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EXISTENCE OF SOLUTIONS TO QUASILINEAR ELLIPTIC SYSTEMS WITH COMBINED CRITICAL SOBOLEV-HARDY TERMS

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Abstract. This article is devoted to the study of multiple positive solutions to a singular elliptic system where the nonlinearity involves a combination of concave and convex terms. Using the effect of the coefficient of the critical nonlinearity, and a variational method, we establish the main result which is based on a compactness argument.

1. Introduction

The aim of this paper is to establish the existence of nontrivial solution to the elliptic system

$$\begin{gathered}
-\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(s_1)-2}u}{|x|^{s_1}} + \frac{\alpha}{\alpha+\beta}Q(x)\frac{|u|^{\alpha-2}|v|^{\beta}u}{|x-x_0|^t} + \lambda h(x)\frac{|u|^{q-2}u}{|x|^s},\\
-\Delta_p v - \mu\frac{|v|^{p-2}v}{|x|^p} = \frac{|v|^{p^*(s_2)-2}v}{|x|^{s_2}} + \frac{\beta}{\alpha+\beta}Q(x)\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x-x_0|^t} + \lambda h(x)\frac{|v|^{q-2}v}{|x|^s},\\
x\in\Omega,\\
u = v = 0, \quad x\in\partial\Omega
\end{gathered} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $0\in\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ with smooth boundary $\partial\Omega$, $\lambda>0$ is a parameter, $1\leq q<p$, $1<p<N$, $0\leq\mu<\overline{\mu}\triangleq\big(\frac{N-p}{p}\big)^p$; $Q(x)$ is nonnegative and continuous on $\overline{\Omega}$ satisfying some additional conditions which will be given later, $Q(x_0)=\|Q\|_\infty$ for $0\neq x_0\neq\Omega$, $h(x)\in C(\overline{\Omega})$; $\alpha,\beta>1$, $\alpha+\beta=p^*(t)\triangleq\frac{p(N-t)}{N-p}$, $p^*(s)\triangleq\frac{p(N-s)}{N-p}$ $(0<s,s_1,s_2\leq t<p)$ are critical Sobolev-Hardy exponents. Note that $p^*(0)=p^*:=\frac{Np}{N-p}$ is the critical Sobolev exponent.

We denote by $W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\big(\int_\Omega|\nabla\cdot|^p dx\big)^{1/p}$.

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Problem (1.1) is related to the well known Caffarelli-Kohn-Nirenberg inequality in [3]:

$$\Big(\int_\Omega \frac{|u|^r}{|x|^t}dx\Big)^{p/r} \le C_{r,t,p}\int_\Omega |\nabla u|^p dx, \quad \text{for all } u\in W_0^{1,p}(\Omega), \tag{1.2}$$

where $p\le r<p^*(t)$. When $t=r=p$, the above inequality becomes the well known Hardy inequality [3, 9, 10]:

$$\int_\Omega \frac{|u|^p}{|x|^p}dx \le \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^p dx, \quad \text{for all } u\in W_0^{1,p}(\Omega). \tag{1.3}$$

In the space $W_0^{1,p}(\Omega)$ we use the norm

$$\|u\|_\mu = \|u\|_{D^{1,p}(\Omega)} := \Big(\int_\Omega \Big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\Big)dx\Big)^{1/p}, \quad \mu\in[0,\overline{\mu}).$$

By using the Hardy inequality (1.3) this norm is equivalent to the usual norm $\big(\int_\Omega |\nabla u|^p dx\big)^{1/p}$. The elliptic operator $L := \big(|\nabla\cdot|^{p-2}\nabla\cdot - \mu\frac{|\cdot|^{p-2}}{|x|^p}\big)$ is positive in $W_0^{1,p}(\Omega)$ if $0\le\mu<\overline{\mu}$.

Now, we define the space $W=W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ with the norm

$$\|(u,v)\|^p = \|u\|_\mu^p + \|v\|_\mu^p.$$

Also, by Hardy inequality and Hardy-Sobolev inequality, for $0\le\mu<\overline{\mu}$, $0\le t<p$ and $p\le r\le p^*(t)$ we can define the best Hardy-Sobolev constant:

$$A_{\mu,t,r}(\Omega) = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \Big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\Big)dx}{\Big(\int_\Omega \frac{|u|^r}{|x|^t}dx\Big)^{p/r}}. \tag{1.4}$$

In the important case when $r=p^*(t)$, we simply denote $A_{\mu,t,p^*(t)}$ as $A_{\mu,t}$.

For any $0\le\mu<\overline{\mu}$, $\alpha,\beta>1$ and $\alpha+\beta=p^*(t)$, by (1.2), (1.3), $0<s_1,s_2\le t<p$, Set

$$A_{\mu,s} := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\big)dx}{\big(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}dx\big)^{\frac{p}{p^*(s)}}}\,, \tag{1.5}$$

$$S_{s,\alpha,\beta} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_\Omega \big(|\nabla u|^p + |\nabla v|^p - \mu\frac{|u|^p+|v|^p}{|x|^p}\big)dx}{\big(\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^s}dx\big)^{\frac{p}{\alpha+\beta}}}\,. \tag{1.6}$$

Then we have the following equality (whose proof is the same as that of Theorem 5 in [1])

$$S_{s,\alpha,\beta}(\mu) = \Big(\big(\frac{\alpha}{\beta}\big)^{\frac{\beta}{\alpha+\beta}} + \big(\frac{\beta}{\alpha}\big)^{\frac{\alpha}{\alpha+\beta}}\Big)A_{\mu,s}\,.$$

Throughout this paper, let R_0 be the positive constant such that $\Omega\subset B(0;R_0)$, where $B(0;R_0)=\{x\in\mathbb{R}^N : |x|<R_0\}$. By Holder and Sobolev-Hardy inequalities,

for all $u \in W_0^{1,p}(\Omega)$, we obtain

$$
\begin{aligned}
\int_\Omega \frac{|u|^q}{|x|^s} &\le \Big(\int_{B(0;R_0)} |x|^{-s}\Big)^{\frac{p^*(s)-q}{p^*(s)}} \Big(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\Big)^{\frac{q}{p^*(s)}} \\
&\le \Big(\int_0^{R_0} r^{N-s+1} dr\Big)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} \|u\|^q \\
&\le \Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} \|u\|^q,
\end{aligned} \tag{1.7}
$$

where $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ is the volume of the unit ball in $\mathbb{R}^N$.

Existence of nontrivial non-negative solutions for elliptic equations with singular potentials were recently studied by several authors, but, essentially, only with a solely critical exponent. We refer, e.g., in bounded domains and for $p = 2$ to [4, 10, 11, 13, 14, 15, 17], and for general $p > 1$ to [5, 6, 7, 8, 12, 16, 18, 19, 26] and the references therein. For example, Han and Liu [17] studied the problem

$$
\begin{gathered}
-\Delta u - \mu \frac{u}{|x|^2} = \lambda u + Q(x)|u|^{2^*-2}u, \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega
\end{gathered} \tag{1.8}
$$

where $0 \in \Omega$ is a smooth bounded domain in $\mathbb{R}^N$ $(N \ge 5)$, $\lambda > 0$, $0 \le \mu < \overline{\mu} \triangleq \big(\frac{N-2}{2}\big)^2$, $2^* = \frac{2N}{N-2}$ and $Q(x)$ is nonnegative and continuous on $\overline{\Omega}$ satisfying some suitable conditions. using critical point theory, the authors proved the existence of nontrivial solutions to problem (1.8). Also, by investigating the effect of the coefficient Q, Han [14] studied problem (1.8) and proved that there exists $\lambda_0 > 0$ such that (1.8) has at least k positive solutions for $\lambda \in (0, \lambda_0)$.

