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SPECTRAL MAPPING THEOREM FOR AN EVOLUTION SEMIGROUP ON A SPACE OF VECTOR-VALUED ALMOST-PERIODIC FUNCTIONS

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Abstract. We give some characterizations for exponential stability of a periodic evolution family of bounded linear operators acting on a Banach space in terms of evolution semigroups acting on a special space of almost periodic functions. As a consequence, a spectral mapping theorem is stated.

1. Introduction

In the recent article [5], some connections between exponential stability of a q-periodic evolution family of bounded linear operators acting on a Banach space and spectral properties of the infinitesimal generator of the evolution semigroup associated to the evolution family, was established. There we cannot close the chain of equivalences, as in Theorem 4.1 below, because the state space of functions where the evolution semigroup acts, is not rich enough. The aim of this article is to enlarge the state space of functions, used there, such that the chain to be closed. As consequence, a spectral mapping theorem for the evolution semigroup is obtained.

This article is organized as follows. The next section contains the necessary definitions for the paper to be self-contained. In the third section we introduce the evolution semigroup associated with the periodic evolution family. Section 4 is devoted to prove the main result, while the last section deals with a spectral mapping theorem for the evolution semigroup, which is a consequence of the theoretical result established in the previous section.

2. Notation and preliminary results

Throughout this article X stands for a Banach space and $\mathcal{L}(X)$ denotes the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathcal{L}(X)$ are denoted by the same symbol, namely with $\|\cdot\|$.

Let $q > 0$. Recall that a family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$ is a strongly continuous and q-periodic evolution family on X if:

(1) $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r \geq 0$.
(2) $U(t,t) = I$ for $t \geq 0$, where I is the identity operator of $\mathcal{L}(X)$.

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(3) For each $x \in X$, the map
$$(t,s) \mapsto U(t,s)x : \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\} \to X$$
is continuous.

(4) $U(t+q, s+q) = U(t,s)$ for all pairs (t,s) with $t \geq s \geq 0$.

Clearly, any q-periodic evolution family $\mathcal{U} = \{U(t,s)\}$ defined for the pairs (t,s) with $t \geq s \geq 0$ could be extended to a q-periodic evolution family for all pairs (t,s) with $t \geq s \in \mathbb{R}$, by setting $U(t,s) = U(t+kq, s+kq)$, where k is the smallest positive integer number for which $s + kq \geq 0$. We say that the evolution family $\mathcal{U}$ *has exponential growth* if there exist the constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$, for all $t \geq s$. Every strongly continuous and q-periodic evolution family acting on a Banach space has an exponential growth, [9]. Recall that a one parameter family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup if $T(t+s) = T(t) \circ T(s)$ for all $t \geq s \geq 0$, $T(0) = I$ and for each $x \in X$ the map $t \mapsto T(t)x$ is continuous. If a strongly continuous evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$, verifies the convolution condition, $U(t,s) = U(t-s,0)$, for every pair (t,s) with $t \geq s \geq 0$, then the one parameter family, $\{T(t)\}_{t\geq 0}$, defined by $T(t) := U(t,0)$, is a strongly continuous semigroup. Each strongly continuous semigroup $\mathbf{T}$ has an infinitesimal generator $A : D(A) \subset X \to X$, defined by $Ax := \frac{d}{dt}T(t)x|_{t=0}$. It is well-known that A is linear, densely defined and closed operator. The domain $D(A)$ consists by all $x \in X$ for which the map $t \mapsto T(t)x$ is differentiable at $t = 0$. By $\rho(A)$ is denoted the resolvent set of A, i.e. the set of all complex scalars z for which $zI - A$ is an invertible operator in $\mathcal{L}(X)$. The set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the spectrum of the operator A and the set $s(A) := \sup\{Re(\lambda) : \lambda \in \sigma(A)\}$ is the spectral bound of A. For further details concerning the theory of strongly continuous semigroups we refer to the monographs [12, 1].

