Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 168, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR CONVEX SWEEPING PROCESSES IN p-UNIFORMLY SMOOTH AND q-UNIFORMLY CONVEX BANACH SPACES

MESSAOUD BOUNKHEL

Abstract. We show the existence of at least one Lipschitz solution for extensions of convex sweeping processes in reflexive smooth Banach spaces. Our result is proved under a weaker assumption on the moving set than those in [3], and using a different discretization.

1. Main result

Bounkhel and Al-yusof [3] studied the following extension of the convex sweeping processes from Hilbert spaces H to reflexive smooth Banach spaces X:

(SP) Find $u : [0,T] \to X$ such that $u(t) = u_0 + \int_0^t \dot{u}(s)ds$,

$$-\frac{d}{dt}(J(u(t))) \in N(C(t); u(t)) \text{ a.e. in } [0,T] \text{ and } u(t) \in C(t), \forall t \in [0,T],$$

where $J : X \to X^*$ is the duality mapping defined from X into X^* (see Section 2 for the definition).

Clearly, (SP) coincides with the well known convex sweeping process introduced and studied in [8] in the Hilbert space setting in which J is the identity mapping. The authors in [3] proved the following theorem.

Theorem 1.1. *Let $p, q > 1$, X be a p-uniformly convex and q-uniformly smooth Banach space, $T > 0$, $I = [0,T]$ and $C : I \rightrightarrows X$ be a set-valued mapping closed convex values satisfying for any $t, t' \in I$ and any $x \in X$*

$$|(d^V_{C(t')})^{1/q'}(\psi) - (d^V_{C(t)})^{1/q'}(\phi)| \le \lambda|t' - t| + \gamma\|\psi - \phi\|, \tag{1.1}$$

where $\lambda, \gamma > 0$, and $q' = \frac{q}{q-1}$. Assume that

$$J(C(t)) \subset K, \forall t \in I \text{ for some convex compact set } K \text{ in } X^*. \tag{1.2}$$

Then (SP) *has at least one Lipschitz solution*

They proved the existence of solutions under the Lipschitz continuity of the function $(t, \psi) \mapsto (d^V_{C(t)})^{1/q'}(\psi)$ defined on $I \times X^*$, and under the compactness assumption (1.2). Using a different discretization we prove the previous theorem

2000 *Mathematics Subject Classification.* 34A60, 49J53.
Key words and phrases. Uniformly smooth and uniformly convex Banach spaces; state dependent sweeping process; generalized projection; duality mapping.

Submitted February 14, 2012. Published October 4, 2012.

under the boundedness of C and the compactness of their values which is clearly weaker than the compactness assumption (1.2), and under the Lipschitz continuity of the usual distance function $t \mapsto (d_{C(t)})^{1/q'}(u)$, for all $u \in X$, defined on I which is easier to handle with, than the function used in (1.1). Although, both Lipschitz assumptions coincide in the Hilbert space setting, in the case of Banach spaces the Lipschitz continuity of the distance function is easier to be checked than (1.1).

Before proving our main result in Theorem 3.1, we recall from [3] some needed concepts and results and for more details we refer the reader to [3] and the references therein.

2. Preliminaries

Let X be a Banach space with topological dual space X^*. We denote by d_S the usual distance function to S; i.e., $d_S(x) := \inf_{u \in S} \|x - u\|$. Let S be a nonempty closed convex set of X and $\bar{x}$ be a point in S. The convex normal cone of S at $\bar{x}$ is defined by (see for instance [6])

$$N(S;\bar{x}) = \{\varphi \in X^* : \langle \varphi, x - \bar{x}\rangle \leq 0 \text{ for all } x \in S\}. \tag{2.1}$$

The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) = \{j(x) \in X^* : \langle j(x), x\rangle = \|x\|^2 = \|j(x)\|^2\}.$$

Many properties of the normalized duality mapping J have been studied. For the details, one may see the books [1, 10, 11]. Let $V : X^* \times X \to \mathbb{R}$ be defined by

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x\rangle + \|x\|^2, \text{ for any } \varphi \in X^* \text{ and } x \in X.$$

Based on the functional V, a set $\pi_S(\varphi)$ of generalized projections of $\varphi \in X^*$ onto S is defined as follows (see [2]).