Kang in [18] studied the following elliptic equation via the generalized Mountain-Pass theorem [24],

$$
\begin{gathered}
-\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda \frac{|u|^{p-2}u}{|x|^s}, \quad x \in \Omega, \\
u = 0, \quad x \in \partial\Omega
\end{gathered} \tag{1.9}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < N$, $0 \le s, t < p$ and $0 \le \mu < \overline{\mu} \triangleq \big(\frac{N-p}{p}\big)^p$. Degiovanni and Lancelotti [6] studied problem (1.9) with $\mu = s = t = 0$ and proved that (1.9) has at least one positive solutions for $\lambda \ge \lambda_1 := A_{0,0}$ ($A_{0,0}$ is defined in (1.5)). Indeed, in [6] the much more difficult case $\lambda \ge \lambda_1$ is treated.

The authors in [8], via the Mountain-Pass Theorem of Ambrosetti and Rabinowitz [2], proved that

$$
-\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = |u|^{p^*-1} + \frac{u^{p^*(s)-1}}{|x|^s}, \quad \text{in } \mathbb{R}^N,
$$

admits a positive solution in $\mathbb{R}^N$, whenever $\mu < \overline{\mu} \triangleq \big(\frac{N-p}{p}\big)^p$ and $0 < s < p$.

Recently, in [26] the author studied the following equation via the Mountain-Pass theorem,

$$
-\operatorname{div}\Big(\frac{|Du|^{p-2}Du}{|x|^{ap}}\Big) - \mu \frac{|u|^{p-2}u}{|x|^{(a+1)p}} = \frac{|u|^{p^*(b)-2}u}{|x|^{bp^*}} + \frac{|u|^{p^*(c)-2}u}{|x|^{cp^*}}, \quad \text{in } \mathbb{R}^N
$$

where $1 < p < N$, $0 \le \mu < \overline{\mu} \triangleq \big(\frac{N-(a+1)p}{p}\big)^p$, $0 \le a < \frac{N-p}{p}$, $a \le b, c < a+1$, $p^*(b) = \frac{Np}{N-(a+1-b)p}$ and $p^*(c) = \frac{Np}{N-(a+1-c)p}$.

Zhang and Wei [27] studied the existence of multiple positive solutions for (1.1) with $t = s = 0$, $Q(x) = f(x)$ and $h(x) = 1$. Set $s_1 = s_2 = t$, $s = t$, $x_0 = 0$ and $Q(x) = h(x) \equiv 1$, then problem (1.1) reduces to the quasilinear elliptic system

$$\begin{gathered} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} + \frac{\eta\alpha}{\alpha+\beta} \frac{|u|^{\alpha-2}|v|^{\beta}u}{|x|^t} + \lambda \frac{|u|^{q-2}u}{|x|^s}, \\ -\Delta_p v - \mu \frac{|v|^{p-2}v}{|x|^p} = \frac{|v|^{p^*(t)-2}v}{|x|^t} + \frac{\eta\beta}{\alpha+\beta} \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^t} + \theta \frac{|v|^{q-2}v}{|x|^s}, \\ x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.10}$$

where $\lambda > 0$, $\theta > 0$, $0 < \eta < \infty$, $1 < p < N$, $0 \le \mu < \overline{\mu} \triangleq \big(\frac{N-p}{p}\big)^p$, $0 \le s, t < p$, $1 \le q < p$, $\alpha + \beta = p^*(t) \triangleq \frac{p(N-t)}{N-p}$ is the Hardy- Sobolev critical exponent. The author [23] have studied (1.10) via the Nehari manifold. In [20], Li et al. studied the following quasilinear elliptic problem

$$\begin{gathered} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = K(x) \frac{|u|^{p^*(s)-2}u}{|x|^s} + Q(x) \frac{|u|^{p^*(t)-2}}{|x-x_0|^t} + \lambda f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega \end{gathered} \tag{1.11}$$

where $1 < p < N$, $K(x), Q(x)$ are nonnegative continuous functions on $\overline{\Omega}$, f satisfying some suitable conditions and obtained the existence of solutions via variational methods. For $p = 2$, $x_0 = 0$, $K(x) \equiv 1$ and $Q(x) \equiv 0$, the problem (1.11) has been studied.

Motivated by the above works we study problem (1.1) by using the Mountain-Pass Theorem of Ambrosetti and Rabinowitz. We shall show that system (1.1) has at least two positive weak solutions.

In this article, we assume that $0 < s_1, s_2 \le t < p$, $\alpha, \beta > 1$ and $\alpha + \beta = p^*(t)$. For $0 \le \mu < \overline{\mu}$, we set

$$\begin{gathered} \theta(\mu, s) := \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}, \\ \theta^* := \big\{\theta(\mu, s_1), \theta(\mu, s_2), \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} S_{t,\alpha,\beta}^{\frac{N-t}{p-t}}\big\}. \end{gathered}$$

Moreover, we assume that $Q(x)$ satisfies some of the following assumptions:

- (H1) $Q \in C(\overline{\Omega})$, $Q(x) \ge 0$ and $\text{meas}(\{x \in \Omega,\ h(x) > 0\}) > 0$.
- (H2) There exist $\vartheta > 0$ such that $Q(x_0) = \|Q\|_\infty > 0$ and $Q(x) = Q(x_0) + O(|x - x_0|^{\varrho})$, as $x \to x_0$.
- (H3) There exist β_0 and $\rho > 0$ such that $B_{2\rho_0}(x_0) \subset \Omega$ and $h(x) \ge \beta_0$ for all $x \in B_{2\rho_0}(x_0)$.

Set

$$h_+ := \max\{h, 0\}, \quad h_- := \max\{-h, 0\}.$$

The main results of this article are stated in the following two theorems.

Theorem 1.1. *Assume that $N \geq 3, \mu \in [0, \overline{\mu})$, $1 < q < p$ and* (H1)*. Then there exists $\Lambda_{11}^* > 0$, such that for $0 < \lambda < \Lambda_{11}^*$ problem* (1.1) *has at lest one positive solutions.*

Theorem 1.2. *Assume that $N \geq p^2$, $0 \leq \mu < \overline{\mu}$, $\theta^* = \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} S_{t,\alpha,\beta}^{\frac{N-t}{p-t}}$, (H1)-(H3), $Q(0) = 0$, $\varrho > b(\mu)p + p - N + t$ and $\frac{N-s}{b(\mu)} < q < p$ hold, and $b(\mu)$ is the constant defined as in Lemma 2.4. Then there exists $\Lambda^{**} > 0$, such that for $0 < \lambda < \Lambda^{**}$, problem* (1.1) *has at least two positive solutions.*

This article is divided into three sections, organized as follows. In Section 2, we establish some elementary results. In Section 3, we prove our main results (Theorems 1.1 and 1.2).

2. Preliminary lemmas

The corresponding energy functional of problem (1.1) is defined by

$$\begin{aligned} J(u,v) = &\frac{1}{p} \int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu \frac{|v|^p}{|x|^p} \Big) dx - \frac{\lambda}{q} \int_\Omega h(x) (\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s}) dx \\ &- \frac{1}{p^*(s_1)} \int_\Omega \frac{|u|^{p^*(s_1)}}{|x|^{s_1}} dx - \frac{1}{p^*(s_2)} \int_\Omega \frac{|v|^{p^*(s_2)}}{|x|^{s_2}} dx \\ &- \frac{1}{\alpha+\beta} \int_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x-x_0|^t} dx, \end{aligned}$$

for each $(u,v) \in W$. Then $J \in C^1(W, \mathbb{R})$.