Proposition 2.1. *Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and q-periodic evolution family acting on the Banach space X. The following four statements are equivalent:*

(1) *The family $\mathcal{U}$ is uniformly exponentially stable.*

(2) *There exist two positive constants N and ν such that*
$$\|U(t,0)\| \leq Ne^{-\nu t}, \text{ for all } t \geq 0.$$

(3) *The spectral radius of $U(q,0)$ is less than one; i.e.,*
$$r(U(q,0)) := \sup\{|\lambda|, \lambda \in \sigma(U(q,0))\} = \lim_{n\to\infty} \|U(q,0)^n\|^{\frac{1}{n}} < 1.$$

(4) *For each $\mu \in \mathbb{R}$, one has*
$$\sup_{n\geq 1} \|\sum_{k=1}^{n} e^{-i\mu k} U(q,0)^k\| := M(\mu) < \infty.$$

The proof of the implications (1) ⇒ (2) ⇒ (3) ⇒ (4) is obvious. The proof of (4) ⇒ (1) can be found in [4, Lemma 1].

3. An evolution semigroup

In this section we consider a space of X-valued functions and define an evolution semigroup acting on it. For this purpose, we need the following spaces:

- $BUC(\mathbb{R}, X)$ which is the space of all X-valued bounded uniformly continuous functions defined on $\mathbb{R}$, endowed with the "sup" norm $\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f\|$.
- $P_q(\mathbb{R}, X)$ which is the subspace of $BUC(\mathbb{R}, X)$ consisting of all functions F such that $F(t+q) = F(t)$ for all $t \in \mathbb{R}$.
- $AP_1(\mathbb{R}, X)$ which is the space of all X-valued functions defined on $\mathbb{R}$ representable in the form $f(t) = \sum_{k=-\infty}^{k=\infty} e^{i\mu_k t} c_k(f)$ for all $t \in \mathbb{R}$, where $\mu_k \in \mathbb{R}$, $c_k(f) \in X$ and
$$\|f\|_1 := \sum_{k=-\infty}^{k=\infty} \|c_k(f)\| < \infty.$$
Further details about the space $AP_1(\mathbb{R}, X)$ can be found in [10].

For an arbitrary $t \geq 0$, we denote by $\mathcal{A}_t$ the set of all X-valued functions defined on $\mathbb{R}$ such that there exists a function F in $P_q(\mathbb{R}, X) \cap AP_1(\mathbb{R}, X)$ with $F(t) = 0$, $f = F_{|_{[t,\infty)}}$ and $f(s) = 0$, for all $s < t$. Set $\mathcal{A} := \{e^{i\mu\cdot} \otimes f : \mu \in \mathbb{R} \text{ and } f \in \cup_{t\geq 0}\mathcal{A}_t\}$ and let $E(\mathbb{R}, X) := \operatorname{span}(\mathcal{A})$. Consider the space $\widetilde{E}(\mathbb{R}, X) := \overline{\operatorname{span}}(\mathcal{A})$ which is a closed subspace of $BUC(\mathbb{R}, X)$ endowed with the "sup" norm. The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ associated to a strongly continuous and q-periodic evolution family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ on $\widetilde{E}(\mathbb{R}, X)$ is formally defined by
$$(\mathcal{T}(t)\widetilde{f})(s) := \begin{cases} U(s, s-t)\widetilde{f}(s-t), & s \geq t \\ 0, & s < t \end{cases} \tag{3.1}$$
for $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$.

Proposition 3.1. *The evolution semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, defined in* (3.1)*, acts on $\widetilde{E}(\mathbb{R}, X)$ and is strongly continuous.*

Proof. Let $\widetilde{f}(t) = e^{i\mu t} f(t)$, with $\mu \in \mathbb{R}$ and $f \in \cup_{t\geq 0}\mathcal{A}_t$. Then, there exist $r \geq 0$ and a function $F \in P_q(\mathbb{R}, X) \cap AP_1(\mathbb{R}, X)$ such that $F(r) = 0$, $f(s) = F(s)$ for $s \geq r$ and $f(s) = 0$ for $s < r$. Thus, for each fixed $t \geq 0$ and $s \in \mathbb{R}$, we have
$$(\mathcal{T}(t)\widetilde{f})(s) = \begin{cases} e^{i\mu(s-t)} U(s, s-t) F(s-t), & s \geq t+r \\ 0, & s < t+r. \end{cases}$$
The map $s \mapsto G(s) := e^{-i\mu t} U(s, s-t) F(s-t)$ is q-periodic and belongs to $AP_1(\mathbb{R}, X)$. Moreover
$$\|G(\cdot)\|_1 \leq \|U(s, s-t)\| \|\sum_{k=-\infty}^{k=\infty} e^{i\mu_k(s-t)} c_k(F)\| \leq M e^{\omega t} \|F(\cdot)\|_1 < \infty,$$
for some $M \geq 1$ and $\omega \in \mathbb{R}$. Thus, $\mathcal{T}(t)\widetilde{f} \in \mathcal{A}$.