Definition 2.1. Let S be a nonempty subset of X and $\varphi \in X^*$. If there exists a point $\bar{x} \in S$ satisfying

$$V(\varphi, \bar{x}) = \inf_{x \in S} V(\varphi, x),$$

then $\bar{x}$ is called a generalized projection of φ onto S. The set of all such points is denoted by $\pi_S(\varphi)$. When the space X is not reflexive $\pi_S(\varphi)$ may be empty for some elements $\varphi \in X^*$ even when S is closed and convex (see [7, Example 1.4]).

The two following propositions are needed in the proof of the main theorem. For their proofs we refer the reader to [5, 9] respectively.

Proposition 2.2. *Let S be a nonempty closed convex subset of X and $x \in S$. Then*

$$\partial d_S(x) = N_S(x) \cap \mathbf{B}.$$

Proposition 2.3. *For a nonempty closed convex subset S of a reflexive smooth Banach space X and $u \in S$, the following assertions are equivalent:*

(i) *$\bar{x} \in S$ is a projection of u onto S, that is $\bar{x} \in P_S(u)$;*
(ii) *$\langle J(u - \bar{x}), x - \bar{x}\rangle \leq 0$ for all $x \in S$;*
(iii) *$J(u - \bar{x}) \in N(S;\bar{x})$.*

Assume now that X is p-uniformly convex and q-uniformly smooth Banach space and let S be closed nonempty set in X. Recall the definition of the function $d_S^V : X^* \to [0, \infty[$, given by $d_S^V(\varphi) = \inf_{x \in S} V(\varphi, x)$. Clearly, in Hilbert spaces, d_S^V coincides with d_S^2. We need the two following lemmas proved in [3].

Lemma 2.4. *Let $p, q > 1$, X be a p-uniformly convex and q-uniformly smooth Banach space, and let S be a bounded set. Then there exist two constants $\alpha > 0$ and $\beta > 0$ so that $\alpha\|x-y\|^p \le V(J(x), y) \le \beta\|x-y\|^q$, for all $x, y \in S$.*

Proposition 2.5. *If S is a bounded set in X, then $d_S^V(\varphi) \le \beta(d_S(J^*(\varphi)))^q$, where β depends on the bound of S and on φ. As a consequence, for sets S_1 and S_2 in X and X^* bounded by l_1 and l_2 respectively, we have $d_S^V(\varphi) \le \beta(d_S(J^*(\varphi)))^q$, for all $\varphi \in S_2$, where β depends on l_1 and l_2.*

The following proposition is taken from [1].

Proposition 2.6. *Let $p \ge 2$ and let X be a p-uniformly convex and q-uniformly smooth Banach space. The duality mapping $J : X \to X^*$ is Lipschitz on bounded sets; that is,*

$$\|J(x) - J(y)\| \le C(R)\|x-y\|, \text{ for all } \|x \le R, \|y\| \le R.$$

Here $C(R) := 32Lc_2^2(q-1)^{-1}$ and $c_2 = \max\{1, R\}$ and $1 < L < 1.7$. The Lipschitz continuity on bounded sets of the duality mapping J_* on X^*, follows from the fact that X^* is p'-uniformly convex and q'-uniformly smooth Banach space with p' and q' are the conjugate numbers of p and q respectively; i.e., $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

The following proposition summarizes some results proved in [4, 7].

Proposition 2.7. *Let X be a reflexive Banach space with dual space X^* and S be a nonempty, closed and convex subset of X. The following properties hold:*

(π1) *$\pi_S(\varphi) \ne \emptyset$, for any $\varphi \in X^*$;*
(π2) *If X is also smooth, then $\varphi \in N(S, \bar{x})$, if and only if, there exists $\alpha > 0$ such that $\bar{x} \in \pi_S(J(\bar{x}) + \alpha\varphi)$.*

3. Main result

Now, we are ready to prove our main result in the following theorem.

Theorem 3.1. *Instead of* (1.1) *and* (1.2) *in Theorem 1.1, assume that C is bounded with compact values and that*

$$|(d_{C(t')})^{p/q}(u) - (d_{C(t)})^{p/q}(u)| \le \lambda|t' - t|. \tag{3.1}$$

Then (SP) *has at least one Lipschitz solution.*

Proof. Assume that $T = 1$. Consider $\forall n \in N$ the following partition of I

$$I_{n,i} = (t_{n,i}, t_{n,i+1}], \quad t_{n,i} = \frac{i}{n}, \quad 0 \le i \le n-1, \quad I_{n,0} = \{0\}.$$