Lemma 2.1. *Assume that $N \geq 3$, $0 \leq \mu < \overline{\mu}$, (H1), $h_+ \neq 0$ and (u,v) is a weak solution of problem* (1.1)*. Then there exists a positive constant d depending on $N, |\Omega|, |h_+|_\infty, A_{\mu,s}, s_1, s_2$ and q such that*

$$J(u,v) \geq -d\lambda^{\frac{p}{p-q}}.$$

Proof. Since (u,v) is a weak solution of (1.1), then, Note that $\langle J'(u,v), (u,v) \rangle = 0$, we have

$$\begin{aligned} &\langle J'(u,v), (u,v) \rangle \\ &= \int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu \frac{|v|^p}{|x|^p} \Big) dx - \lambda \int_\Omega h(x) (\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s}) dx \\ &\quad - \int_\Omega \frac{|u|^{p^*(s_1)}}{|x|^{s_1}} dx - \int_\Omega \frac{|v|^{p^*(s_2)}}{|x|^{s_2}} dx - \int_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x-x_0|^t} dx = 0. \end{aligned} \tag{2.1}$$

Now, by using $h_+ \neq 0$, (2.1), (1.7), the Hölder inequality and the Sobolev-Hardy inequality, we have

$$\begin{aligned} J(u,v) \geq &\frac{1}{p} \int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu \frac{|v|^p}{|x|^p} \Big) dx - \frac{\lambda}{q} \int_\Omega h(x) (\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s}) dx \\ &- \frac{1}{p^*(t)} \Big[\int_\Omega \frac{|u|^{p^*(s_1)}}{|x|^{s_1}} dx - \int_\Omega \frac{|v|^{p^*(s_2)}}{|x|^{s_2}} dx - \int_\Omega Q(x) \frac{|u|^\alpha |v|^\beta}{|x-x_0|^t} dx \Big] \\ = &\frac{1}{p} \int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu \frac{|v|^p}{|x|^p} \Big) dx - \frac{\lambda}{q} \int_\Omega h(x) (\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s}) dx \\ &- \frac{1}{p^*(t)} \Big[\int_\Omega \Big(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu \frac{|v|^p}{|x|^p} \Big) dx \end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_\Omega h(x)(\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s})dx\Big] \\
&\geq \Big(\frac{1}{p} - \frac{1}{p^*(t)}\Big) \int_\Omega \Big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p}\Big)dx \\
&- \lambda\Big(\frac{1}{q} - \frac{1}{p^*(t)}\Big) \int_\Omega h(x)(\frac{|u|^q}{|x|^s} + \frac{|v|^q}{|x|^s})dx \\
&\geq \Big(\frac{1}{p} - \frac{1}{p^*(t)}\Big)(\|u\|_\mu^p + \|v\|_\mu^p) \\
&- \lambda\Big(\frac{1}{q} - \frac{1}{p^*(t)}\Big)\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} |h_+|_\infty(\|u\|_\mu^q + \|v\|_\mu^q) \\
&\geq 2\inf_{t\geq 0}\Big[\Big(\frac{1}{p} - \frac{1}{p^*(s)}\Big)t^p - \lambda\Big(\frac{1}{q} - \frac{1}{p^*(s)}\Big)\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}}|h_+|_\infty t^q\Big] \\
&\geq -d\lambda^{\frac{p}{p-q}}.
\end{aligned}
$$

Here $d_\Omega := \sup_{x,y\in\Omega}|x-y|$ is the diameter of Ω and d is a positive constant depending on $N, |\Omega|, |h_+|_\infty, A_{\mu,s}, s_1, s_2$ and q. □

Recall that a sequence $(u_n, v_n)_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence for the functional J if $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0$. If any $(PS)_c$ sequence $(u_n, v_n)_{n\in\mathbb{N}}$ has a convergent subsequence, we say that J satisfies the $(PS)_c$ condition.

Lemma 2.2. *Assume that $N \geq 3$, $0 \leq \mu < \overline{\mu}$,* (H1)*, $h_+ \neq 0$ and $Q(0) = 0$. Then $J(u,v)$ satisfies the $(PS)_c$ condition with c satisfying*

$$
\begin{aligned}
c < c_* := \min\Big\{&\frac{p-s_1}{p(N-s_1)}A_{\mu,s_1}^{\frac{N-s_1}{p-s_1}}, \frac{p-s_2}{p(N-s_2)}A_{\mu,s_2}^{\frac{N-s_2}{p-s_2}}, \\
&\frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}S_{t,\alpha,\beta}^{\frac{N-t}{p-t}}\Big\} - d\lambda^{\frac{p}{p-q}}.
\end{aligned} \tag{2.2}
$$

Proof. It is easy to see that the $(PS)_c$ sequence $(u_n, v_n)_{n\in\mathbb{N}}$ of $J(u,v)$ is bounded in W. Then $(u_n, v_n) \rightharpoonup (u,v)$ weakly in W as $n \to \infty$, which implies $u_n \rightharpoonup u$ weakly and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Passing to a subsequence we may assume that

$$
\begin{gathered}
|\nabla u_n|^p dx \rightharpoonup \overline{\alpha}, \quad |\nabla v_n|^p dx \rightharpoonup \widetilde{\alpha}, \\
\frac{|u_n|^p}{|x|^p}dx \rightharpoonup \overline{\beta}, \quad \frac{|v_n|^p}{|x|^p}dx \rightharpoonup \widetilde{\beta}, \\
\frac{|u_n|^{p^*(s_1)}}{|x|^{s_1}}dx \rightharpoonup \overline{\gamma}, \quad \frac{|v_n|^{p^*(s_2)}}{|x|^{s_2}}dx \rightharpoonup \widetilde{\gamma}, \\
Q(x)\frac{|u_n|^\alpha|v_n|^\beta}{|x-x_0|^t}dx \rightharpoonup \nu
\end{gathered}
$$

weakly in the sense of measures. Using the concentration-compactness principle in [21], there exist an at most countable set I, a set of points $\{x_i\}_{i\in I} \in \Omega \setminus \{0\}$, real numbers $\overline{a}_{x_i}, \widetilde{a}_{x_i}, d_{x_i}$, $i \in I$, $\overline{a}_0, \widetilde{a}_0, \overline{b}_0, \widetilde{b}_0, \overline{c}_0, \widetilde{c}_0$ and d_0, such that

$$
\overline{\alpha} \geq |\nabla u|^p dx + \sum_{i\in I}\overline{a}_{x_i}\delta_{x_i} + \overline{a}_0\delta_0, \tag{2.3}
$$

$$\widetilde{\alpha} \geq |\nabla u|^p dx + \sum_{i\in I} \widetilde{a}_{x_i}\delta_{x_i} + \widetilde{a}_0\delta_0, \tag{2.4}$$

$$\overline{\beta} = \frac{|u|^p}{|x|^p}dx + \overline{b}_0\delta_0, \tag{2.5}$$

$$\widetilde{\beta} = \frac{|v|^p}{|x|^p}dx + \widetilde{b}_0\delta_0, \tag{2.6}$$

$$\overline{\gamma} = \frac{|u|^{p^*(s_1)}}{|x|^{s_1}} + \overline{c}_0\delta_0, \tag{2.7}$$

$$\widetilde{\gamma} = \frac{|v|^{p^*(s_2)}}{|x|^{s_2}}dx + \widetilde{c}_0\delta_0, \tag{2.8}$$

$$\nu = Q(x)\frac{|u|^\alpha |v|^\beta}{|x-x_0|^t}dx + \sum_{i\in I} Q(x_i)d_{x_i}\delta_{x_i} + Q(0)d_0\delta_0, \tag{2.9}$$

where δ_x is the Dirac-mass of mass 1 concentrated at the point x.