As operator from $\widetilde{E}(\mathbb{R}, X)$ to $BUC(\mathbb{R}, X)$, $\mathcal{T}(t)$ is linear. When $\widetilde{f} = \alpha\widetilde{g} + \beta\widetilde{h} \in E(\mathbb{R}, X)$, with $\widetilde{g}, \widetilde{h} \in \mathcal{A}$ and α, β are complex scalars, one has $\mathcal{T}(t)\widetilde{f} = \alpha\mathcal{T}(t)(\widetilde{g}) + \beta\mathcal{T}(t)(\widetilde{h})$. But $\mathcal{T}(t)(\widetilde{g}), \mathcal{T}(t)(\widetilde{h}) \in \mathcal{A}$ and therefore $\mathcal{T}(t)\widetilde{f}$ belongs to $E(\mathbb{R}, X)$. Finally, let $\widetilde{f}$ in $\widetilde{E}(\mathbb{R}, X)$. There exists a sequence $(\widetilde{f}_n)$ in $E(\mathbb{R}, X)$ such that $\sup_{t\geq 0} \|\widetilde{f}_n(t) - \widetilde{f}(t)\| \to 0$ as $n \to \infty$. Hence
$$\sup_{s\geq 0} \|(\mathcal{T}(t)\widetilde{f}_n)(s) - (\mathcal{T}(t)\widetilde{f})(s)\| = \sup_{s\geq t} \|U(s, s-t)\widetilde{f}_n(s-t) - U(s, s-t)\widetilde{f}(s-t)\|$$

$$\le Me^{\omega t} \sup_{s\ge t} \|\widetilde{f}_n(s-t) - \widetilde{f}(s-t)\| \to 0.$$

Thus, the evolution semigroup $\mathcal{T}$ acts on $\widetilde{E}(\mathbb{R}, X)$. In what follows we prove that it is strongly continuous. For each $f \in \mathcal{A}$, we have

$$\begin{aligned}
&\|\mathcal{T}(t)f - f\|_{\widetilde{E}(\mathbb{R},X)}\\
&= \sup_{s\ge t} \|U(s, s-t)f(s-t) - f(s)\|\\
&= \sup_{s\ge t} \|U(s, s-t)f(s-t) - U(s, s-t)f(s) + U(s, s-t)f(s) - f(s)\|\\
&\le \sup_{s\ge t} \|U(s, s-t)\|\|f(s-t) - f(s)\| + \sup_{s\ge t} \|[U(s, s-t) - U(s, s)]f(s)\|\\
&\le Me^{\omega t} \sup_{s\ge t} \|f(s-t) - f(s)\| + \sup_{s\ge t} \|[U(s, s-t) - U(s, s)]f(s)\|.
\end{aligned}$$

The first term of the last line tends to zero as $t \to 0$, because the function $\widetilde{f}$ is uniformly continuous, while the second one tends to zero because the evolution family $\mathcal{U}$ is strongly continuous.