Put $\mu_n = 1/n$. Fix $n \ge 2$. Define by induction

$$u_{n,0} = u_0 \in C(0);$$
$$u_{n,i+1} \in \pi(C(t_{n,i+1}); u_{n,i}), \quad \text{for } 0 \le i \le n-1,$$

and

$$u_n(t) := J^*(u_n^*(t))$$
$$u_n^*(t) := J(u_{n,i}) + \frac{(t - t_{n,i})}{\mu_n}(J(u_{n,i+1}) - J(u_{n,i})), \quad \text{for all } t \in I_{n,i}$$

and $u_n^*(0) = J(u_0)$. The construction is well defined since the generalized projection π exits by Proposition 2.7. Clearly u_n^* and u_n are continuous on all I and u_n^* is differentiable on $I \setminus \{t_{n,i}\}$ and $\dot{u}_n^*(t) = \frac{J(u_{n,i+1})-J(u_{n,i})}{\mu_n}$, for all $t \in I \setminus \{t_{n,i}\}$.

Let us find an upper bound estimate for the expression $\|J(u_{n,i+1}) - J(u_{n,i})\|$. First, we have to point out that the sequence u_i^n is bounded by some l because the set-valued mapping C is bounded. Now, since X is q-uniformly smooth and p-uniformly convex and the sequence u_i^n is bounded by l, there exist some constants α and β depending on l such that

$$\alpha\|u_{n,i+1} - u_{n,i}\|^p \le V(J(u_{n,i}), u_{n,i+1}) \le \beta\|u_{n,i+1} - u_{n,i}\|^q,$$

and so by the construction of the sequence u_i^n and Proposition 2.5 we obtain

$$\alpha\|u_{n,i+1}) - u_{n,i}\|^p \le d^V_{C(t_{n,i+1})}(J(u_{n,i})) \le \beta d^q_{C(t_{n,i+1})}(u_{n,i})$$

and so by the Lipschitz continuity in (3.1) we obtain

$$\begin{aligned}(\frac{\alpha}{\beta})^{\frac{1}{p}}\|u_{n,i+1}) - u_{n,i}\| &\le d^{q/p}_{C(t_{n,i+1})}(u_{n,i}) - d^{q/p}_{C(t_{n,i})}(u_{n,i})\\ &\le \lambda|t_{n,i+1} - t_{n,i}| = \lambda\mu_n,\end{aligned}$$

and so

$$\|u_{n,i+1}) - u_{n,i}\| \le \bar{\lambda}\mu_n,$$

where $\bar{\lambda} = (\frac{\beta}{\alpha})^{\frac{1}{p}}\lambda$. Using now the Lipschitz property of the duality mapping J in Proposition 2.6, we can write

$$\|J(u_{n,i+1}) - J(u_{n,i})\| \le C(l)\|u_{n,i+1} - u_{n,i}\| \le C(l)\bar{\lambda}\mu_n.$$

This inequality ensures the Lipschitz continuity of u_n^* on all I with ratio $\delta := C(l)\bar{\lambda}$. Using the characterization of the normal cone, in terms of the generalized projection π projection operator stated in Proposition 2.7, we can write for a.e. $t \in I$

$$J(u_{n,i+1}) - J(u_{n,i}) \in -N(C(t_{n,i+1}); u_{n,i+1}),$$

which ensures together with Proposition 2.2 that

$$-\frac{J(u_{n,i+1}) - J(u_{n,i})}{\mu_n} \in \delta\partial d_{C(t_{n,i+1})}(u_{n,i+1}).$$

Define now on $I_{n,i}$ the functions $\theta_n : I \to I$ by $\theta_n(0) = 0$, and

$$\theta_n(t) = t_{n,i+1}, \quad \text{for all } t \in I_{n,i}.$$

Then the above inclusion becomes

$$-\dot{u}_n^*(t) \in \delta\partial d_{C(\theta_n(t))}(u_n(\theta_n(t))). \tag{3.2}$$

Now, let us prove that the sequence (u_n) has a convergent subsequence. Clearly, we have $B = \{u_n; n \ge 2\}$ is equi-Lipschitz and bounded. So it remains to prove that $B(t) = \{u_n(t); n \ge 2\}$ is relatively compact in X, for all $t \in I$. By construction we have

$$u_n(\theta_n(t)) \in C(\theta_n(t)), \quad \forall t \in I \text{ and all } n \ge 2, \tag{3.3}$$

and hence by the Lipschitz property of $d_C^{p/q}$ and the equi-Lipschitz property of u_n we can write

$$\begin{aligned}d^{p/q}_{C(t)}(u_n(t)) &= d^{p/q}_{C(t)}(u_n(t)) - d^{p/q}_{C(\theta_n(t))}(u_n(\theta_n(t)) \le \lambda\mu_n + \|u_n(\theta_n(t)) - u_n(t)\|\\ &\le (\lambda + \delta)\mu_n.\end{aligned}$$