First, we consider the possibility of the concentration at $\{x_i\}_{i\in I} \in \Omega \setminus \{0\}$. Let $\epsilon > 0$ be small enough, take $\eta_{x_i} \in C_c^\infty(B_{2\varepsilon}(x_i))$, such that $\eta_{x_i}|_{B_\varepsilon(x_i)} = 1$, $0 \leq \eta_{x_i} \leq 1$ and $|\nabla \eta_{x_i}(x)| \leq \frac{C}{\varepsilon}$. Then

$$\begin{aligned}
o(1) &= \langle J'(u_n,v_n), (\eta_{x_i}^p u_n, \eta_{x_i}^p v_n)\rangle \\
&= \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n \nabla(\eta_{x_i}^p u_n) + |\nabla v_n|^{p-2}\nabla v_n\nabla(\eta_{x_i}^p v_n)\big)\,dx \\
&\quad - \int_\Omega Q(x)\frac{|u_n|^\alpha |v_n|^\beta}{|x-x_0|^t}\eta_{x_i}^p dx - \mu\int_\Omega \Big(\frac{|u_n|^p}{|x|^p}\eta_{x_i}^p + \frac{|v_n|^p}{|x|^p}\eta_{x_i}^p\Big)dx \\
&\quad - \lambda\int_\Omega h(x)\Big(\frac{|u_n|^q}{|x|^s}\eta_{x_i}^p + \frac{|v_n|^q}{|x|^s}\eta_{x_i}^p\Big)dx \\
&\quad - \int_\Omega \frac{|u_n|^{p^*(s_1)}}{|x|^{s_1}}\eta_{x_i}^p dx - \int_\Omega \frac{|v_n|^{p^*(s_2)}}{|x|^{s_2}}\eta_{x_i}^p dx.
\end{aligned}$$

From (2.5)-(2.9), one can obtain

$$\begin{gathered}
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \Big(\frac{|u_n|^p}{|x|^p}\eta_{x_i}^p + \frac{|v_n|^p}{|x|^p}\eta_{x_i}^p\Big)dx = \lim_{\varepsilon\to 0}\Big(\int_\Omega \eta_{x_i}^p d\overline{\beta} + \int_\Omega \eta_{x_i}^p d\widetilde{\beta}\Big) = 0, \qquad (2.10)\\
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \Big(\frac{|u_n|^{p^*(s_1)}}{|x|^{s_1}}\eta_{x_i}^p + \frac{|v_n|^{p^*(s_2)}}{|x|^{s_2}}\eta_{x_i}^p\Big)dx = \lim_{\varepsilon\to 0}\Big(\int_\Omega \eta_{x_i}^p d\overline{\gamma} + \int_\Omega \eta_{x_i}^p d\widetilde{\gamma}\Big) = 0,\\
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega h(x)\Big(\frac{|u_n|^q}{|x|^s}\eta_{x_i}^p + \frac{|v_n|^q}{|x|^s}\eta_{x_i}^p\Big)dx = 0,\\
\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega Q(x)\frac{|u_n|^\alpha|v_n|^\beta}{|x-x_0|^t}\eta_{x_i}^p dx = \lim_{\varepsilon\to 0}\int_\Omega \eta_{x_i}^p d\nu = Q(x_i)dx_i.
\end{gathered}$$

Thus,

$$\begin{aligned}
0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \Big(&|\nabla u_n|^{p-2}\nabla u_n\nabla(\eta_{x_i}^p u_n) \\
&+ |\nabla v_n|^{p-2}\nabla v_n\nabla(\eta_{x_i}^p v_n)\Big)dx - Q(x_i)dx_i.
\end{aligned} \tag{2.11}$$

Moreover, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\eta_{x_i}^p dx\Big| \\
&\le \lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\int_\Omega |u_n|^p|\nabla\eta_{x_i}^p|^p dx\Big)^{1/p}\Big(\int_\Omega|\nabla u_n|^p dx\Big)^{\frac{p-1}{p}} \\
&\le C\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big(\int_\Omega |u_n|^p|\nabla\eta_{x_i}^p|^p dx\Big)^{1/p} \\
&= C\lim_{\varepsilon\to 0}\Big(\int_\Omega |u|^p|\nabla\eta_{x_i}^p|^p dx\Big)^{1/p} \\
&\le C\lim_{\varepsilon\to 0}\Big(\int_{B_\varepsilon(x_i)}|\nabla\eta_{x_i}^p|^N dx\Big)^{1/N}\Big(\int_{B_\varepsilon(x_i)}|u|^{p^*}dx\Big)^{1/P^*} \\
&\le C\lim_{\varepsilon\to 0}\Big(\int_{B_\varepsilon(x_i)}|u|^{p^*}dx\Big)^{1/P^*}=0.
\end{aligned} \tag{2.12}$$

Similarly,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\Big|\int_\Omega v_n|\nabla v_n|^{p-2}\nabla v_n\nabla\eta_{x_i}^p dx\Big|=0. \tag{2.13}$$

Combining (2.11)-(2.13), there holds

$$\begin{aligned}
0&=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega(|\eta_{x_i}\nabla u_n|^p+|\eta_{x_i}\nabla v_n|^p)dx-Q(x_i)d_{x_i} \\
&=\lim_{\varepsilon\to 0}\int_\Omega\big(\eta_{x_i}^p d\overline{\alpha}+\eta_{x_i}d\widetilde{\alpha}\big)-Q(x_i)d_{x_i}.
\end{aligned} \tag{2.14}$$

On the other hand, (1.6) implies

$$\begin{aligned}
&\frac{1}{\|Q\|_\infty^{\frac{p}{p^*(t)}}}S_{t,\alpha,\beta}\Big(\int_\Omega Q(x)\frac{|\eta_{x_i}u_n|^\alpha|\eta_{x_i}v_n|^\beta}{|x-x_0|^t}dx\Big)^{\frac{p}{p^*(t)}} \\
&\le\int_\Omega\Big(|\nabla(\eta_{x_i}u_n)|^p+|\nabla(\eta_{x_i}v_n)|^p-\mu\frac{|\eta_{x_i}u_n|^p+|\eta_{x_i}v_n|^p}{|x|^p}\Big)dx.
\end{aligned} \tag{2.15}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla\eta_{x_i}|^p|u_n|^p dx=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla\eta_{x_i}|^p|v_n|^p dx=0.$$

From this equality, (2.12) and (2.13), we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\eta_{x_i}\nabla u_n|^p dx=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla(\eta_{x_i}u_n)|^p dx, \tag{2.16}$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\eta_{x_i}\nabla v_n|^p dx=\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega|\nabla(\eta_{x_i}v_n)|^p dx. \tag{2.17}$$

Relations (2.9), (2.10) and (2.15)-(2.17) imply

$$\frac{1}{\|Q\|_\infty^{\frac{p}{p^*(t)}}}S_{t,\alpha,\beta}\big(Q(x_i)d_{x_i}\big)^{\frac{p}{p^*(t)}}\le\lim_{\varepsilon\to 0}\int_\Omega|\eta_{x_i}|^p d\overline{\alpha}+\lim_{\varepsilon\to 0}\int_\Omega|\eta_{x_i}|^p d\widetilde{\alpha}. \tag{2.18}$$

Combining (2.14) and (2.18),

$$\frac{1}{\|Q\|_\infty^{\frac{p}{p^*(t)}}}S_{t,\alpha,\beta}\big(Q(x_i)d_{x_i}\big)^{\frac{p}{p^*(t)}}\le Q(x_i)d_{x_i}, \tag{2.19}$$

which implies that either

$$Q(x_i)d_{x_i} = 0, \quad \text{or} \quad Q(x_i)d_{x_i} \geq \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} S_{t,\alpha,\beta}^{\frac{N-t}{p-t}}. \tag{2.20}$$

Now, we consider the possibility of the concentration at 0. For $\epsilon > 0$ be small enough, take $\eta_0 \in C_c^\infty(B_{2\varepsilon}(0))$, such that $\eta_0|_{B_\varepsilon(0)} = 1$, $0 \leq \eta_0 \leq 1$ and $|\nabla \eta_0(x)| \leq \frac{C}{\varepsilon}$. Then

$$\begin{aligned}
o(1) &= \langle J'(u_n, v_n), (\eta_0^p u_n, 0)\rangle \\
&= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(\eta_0^p u_n)dx - \mu \int_\Omega \frac{|u_n|^p}{|x|^p}\eta_0^p dx - \lambda \int_\Omega h(x)\frac{|u_n|^q}{|x|^s}\eta_0^p dx \\
&\quad - \int_\Omega \frac{|u_n|^{p^*(s_1)}}{|x|^{s_1}}\eta_0^p dx - \frac{\alpha}{\alpha+\beta}\int_\Omega Q(x)\frac{|u_n|^\alpha |v_n|^\beta}{|x-x_0|^t}\eta_0^p dx.
\end{aligned}$$