Let $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ and let (f_n) be a sequence in $E(\mathbb{R}, X)$ converging to $\widetilde{f}$. Choose a positive integer number N such that $\|\widetilde{f} - f_N\|_\infty$ is sufficiently small. Then,

$$\begin{aligned}
&\|\mathcal{T}(t)\widetilde{f} - \widetilde{f}\|_{\widetilde{E}(\mathbb{R},X)}\\
&\le \|\mathcal{T}(t)\widetilde{f} - \mathcal{T}(t)f_N\|_{\widetilde{E}(\mathbb{R},X)} + \|\mathcal{T}(t)f_N - f_N\|_{\widetilde{E}(\mathbb{R},X)} + \|f_N - \widetilde{f}\|_\infty\\
&\le Me^{\omega t}\|\widetilde{f} - f_N\|_\infty + \|\mathcal{T}(t)f_N - f_N\| + \|f_N - \widetilde{f}\|_\infty.
\end{aligned}$$

The middle term tends to zero as $t \to 0^+$ as is shown before. □

4. Results

In the next theorem we collect some characterizations for uniform exponential stability of a q-period evolution family in terms of evolution semigroups and admissibility related to the spaces $\widetilde{E}(\mathbb{R}, X)$ and $BUC(\mathbb{R}, X)$. Similar results to Theorem 4.1 was stated in [5], but there the chain of equivalences was not closed. Our space $\widetilde{E}(\mathbb{R}, X)$ is rich enough and this property allow us to prove (5) ⇒ (1) under the assumption that there is a dense set $D \subset X$ such that the map $U(\cdot, 0)x$ satisfy a Lipschitz condition on $\mathbb{R}_+ := (0, \infty)$.

Theorem 4.1. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on a Banach space X and let $\mathcal{T}$ be its associated evolution semigroup on $\widetilde{E}(\mathbb{R}, X)$. Denote by G its infinitesimal generator. Consider the statements:*

(1) *$\mathcal{U}$ is uniformly exponentially stable.*
(2) *$\mathcal{T}$ is uniformly exponentially stable.*
(3) *G is invertible.*
(4) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map*

$$t \mapsto g_{\widetilde{f}}(t) := \int_0^t U(t, s)\widetilde{f}(s)ds$$

belongs to $\widetilde{E}(\mathbb{R}, X)$.

(5) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $g_{\widetilde{f}}$ belongs to $BUC(\mathbb{R}_+, X)$.*

Then,

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5).$$

In addition, if there is a dense subset D of X such that for each $x \in D$ the map $s \mapsto U(s,0)x : \mathbb{R}_+ \to X$ satisfy a Lipschitz condition on $\mathbb{R}_+$, then (5) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2). Let N and ν be two positive constants such that $\|U(t,s)\| \leq Ne^{-\nu(t-s)}$ for all $t \geq s$. Then, for all $t \geq 0$ and any $\widetilde{f}$ belonging to $\widetilde{E}(\mathbb{R},X)$, one has

$$\begin{aligned}
\|\mathcal{T}(t)\widetilde{f}\|_{\widetilde{E}(\mathbb{R},X)} &= \sup_{s\geq t} \|U(s,s-t)\widetilde{f}(s-t)\| \\
&\leq Ne^{-\nu t} \sup_{s\geq t} \|\widetilde{f}(s-t)\| \\
&= Ne^{-\nu t}\|\widetilde{f}\|_{\widetilde{E}(\mathbb{R},X)}.
\end{aligned}$$

(2) $\Rightarrow$ (1). Let g be a q-periodic function given on $[0,q]$ by $g(s) := \frac{4}{q^2}\left(s - \frac{q}{2}\right)^2$. It is obviously that g belongs to $AP_1(\mathbb{R},\mathbb{C})$. Then, for each nonzero $x \in X$, the map

$$s \mapsto f_x(s) : \begin{cases} e^{i\mu s} g(s)x, & s \geq \frac{q}{2} \\ 0, & s < \frac{q}{2}, \end{cases}$$

belongs to $\widetilde{E}(\mathbb{R},X)$. By assumption, there exist two positive constants N and ν such that for all $s \geq t \geq 0$, have that

$$Ne^{-\nu t}\|\widetilde{f}_x\|_{\widetilde{E}(\mathbb{R},X)} \geq \|(\mathcal{T}(t)\widetilde{f}_x)(s)\| \geq g(s-t)\|U(s,s-t)x\|.$$

For $s = t + q$ this yields

$$Ne^{-\nu t}\|x\| \geq Ne^{-\nu t}\|\widetilde{f}_x\|_{\widetilde{E}(\mathbb{R},X)} \geq g(q)\|U(t+q,q)x\| = \|U(t,0)x\|.$$

The assertion follows by applying Proposition 2.1.