Assume by contradiction that $B(t_0)$ is not relatively compact in X for some $t_0 \in I$. So, $\gamma(B(t_0)) \geq 2\bar{\delta} > 0$, for some $\bar{\delta} \in (0,1]$. Fix now $n_0 \in \mathbb{N}$ such that $\mu_n \leq \mu_{n_0} < \frac{(\frac{\bar{\delta}}{2})^{p/q}}{\lambda+\delta}$, for all $n \geq n_0$. So

$$u_n(t) \in C(t) + (\lambda+\delta)^{q/p}\mu_{n_0}^{q/p}\mathbb{B}, \quad \text{for all } n \geq n_0 \text{ and all } t \in I,$$

which implies

$$B(t) \subset C(t) + (\lambda+\delta)^{q/p}\mu_{n_0}^{q/p}\mathbb{B}, \quad \text{for all } t \in I.$$

Then the properties of γ and the compactness of the values of C imply

$$\begin{aligned}\gamma(B(t_0)) &= \gamma(\{u_n(t_0) : n \geq n_0\}) \leq \gamma((C(t_0)) + \gamma((\lambda+\delta)^{q/p}\mu_{n_0}^{q/p}\mathbb{B}) \\ &\leq 2(\lambda+\delta)^{q/p}\mu_{n_0}^{q/p} < \bar{\delta},\end{aligned}$$

which is a contradiction. Therefore, the set $B(t)$ is relatively compact in X for any $t \in I$. Thus, Arzela-Ascoli theorem concludes that (u_n) has a subsequence (still denoted u_n) converging uniformly to some u. Since $\lim_n \theta_n(t) = t$, we can write $\lim_n u_n(\theta_n(t)) = \lim_n u_n(t) = u(t)$ uniformly on I. So the sequence $u_n^* = J(u_n)$ will converge uniformly to $u^* = J(u)$ on I, since J is uniformly continuous on bounded sets. We also have $(\dot{u}_n^*)$ converges weakly star in $L^\infty(I, X^*)$ to some w. So, by the reflexivity and the separability of the space X, we can write

$$u^*(t) = J(u(t)) = \lim_n u_n^*(t) = \lim_n \Big(u_n^*(0) + \int_0^t \dot{u}_n^*(s)ds\Big) = u_0 + \int_0^t w(s)ds.$$

Hence $\dot{u}^*(t) = \frac{d}{dt}J(u(t)) = w(t)$ a.e. on I. Let us prove that u is the solution of our problem. First, we have to prove that $u(t) \in C(t)$, for all $t \in I$. Using now the Lipschitz property of the function $t \mapsto d_{C(t)}^{q/p}$ to write for all $t \in I$

$$\begin{aligned}d_{C(t)}^{q/p}(u_n(\theta_n(t)) &= d_{C(t)}^{q/p}(u_n(\theta_n(t)) - d_{C(\theta_n(t))}^{q/p}(u_n(\theta_n(t)) \\ &\leq \lambda|\theta_n(t) - t| \leq \lambda\mu_n,\end{aligned}$$

and so

$$\begin{aligned}d_{C(t)}(u(t)) &= d_{C(t)}(u_n(\theta_n(t)) + \|u_n(\theta_n(t) - u(t)\| \\ &\leq (\lambda\mu_n)^{p/q} + \|u_n(\theta_n(t)) - u(t)\| \to 0,\end{aligned}$$

as $n \to \infty$, by the fact that $\lim_n u_n(\theta_n(t)) = u(t)$ uniformly on I. So the closedness of the set $C(t)$ ensures $u(t) \in C(t)$, for all $t \in I$. Going back to (3.2) we have

$$-\dot{u}_n^*(t)) \in N(C(\theta_n(t)); u_n(\theta_n(t))), \quad \text{a.e. on } I.$$

So, Proposition 2.2 ensures for a.e. $t \in I$,

$$\langle -\dot{u}_n^*(t)); x - u_n(\theta_n(t))\rangle \leq 0, \quad \forall x \in C(\theta_n(t)). \tag{3.4}$$