From (2.5), (2.7), (2.9) and $Q(0) = 0$, we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^p}{|x|^p}\eta_0^p dx = \bar{b}_0, \quad \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p^*(s_1)}}{|x|^{s_1}}\eta_0^p dx = \bar{c}_0,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega Q(x)\frac{|u_n|^\alpha|v_n|^\beta}{|x-x_0|^t}\eta_0^p dx = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega h(x)\frac{|u_n|^q}{|x|^s}\eta_0^p dx = 0.$$

Thus,

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(\eta_0^p u_n)dx - \mu\bar{b}_0 - \bar{c}_0. \tag{2.21}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n \nabla\eta_0^p dx = 0.$$

This equality and (2.21) yield

$$\lim_{\varepsilon\to 0}\int_\Omega \eta_0^p d\bar{\alpha} - \mu\bar{b}_0 = \bar{c}_0. \tag{2.22}$$

On the other hand, (1.5) implies

$$A_{\mu,s_1}\Big(\int_\Omega \frac{|\eta_0 u_n|^{p^*(s_1)}}{|x|^{s_1}}dx\Big)^{\frac{p}{p^*(s_1)}} \leq \int_\Omega \Big(|\nabla(\eta_0 u_n)|^p - \mu\frac{|\eta_0 u_n|^p}{|x|^p}\Big)dx.$$

Thus

$$A_{\mu,s_1}\bar{c}_0^{\frac{p}{p^*(s_1)}} \leq \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla(\eta_0 u_n)|^p dx - \mu\bar{b}_0. \tag{2.23}$$

Note that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\eta_0\nabla u_n|^p dx = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega |\nabla(\eta_0 u_n)|^p dx,$$

which together with (2.23) imply

$$A_{\mu,s_1}\bar{c}_0^{\frac{p}{p^*(s_1)}} \leq \lim_{\varepsilon\to 0}\int_\Omega |\eta_0|^p d\bar{\alpha} - \mu\bar{b}_0. \tag{2.24}$$

Therefore, from (2.22) and (2.24),

$$A_{\mu,s_1}\bar{c}_0^{\frac{p}{p^*(s_1)}} \leq \bar{c}_0, \tag{2.25}$$

which implies that either

$$\bar{c}_0 = 0, \quad \text{or} \quad \bar{c}_0 \geq A_{\mu,s_1}^{\frac{N-s_1}{p-s_1}}. \tag{2.26}$$

Similarly, either

$$\bar{c}_0 = 0, \quad \text{or} \quad \bar{c}_0 \geq A_{\mu,s_2}^{\frac{N-s_2}{p-s_2}}. \tag{2.27}$$

Recall that $u_n \rightharpoonup u$ weakly and $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} & c + o(1) \\ & = J(u_n, v_n) \\ & = \frac{1}{p} \int_\Omega \Big(|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^p}{|x|^p} + |\nabla v_n - \nabla v|^p - \mu \frac{|v_n - v|^p}{|x|^p} \Big) dx \\ & \quad - \frac{1}{p^*(s_1)} \int_\Omega \frac{|u_n - u|^{p^*(s_1)}}{|x|^{s_1}} dx - \frac{1}{p^*(s_2)} \int_\Omega \frac{|v_n - v|^{p^*(s_2)}}{|x|^{s_2}} dx \\ & \quad - \frac{1}{p^*(t)} \int_\Omega Q(x) \frac{|u_n - u|^\alpha |v_n - v|^\beta}{|x - x_0|^t} dx + J(u, v). \end{aligned} \tag{2.28}$$

On the other hand, from $o(1) = J'(u_n, v_n)$, we obtain

$$J'(u_n, v_n) = 0. \tag{2.29}$$

Thus, $0 = \langle J'(u, v), (u, v) \rangle$, which together with $o(1) = \langle J'(u_n, v_n), (u_n, v_n) \rangle$ imply

$$\begin{aligned} o(1) = & \int_\Omega \Big(|\nabla u_n - \nabla u|^p - \mu \frac{|u_n - u|^p}{|x|^p} + |\nabla v_n - \nabla v|^p - \mu \frac{|v_n - v|^p}{|x|^p} \Big) dx \\ & - \int_\Omega \frac{|u_n - u|^{p^*(s_1)}}{|x|^{s_1}} dx - \int_\Omega \frac{|v_n - v|^{p^*(s_2)}}{|x|^{s_2}} dx \\ & - \int_\Omega Q(x) \frac{|u_n - u|^\alpha |v_n - v|^\beta}{|x - x_0|^t} dx. \end{aligned} \tag{2.30}$$

From (2.28)-(2.30) and Lemma 2.1,

$$\begin{aligned} c + o(1) \geq & \frac{p - s_1}{p(N - s_1)} \int_\Omega \frac{|u_n - u|^{p^*(s_1)}}{|x|^{s_1}} dx + \frac{p - s_2}{p(N - s_2)} \int_\Omega \frac{|v_n - v|^{p^*(s_2)}}{|x|^{s_2}} dx \\ & + \frac{p - t}{p(N - t)} \int_\Omega Q(x) \frac{|u_n - u|^\alpha |v_n - v|^\beta}{|x - x_0|^t} dx - d\lambda^{\frac{p}{p-q}}. \end{aligned} \tag{2.31}$$

Passing to the limit in (2.31) as $n \to \infty$, we have

$$c \geq \frac{p - s_1}{2(N - s_1)} \bar{c}_0 + \frac{p - s_2}{p(N - s_2)} \widetilde{c}_0 + \frac{p - t}{p(N - t)} \sum_{i \in I} Q(x_i) d_{x_i} - d\lambda^{\frac{p}{p-q}}. \tag{2.32}$$

By the assumption $c < c_*$ and in view of (2.20), (2.26) and (2.27), there holds $\bar{c}_0 = \widetilde{c}_0 = 0$, $Q(x_i) d_{x_i} = 0$, $i \in I$. Up to a subsequence, $(u_n, v_n) \to (u, v)$ strongly in W as $n \to \infty$. □

When the restriction $Q(0) = 0$ is removed, we establish the following version of Lemma 2.2.

Lemma 2.3. *Assume that $N \geq 3$, $0 \leq \mu < \overline{\mu}$, (H1) and $h_+ \neq 0$. Then $J(u,v)$ satisfies the $(PS)_c$ condition with c satisfying*

$$\begin{aligned} c < c_0 := \min\Big\{ &\frac{p-s_1}{p(N-s_1)}\Big(\frac{1}{p}A_{\mu,s_1}\Big)^{\frac{N-s_1}{p-s_1}}, \frac{p-s_2}{p(N-s_2)}\Big(\frac{1}{p}A_{\mu,s_2}\Big)^{\frac{N-s_2}{p-s_2}}, \\ &\frac{p-t}{p(N-t)}\frac{\Big(\frac{1}{p}S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}}}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\Big\} - d\lambda^{\frac{p}{p-q}}. \end{aligned} \tag{2.33}$$

The proof of the above lemma is similar to Lemma 2.2 and is omitted.

