ is contained in the set $N_\varphi = \{\lambda : |\arg\lambda| > \pi - \varphi\}$ where $\varphi = \arctan(NM/(2\mu))$, $0 < \varphi < \pi/2$. Choosing $\varphi < \theta < \pi/2$ and denoting

$$\mathcal{S}_\theta = \{\lambda : |\arg\lambda| < \pi - \theta\} \tag{3.16}$$

It follows that there is a constant $C_\theta = \sin(\theta - \varphi) > 0$ for which the distance of λ from $\mathcal{N}(-A)$

$$d(\lambda, \overline{\mathcal{N}(-A)}) \geq C_\theta|\lambda|, \quad \text{for } \lambda \in \mathcal{S}_\theta$$

Since $\lambda > 0$ is in the resolvent set $\rho(-A)$ of the operator $-A$ by Theorem 3.1, it follows from [5, Theorem 1.3.9] that $S_\theta \subset \rho(-A)$ and that

$$\|(\lambda I + A)^{-1}\| \leq \frac{1}{C_\theta|\lambda|}, \quad \text{for all } \lambda \in \mathcal{S}_\theta \tag{3.17}$$

Whence by [5, Theorem 2.5.2], the operator $-A$ is the infinitesimal generator of an analytic semigroup on the space $X = L^2(\Omega) \times L^2(\Omega)$. □

4. Generalization

The above results are also true for the operator

$$A(x,D) = \begin{pmatrix} A_{11}(x,D) & A_{12}(x,D) & \cdots & A_{1n}(x,D) \\ A_{21}(x,D) & A_{22}(x,D) & \cdots & A_{2n}(x,D) \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1}(x,D) & A_{n2}(x,D) & \cdots & A_{nn}(x,D) \end{pmatrix},$$

where

$$A_{hl}(x,D)u = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\Big(a_{j,k}^{hl}(x)\frac{\partial u}{\partial x_k}\Big), \quad x \in \overline{\Omega},\ h,l = 1,\dots,N,$$

under the following assumptions:

(A1) The operators A_{hh} $(h = 1,\dots,n)$ are symmetric; i.e.,
$$a_{kj}^{hh}(x) = a_{jk}^{hh}(x), x \in \overline{\Omega}, \quad \text{for all } j,k = 1,\dots,N.$$

(A2) The operators A_{hl} $(h = 1,\dots,n)$ are regular; i.e., for all $h,l = 1,\dots,n$
$$a_{jk}^{hl}(x) \in C^1(\overline{\Omega};\mathbb{R}), \quad j,k = 1,\dots,N.$$

(A3) There exists $m \in \{1,\dots,n\}$ such that the diagonal operator A_{mm} is strongly elliptic; i.e., there is a constant $\mu > 0$ such that for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$,
$$\sum_{j,k=1}^{N} a_{jk}^{mm}(x)\xi_j\xi_k \geq \mu\sum_{j=1}^{N}\xi_j^2 = \mu|\xi|^2$$

(A4) The other diagonal operators A_{ll} $(l \neq m)$ are weakly elliptic; i.e., for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$
$$\sum_{j,k=1}^{N} a_{jk}^{ll}(x)\xi_j\xi_k \geq 0, \quad \text{for all } l \neq m.$$

(A5) The operators sums $A_{hl} + A_{lh}$ $(h \neq l)$ are weakly elliptic; i.e., for all $\xi = (\xi_j)_{j=1}^N \in \mathbb{R}^N$ and all $x \in \Omega$

$$\sum_{j,k=1}^{N} (a_{jk}^{hl} + a_{jk}^{lh})(x)\xi_j\xi_k \geq 0.$$

By examining the proof of the Theorem 3.5, we note that the above results remain true if we assume only that one of the operators A_{hh} $(h = 1, \dots, n)$, $A_{hl}+A_{lh}$ $(h \neq l$, $h, l = 1, \dots, n)$ is strongly elliptic and the rest of them are all weakly elliptic.

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