Using the fact that $\dot{u}_n^*$ converges to $\frac{d}{dt}J(u(\cdot))$ in the weak star topology of $L^\infty(I, X^*)$, we can pass to the limit in (3.4) to obtain

$$\langle -\frac{d}{dt}J(u(t)); x - u(t)\rangle \leq 0, \quad \forall x \in C(t), \text{ a.e. on } I. \tag{3.5}$$

Indeed, fix $t \in I$, for which $\dot{u}_n^*(t)$ exists and converges weakly to $\frac{d}{dt}J(u(t))$, and let x be any element in $C(t)$. Then, we have

$$x \in C(\theta_n(t)) + (\lambda\mu_n)^{q/p}\mathbb{B};$$

that is, $x = y_n(t) + (\lambda\mu_n)^{q/p} b_n$, with $b_n \in \mathbb{B}$ and $y_n(t) \in C(\theta_n(t))$. Hence (3.4) yields

$$
\begin{aligned}
&\langle -\frac{d}{dt} J(u(t)), x - u(t)\rangle \\
&= \langle -\frac{d}{dt} J(u(t)) + \dot{u}_n^*(t)), x - u(t)\rangle + \langle -\dot{u}_n^*(t)), x - u(t)\rangle \\
&= \langle -\frac{d}{dt} J(u(t)) + \dot{u}_n^*(t)), x - u(t)\rangle + \langle -\dot{u}_n^*(t)), u_n(\theta_n(t)) - u(t)\rangle \\
&\quad + \langle -\dot{u}_n^*(t), y_n(t) - u_n(\theta_n(t))\rangle + \langle -\dot{u}_n^*(t)), (\lambda\mu_n)^{q/p} b_n\rangle \\
&\le \langle \dot{u}_n^*(t)) - \frac{d}{dt} J(u(t)), x - u(t)\rangle + \lambda(\lambda\mu_n)^{q/p} + \lambda \| u_n\big(\theta_n(t) - u(t)\big)\| \to 0
\end{aligned}
$$

as $n \to \infty$. So,

$$\langle -\frac{d}{dt} J(u(t)), x - u(t)\rangle \le 0, \quad \text{for all } x \in C(t), \tag{3.6}$$

which by Proposition 2.2 gives

$$-\frac{d}{dt} J(u(t)) \in N(C(t); u(t)), \quad \text{a.e. on } I \tag{3.7}$$

and hence the proof is complete. □

Acknowledgments. The author extends his appreciation to the Deanship of Scientific Research at King Saud University for funding the work through the research group project No. RGP-VPP-024.

References

[1] Y. Alber, I. Ryazantseva; *Nonlinear ill-posed problems of monotone type.* Springer, Dordrecht, 2006.

[2] Y. Alber; *Generalized Projection Operators in Banach Spaces: Properties and Applications,* Funct. Different. Equations 1 (1), 1-21 (1994).

[3] M. Bounkhel, R. Al-yusof; *First and second order Convex Sweeping Processes in Reflexive smooth Banach spaces*, Set-Valued Var. Anal. 18 (2010), no. 2, 151–182.

[4] M. Bounkhel, R. Al-yusof; *Proximal Analysis in reflexive smooth Banach spaces*, Nonlinear Analysis: Theory, Methods & Applications, Volume 73, Issue 7, 1 October 2010, Pages 1921-1939.

[5] M. Bounkhel, L. Thibault; *On various notions of regularity of sets*, Nonlinear Anal.: Theory, Methods and Applications, Vol. **48** (2002), No. 2, 223-246.

[6] F. H. Clarke; *Optimization and Nonsmooth Analysis*, John Wiley & Sons Inc., New York, 1983.

[7] J. Li; *The generalized projection operator on reflexive Banach spaces and its applications*, J. Math. Anal. Appl. Vol. 306. 2005, 55-71.

[8] J. J. Moreau; *Evolution problem associated with a moving convex set in a Hilbert space*, J. Diff. Eqs., 26, 347-374 (1977).

[9] J. P. Penot, R. Ratsimahalo; *Characterizations of metric projections in Banach spaces and applications,* Abstr. Appl. 3(1998), no. 1-2, 85-103.

[10] W. Takahashi; *Nonlinear Functional Analysis*, 2000. Yokohama Publishers.

[11] M. M. Vainberg; *Variational methods and method of monotone operators in the theory of nonlinear equations*, John Wiley, New York, 1973.

Messaoud Bounkhel

King Saud University, Department of Mathematics, P.O. Box 2455, Riyadh 11451, Riyadh, Saudi-Arabia

E-mail address: bounkhel@ksu.edu.sa, bounkhel_messaoud@yahoo.fr