Lemma 2.4 ([18])**.** *Assume that $1 < p < N$, $0 \leq t < p$ and $0 \leq \mu < \overline{\mu}$. Then the limiting problem*

$$\begin{gathered} -\Delta_p u - \mu\frac{|u|^{p-1}}{|x|^p} = \frac{|u|^{p^*(t)-1}}{|x|^t}, \quad \text{in } \mathbb{R}^N \setminus \{0\}, \\ u \in D^{1,p}(\mathbb{R}^N), \quad u > 0, \quad \text{in } \mathbb{R}^N \setminus \{0\}, \end{gathered}$$

has positive radial ground states

$$V_\epsilon(x) \triangleq \epsilon^{\frac{p-N}{p}} U_{p,\mu}(\frac{x}{\epsilon}) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}(\frac{|x|}{\epsilon}), \quad \forall \epsilon > 0, \tag{2.34}$$

that satisfy

$$\int_\Omega \Big(|\nabla V_\epsilon(x)|^p - \mu\frac{|V_\epsilon(x)|^p}{|x|^p}\Big)dx = \int_\Omega \frac{|V_\epsilon(x)|^{p^*(t)}}{|x|^t}dx = (A_{\mu,t})^{\frac{N-t}{p-t}},$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the limiting problem with

$$U_{p,\mu}(1) = \Big(\frac{(N-t)(\overline{\mu}-\mu)}{N-p}\Big)^{\frac{1}{p^*(t)-p}}.$$

Furthermore, $U_{p,\mu}$ has the following properties:

$$\begin{gathered} \lim_{r\to 0} r^{a(\mu)}U_{p,\mu}(r) = C_1 > 0, \quad \lim_{r\to+\infty} r^{b(\mu)}U_{p,\mu}(r) = C_2 > 0, \\ \lim_{r\to 0} r^{a(\mu)+1}|U'_{p,\mu}(r)| = C_1 a(\mu) \geq 0, \\ \lim_{r\to+\infty} r^{b(\mu)+1}|U'_{p,\mu}(r)| = C_2 b(\mu) > 0, \end{gathered}$$

where C_i $(i = 1,2)$ are positive constants and $a(\mu)$ and $b(\mu)$ are zeros of the function

$$f(\zeta) = (p-1)\zeta^p - (N-p)\zeta^{p-1} + \mu, \quad \zeta \geq 0,\ 0 \leq \mu < \overline{\mu},$$

that satisfy

$$0 \leq a(\mu) < \frac{N-p}{p} < b(\mu) \leq \frac{N-p}{p-1}.$$

Lemma 2.5. *Under the assumptions of Theorem 1.2, there exists $(u_1, v_1) \in W \setminus \{(0,0)\}$ and $\Lambda_1 > 0$, such that for $0 < \lambda < \Lambda_1$, there holds*

$$\sup_{t\geq 0} J(tu_1, tv_1) < \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} S_{t,\alpha,\beta}^{\frac{N-t}{p-t}}(\mu) - d\lambda^{\frac{p}{p-q}}. \tag{2.35}$$

Proof. First, we give some estimates on the extremal function $V_\epsilon(x)$ defined in (2.34). For $m \in \mathbb{N}$ large, choose $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $|x| \le \frac{1}{2m}$, $\varphi(x) = 0$ for $|x| \ge \frac{1}{m}$, $\|\nabla\varphi(x)\|_{L^p(\Omega)} \le 4m$, set $u_\epsilon(x) = \varphi(x)V_\epsilon(x)$. For $\epsilon \to 0$, the behavior of u_ϵ has to be the same as that of V_ϵ, but we need precise estimates of the error terms. For $1 < p < N$, $0 \le t < p$ and $1 < q < p^*(s)$, we have the following estimates [4]:

$$\int_\Omega \Big(|\nabla u_\epsilon|^p - \mu\frac{|u_\epsilon|^p}{|x|^p}\Big)dx = (A_{\mu,t})^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}), \tag{2.36}$$

$$\int_\Omega \frac{|u_\epsilon|^{p^*(t)}}{|x|^t}dx = (A_{\mu,t})^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p^*(t)-N+t}), \tag{2.37}$$

$$\int_\Omega \frac{|u_\epsilon|^q}{|x|^s}dx \ge \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \frac{N-s}{b(\mu)},\\ C\epsilon^{N-s+(1-\frac{N}{p})q}|\ln\epsilon|, & q = \frac{N-s}{b(\mu)},\\ C\epsilon^{q(b(\mu)+1-\frac{N}{p})q}, & q < \frac{N-s}{b(\mu)}. \end{cases} \tag{2.38}$$

Now, we consider the functional $I : W \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{p}\int_\Omega \Big(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p} + |\nabla v|^p - \mu\frac{|v|^p}{|x|^p}\Big)dx - \frac{1}{p^*(t)}\int_\Omega Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^t}dx.$$

Let $u_1 = \alpha^{\frac{1}{p}}u_\epsilon$, $v_1 = \beta^{\frac{1}{p}}u_\epsilon$ and define the function $g_1(t) := J(tu_1, tv_1)$, $t \ge 0$. Note that $\lim_{t\to+\infty} g_1(t) = -\infty$ and $g_1(t) > 0$ as t is close to 0. Thus $\sup_{t\ge0} g_1(t)$ is attained at some finite $t_\epsilon > 0$ with $g_1'(t_\epsilon) = 0$. Furthermore, $C' < t_\epsilon < C''$; where C' and C'' are the positive constants independent of ϵ. We have

$$I(tu_1,tv_1) = y(tu_1,tv_1) - \frac{t^{p^*(t)}}{p^*(t)}(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})\int_\Omega (Q(x)-Q(x_0))\frac{|u_\epsilon|^{p^*(t)}}{|x-x_0|^t}dx. \tag{2.39}$$

where

$$\begin{aligned} &y(tu_1,tv_1)\\ &:= \Big[\frac{t^p}{p}(\alpha+\beta)\int_\Omega\Big(|\nabla u_\epsilon|^p - \mu\frac{|u_\epsilon|^p}{|x|^p}\Big)dx - \frac{t^{p^*(t)}}{p^*(t)}(\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})\int_\Omega Q(y_0)\frac{|u_\epsilon|^{p^*(t)}}{|x-x_0|^t}dx\Big]. \end{aligned}$$

Note that

$$\sup_{t\ge0}\big(\frac{t^p}{p}A - \frac{t^{p^*(t)}}{p^*(t)}B\big) = \big(\frac{1}{p} - \frac{1}{p^*(t)}\big)\big(\frac{A}{B^{\frac{p}{p^*(t)}}}\big)^{\frac{p^*(t)}{p^*(t)-p}}, \quad A,B>0. \tag{2.40}$$

From (H2), (2.36), (2.37) and (2.40) it follows that

$$\begin{aligned} &\sup_{t\ge0} y(tu_1,tv_1)\\ &= y(t_\epsilon u_1, t_\epsilon v_1)\\ &\le \frac{p-t}{p(N-t)}\Big(\frac{(\alpha+\beta)\int_\Omega\Big(|\nabla u_\epsilon|^p - \mu\frac{|u_\epsilon|^p}{|x|^p}\Big)dx}{((\alpha^{\frac{\alpha}{p}}\beta^{\frac{\beta}{p}})\|Q\|_\infty\int_\Omega\frac{|u_\epsilon|^{p^*(t)}}{|x-x_0|^t}dx)^{\frac{N-p}{N-t}}}\Big)^{\frac{N-t}{p-t}}\\ &\le \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\Big[\Big(\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}}\Big)\frac{(A_{\mu,t})^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N})}{(A_{\mu,t})^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p^*(t)-N+t})}\Big]^{\frac{N-t}{p-t}}\\ &\le \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\Big[\Big(\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{\alpha+\beta}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{\alpha+\beta}}\Big)(A_{\mu,t})^{\frac{N-t}{p-t}}\Big]^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}) \end{aligned}$$

$$\leq \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} \Big(S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}). \tag{2.41}$$

On the other hand, (H2) implies that there exists $r_1 < r$, such that for $x \in B_{r_1}(y_0)$, $|Q(x) - Q(x_0)| \leq C|x - x_0|^\vartheta$. Thus

$$\begin{aligned} \Big| \int_\Omega (Q(x) - Q(x_0)) \frac{|u_\epsilon|^{p^*(t)}}{|x-x_0|^t} dx \Big| &\leq C \int_\Omega |Q(x) - Q(x_0)| \frac{|u_\epsilon|^{p^*(t)}}{|x-x_0|^t} dx \\ &= C \int_{B_{2r}(x_0)} \frac{|x-x_0|^\vartheta |u_\epsilon|^{p^*(t)}}{|x-x_0|^t} dx = O(\epsilon^{\vartheta - t}) \end{aligned} \tag{2.42}$$

From (2.39), (2.41) and (2.42), we conclude that

$$\begin{aligned} &\sup_{t\geq 0} I(tu_1, tv_1) = I(t_\epsilon u_1, t_\epsilon v_1) \\ &\leq \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} \Big(S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}). \end{aligned} \tag{2.43}$$

Observe that there exists $\Lambda_1^* > 0$, such that for $0 < \lambda < \Lambda_1^*$ and

$$\frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} \Big(S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}} - d\lambda^{\frac{p}{p-q}} > 0.$$

Then for $0 < \lambda < \Lambda_1^*$, there exists $t_1 \in (0,1)$, such that

$$\begin{aligned} \sup_{0\leq t\leq t_1} J(tu_1, tv_1) &\leq \sup_{0\leq t\leq t_1} \frac{1}{p} t^p \int_\Omega \Big(|\nabla u_1|^p + |\nabla v_1|^p\Big) dx \\ &< \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} \Big(S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}} - d\lambda^{\frac{p}{p-q}}. \end{aligned} \tag{2.44}$$

On the other hand,

$$\begin{aligned} &\sup_{t\geq t_1} J(tu_1, tv_1) \\ &\leq \sup_{t\geq t_1} \Big[I(tu_1, tv_1) - \frac{\lambda}{q} t^q \int_\Omega h(x) \frac{|u_1|^q}{|x|^s} dx - \frac{1}{p^*(s_1)} t^{p^*(s_1)} \int_\Omega \frac{|u_1|^{p^*(s_1)}}{|x|^{s_1}} dx \Big] \\ &\leq \sup_{t\geq t_1} I(tu_1, tv_1) - \frac{\lambda}{q} t_1^q \int_\Omega h(x) |u_1|^q dx - \frac{1}{p^*(s_1)} t_1^{p^*(s_1)} \int_\Omega \frac{|u_1|^{p^*(s_1)}}{|x|^{s_1}} dx \Big] \\ &\leq \frac{p-t}{p(N-t)} \frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}} \Big(S_{t,\alpha,\beta}\Big)^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}) \\ &\quad - C \int_\Omega \frac{|u_\epsilon|^{p^*(s_1)}}{|x|^{s_1}} dx - \lambda C \int_\Omega h(x) \frac{|u_\epsilon|^q}{|x|^s} dx \end{aligned} \tag{2.45}$$

From (2.37),

$$\int_\Omega \frac{|u_\epsilon|^{p^*(s_1)}}{|x|^{s_1}} dx \geq O(\epsilon^{b(\mu)p^*(s_1) - N + s_1}). \tag{2.46}$$

Also, from (2.38), it follows that

$$\int_\Omega h(x)\frac{|u_\epsilon|^q}{|x|^s}dx \ge \beta_0 \int_\Omega \frac{|u_\epsilon|^q}{|x|^s}dx \ge \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \frac{N-s}{b(\mu)},\\ C\epsilon^{N-s+(1-\frac{N}{p})q}|\ln\epsilon|, & q = \frac{N-s}{b(\mu)},\\ C\epsilon^{q(b(\mu)+1-\frac{N}{p})q}, & q < \frac{N-s}{b(\mu)}. \end{cases} \tag{2.47}$$

Since $q \ge \frac{N-s}{b(\mu)}$, by (2.45)-(2.47) we have

$$\begin{aligned} \sup_{t\ge t_1} J(tu_1, tv_1) &\le \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\big(S_{t,\alpha,\beta}\big)^{\frac{N-t}{p-t}} + O(\epsilon^{b(\mu)p+p-N}) \\ &\quad + O(\epsilon^{b(\mu)p^*(s_1)-N+s_1}) - \lambda \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \frac{N-s}{b(\mu)},\\ C\epsilon^{N-s+(1-\frac{N}{p})q}|\ln\epsilon|, & q = \frac{N-s}{b(\mu)}. \end{cases} \end{aligned}$$

Note that $b(\mu)p + p - N < b(\mu)p^*(s_1) - N + s_1$, then we have

$$\begin{aligned} \sup_{t\ge t_1} J(tu_1, tv_1) &\le \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\big(S_{t,\alpha,\beta}\big)^{\frac{N-t}{p-t}} \\ &\quad + O(\epsilon^{b(\mu)p^*(s_1)-N+s_1}) - \lambda \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \frac{N-s}{b(\mu)},\\ C\epsilon^{N-s+(1-\frac{N}{p})q}|\ln\epsilon|, & q = \frac{N-s}{b(\mu)}. \end{cases} \end{aligned} \tag{2.48}$$

Note that $N > p^2$, $b(\mu) \ge \frac{N-s}{q}$. Thus

$$\big[N - s + (1-\frac{N}{p})q\big]\frac{p-q}{q} < b(\mu)p + p - N - \big[N - s + (1-\frac{N}{p})q\big].$$

Choose $\lambda = \epsilon^\tau$, where $\big[N-s+(1-\frac{N}{p})q\big]\frac{p-q}{q} < \tau < b(\mu)p+p-N-\big[N-s+(1-\frac{N}{p})q\big]$. Then

$$\lambda O(\epsilon^{N-s+(1-\frac{N}{p})q}) = O(\epsilon^{\tau+N-s+(1-\frac{N}{p})q}), \quad d\lambda^{\frac{p}{p-q}} = O(\epsilon^{\frac{p\tau}{p-q}}).$$

Since $\tau + N - s + (1-\frac{N}{p})q < \frac{p\tau}{p-q}$, $\tau + N - s + (1-\frac{N}{p})q < b(\mu)p + p - N$, taking ϵ small enough we deduce that there exists $\delta > 0$, such that

$$O(\epsilon^{b(\mu)p^*(s_1)-N+s_1}) - \lambda O(\epsilon^{N-s+(1-\frac{N}{p})q}) < -d\lambda^{\frac{p}{p-q}}, \quad \forall\lambda : 0 < \lambda^{\frac{p}{p-q}} < \delta. \tag{2.49}$$

Choose $\Lambda_1 = \min\{\Lambda_1^*, \frac{p-q}{p}\delta\}$. Then for all $\lambda \in (0, \Lambda_1)$ we have

$$\sup_{t\ge t_1} J(tu_1, tv_1) \le \frac{p-t}{p(N-t)}\frac{1}{\|Q\|_\infty^{\frac{N-p}{p-t}}}\big(S_{t,\alpha,\beta}\big)^{\frac{N-t}{p-t}} - d\lambda^{\frac{p}{p-q}}.$$

Together with (2.44), we get the conclusion of Lemma 2.5. □

3. Proof of the main results

Proof of Theorem 1.1. Let

$$r := \|(u,v)\|,$$

$$f(r) := \frac{1}{p}r^p - \frac{1}{p^*(s_1)}A_{\mu,s_1}^{-\frac{p^*(s_1)}{p}} r^{p^*(s_1)} - \frac{1}{p^*(s_2)}A_{\mu,s_2}^{-\frac{p^*(s_2)}{p}} r^{p^*(s_2)} - \frac{1}{p^*(t)}S_{t,\alpha,\beta}^{-\frac{p^*(t)}{p}}\|Q\|_\infty,$$

$$h(r) := \frac{\lambda}{q}\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} r^q.$$

From (1.5), (1.6) and (1.7),

$$J(u,v) \geq f(r) - h(r).$$

Note that $p < p^*(s_1)$, $p^*(s_2)$, $p^*(t)$, it is easy to see that there exists $\varrho > 0$ such that $f(r)$ achieves its maximum at ϱ and $f(\varrho) > 0$. Therefore, there exists $\Lambda_{11} > 0$, such that for $0 < \lambda < \Lambda_{11}$,

$$\inf_{\|(u,v)\|=\varrho} I(u,v) \geq f(\varrho) - h(\varrho) > 0. \tag{3.1}$$

On the other hand, set $B_\varrho = \{(u,v);\ \|(u,v)\| \leq \varrho\}$. For $(u,v) \neq (0,0)$, we can choose $d > 0$ small enough, such that $(du, dv) \in B_\varrho$ and

$$I(du, dv) < 0. \tag{3.2}$$

Thus,

$$-\infty < \inf_{(u,v)\in B_\varrho} I(u,v) < 0. \tag{3.3}$$

Now we can apply the Ekeland variational principle in [22] and obtain $\{(u_n, v_n)\} \subset B_\varrho$, such that

$$I(u_n, v_n) \leq \inf_{(u,v)\in B_\varrho} I(u,v) + \frac{1}{n}, \tag{3.4}$$

$$I(u_n, v_n) \leq I(u,v) + \frac{1}{n}\|(u_n - u, v_n - v)\|, \tag{3.5}$$

for all $(u,v) \in B_R$. Define

$$J_1(u,v) := J(u,v) + \frac{1}{n}\|(u_n - u, v_n - v)\|. \tag{3.6}$$

By (3.5), we have (u_n, v_n) is the minimizer of $J_1(u,v)$ on B_ϱ. (3.1), (3.3) and (3.4) imply that there exists $\epsilon > 0$ and $k \in N$, such that for $n \geq k$, $\{(u,v),\ \|(u,v)\| \leq \varrho - \epsilon\}$. Therefore, for $n \geq k$ and $(\phi, \varphi) \in W$, we can choose $t > 0$ small enough, such that $(u_n + t\phi, v_n + t\varphi) \in B_\varrho$ and

$$\frac{J_1(u_n + t\phi, v_n + t\varphi) - J_1(u_n, v_n)}{t} \geq 0.$$

That is,

$$\frac{J(u_n + t\phi, v_n + t\varphi) - J(u_n, v_n)}{t} + \frac{1}{n}\|(\phi, \varphi)\| \geq 0. \tag{3.7}$$

Passing to the limit in (3.7) as $n \to 0$, one can obtain

$$\langle J'(u_n, v_n), (\phi, \varphi)\rangle \geq -\frac{1}{n}\|(\phi, \varphi)\|,$$

which implies

$$\|J'(u_n, v_n)\| \leq \frac{1}{n}. \tag{3.8}$$

Combining (3.4) and (3.8), there holds

$$\lim_{n\to\infty} J(u_n, v_n) = \inf_{(u,v)\in B_\varrho} J(u,v) < 0, \tag{3.9}$$

$$\lim_{n\to\infty} J'(u_n, v_n) = 0. \tag{3.10}$$

We note that there exists $\Lambda_{11}^* \in (0, \Lambda_{11})$, such that for $0 < \lambda < \Lambda_{11}^*$, and $c_0 > \inf_{(u,v)\in B_\varrho} I(u,v)$, where c_0 is defined in Lemma 2.3. Thus, (3.9) and (3.10) and Lemma Lemma 2.3 imply that for $0 < \lambda < \Lambda_{11}^*$, $(u_n, v_n) \to (u,v)$ strongly in

W. Therefore, (u,v) is a nontrivial solution of problem (1.1) satisfying $J(u,v) = \inf_{(u,v)\in B_\varrho} J(u,v) < 0$. Note that $J(u,v) = J(|u|,|v|)$ and

$$(|u|,|v|) \in \{(u,v),\ \|(u,v)\| \le \varrho - \epsilon\},$$

we have $I(|u|,|v|) = \inf_{(u,v)\in B_\varrho} J(u,v)$ and $J'(|u|,|v|) = 0$. Then problem (1.1) has a nontrivial nonnegative solution. By the strongly maximum principle, we get the conclusion of Theorem 1.1. □

Proof of Theorem 1.2. In view of the proof of Theorem 1.1, we know that for $0 < \lambda < \Lambda_{11}$, there exists $\varrho > 0$, such that $\inf_{\|(u,v)\|=\varrho} I(u,v) \ge \vartheta^* > 0$. Moreover, (3.9) and (3.10) hold. We note that there exists $\Lambda_{12} \in (0,\Lambda_{11})$, such that for $0 < \lambda < \Lambda_{12}$, $c_* > \inf_{(u,v)\in B_\varrho} J(u,v)$, where c_* is defined in Lemma 2.2. Thus (3.9) and (3.10) and Lemma 2.2 imply that $(u_n,v_n) \to (u,v)$ strongly in W. Standard argument shows that for $0 < \lambda < \Lambda_{12}$, problem (1.1) has at least one positive solution satisfying $J(u,v) < 0$ and $J'(u,v) = 0$.

Now we prove a second positive solution. It is easy to see $J(0,0) = 0$. Set $\Lambda^{**} = \min\{\Lambda_{12},\Lambda_1\}$, where Λ_1 is given in Lemma 2.5. Then it follows from Lemma 2.5 there exists $(u',v') \in W \setminus \{0\}$, such that for $0 < \lambda < \Lambda^{**}$,

$$\sup_{t\ge 0} J(tu',tv') < c_*.$$

On the other hand we obtain that $\lim_{l\to\infty} J(lu',lv') = -\infty$. Thus there exists $l' > 0$ such that $\|(l'u',l'v')\| > \varrho$ and $J(l'u',l'v') < 0$. Let

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C^0([0,1],W) : \gamma(0) = (0,0),\ \gamma(1) = (l'u',l'v')\}.$$

Thus, it follows from the mountain pass lemma in [2] that there exists a sequence $(u_n,v_n) \in W$ such that

$$\lim_{n\to\infty} J(u_n,v_n) = c,$$
$$\lim_{n\to\infty} J'(u_n,v_n) = 0.$$

Moreover, $c \in (0,c_*)$. From Lemma 2.2, $(u_n,v_n) \to (\overline{u},\overline{v})$ strongly in W, which implies that $J(\overline{u},\overline{v}) = c$ and $J'(\overline{u},\overline{v}) = 0$, Therefore, $(\overline{u},\overline{v})$ is a second nontrivial solution of (1.1). Set $u^+ = \max\{u,0\}, v^+ = \max\{v,0\}$. Replacing

$$\int_\Omega \frac{|u|^q}{|x|^s}dx, \quad \int_\Omega \frac{|v|^q}{|x|^s}dx, \quad \int_\Omega \frac{|u|^{p^*(s_1)}}{|x|^{s_1}}dx, \quad \int_\Omega \frac{|v|^{p^*(s_2)}}{|x|^{s_2}}dx, \quad \int_\Omega Q(x)\frac{|u|^\alpha|v|^\beta}{|x-x_0|^t}dx$$

by

$$\int_\Omega \frac{(u^+)^q}{|x|^s}dx, \quad \int_\Omega \frac{(v^+)^q}{|x|^s}dx, \quad \int_\Omega \frac{(u^+)^{p^*(s_1)}}{|x|^{s_1}}dx,$$
$$\int_\Omega \frac{(v^+)^{p^*(s_2)}}{|x|^{s_2}}dx, \quad \int_\Omega Q(x)\frac{(u^+)^\alpha(v^+)^\beta}{|x-x_0|^t}dx$$

and repeating the above process, we have a nonnegative solution $(\widetilde{u},\widetilde{v})$ of problem (1.1) satisfying $J(\widetilde{u},\widetilde{v}) > 0$. Then by the strongly maximum principle, we have a second positive solution. □

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