(2) $\Rightarrow$ (3). The growth bound $\omega_0(\mathcal{T})$ is negative and $s(G) \leq \omega_0(\mathcal{T})$. Hence 0 belongs to $\rho(G)$.

(3) $\Rightarrow$ (4). We need de following lemma.

Lemma 4.2. *Let $\widetilde{f}, \widetilde{u} \in \widetilde{E}(\mathbb{R},X)$. The following two statements are equivalent:*

- $\widetilde{u} \in D(G)$ *and* $G\widetilde{u} = -\widetilde{f}$.
- $\widetilde{u}(t) = \int_0^t U(t,s)\widetilde{f}(s)ds$ *for all* $t \geq 0$.

The proof of this lemma is similar to [13, Lemma 1.1], an it is omitted. Let $\widetilde{f} \in \widetilde{E}(\mathbb{R},X)$. The surjectivity of G yields the existence of u in $D(G)$ subset of $E(\mathbb{R},X)$ such that $Gu = -\widetilde{f}$. The assertion follows by applying Lemma 4.2.

(4) $\Rightarrow$ (5). The map $g_{\widetilde{f}}$ is bounded because it belongs to $\widetilde{E}(\mathbb{R},X)$ which is a subset of $BUC(\mathbb{R},X)$.

(5) $\Rightarrow$ (1). Let consider the map $H : \widetilde{E}(\mathbb{R},X) \to BUC(\mathbb{R},X)$, given by

$$H(\widetilde{f}) := \int_0^{\cdot} U(\cdot,s)\widetilde{f}(s)ds.$$

Let $\widetilde{f}, \widetilde{h}, \widetilde{f}_n \in \widetilde{E}(\mathbb{R},X)$ with $\widetilde{f}_n \to \widetilde{f}$ and $H(\widetilde{f}_n) \to \widetilde{h}$. Then for each fixed $t \geq 0$, one has

$$\left(H(\widetilde{f}_n) - H(\widetilde{f})\right)(t) = \int_0^t \left(U(t,s)(\widetilde{f}_n(s) - \widetilde{f}(s))\right) ds \to 0,$$

when $n \to \infty$. Therefore, $\widetilde{h} = H(\widetilde{f})$; i.e., the operator H is closed, and then it is bounded. As a consequence there exists a positive constant k, such that

$$\|H(\widetilde{f})\|_\infty \le k\|\widetilde{f}\|_\infty, \quad \forall \widetilde{f} \in \widetilde{E}(\mathbb{R}, X).$$

Let g in $P_q(\mathbb{R},\mathbb{R})$ satisfying a Lipschitz condition on $\mathbb{R}$ and having the properties that $g(0) = g(q) = 0$, $\sup_{s\in[0,q]} g(s) := k_1 < \infty$ and $m(\mu) := \int_0^q e^{i\mu s} g(s)ds \neq 0$ for each real number μ. For an arbitrary x in D let us consider the map $g_x \in P_q(\mathbb{R}_+, X)$ given on $[0,q]$ by $g_x(s) := g(s)U(s,0)x$. Obviously, g_x satisfies a Lipschitz condition on $\mathbb{R}$, and by [2, Lemma 3.3], it belongs to $AP_1(\mathbb{R}, X)$. Now, consider the map

$$s \mapsto \widetilde{f}_{x,\mu}(s) := \begin{cases} e^{i\mu s} g_x(s), & s \ge 0 \\ 0, & s < 0. \end{cases}$$

Clearly, $\widetilde{f}_{x,\mu}$ belongs to $\widetilde{E}(\mathbb{R}, X)$. Then,

$$H(\widetilde{f}_{x,\mu})(t) = \int_0^t U(t, t-s)\widetilde{f}_{x,\mu}(t-s)ds = \int_0^t U(t,\rho)\widetilde{f}_{x,\mu}(\rho)d\rho.$$

Let $t = nq$, for $n = 0, 1, 2, \dots$. Then

$$\begin{aligned}
\int_0^{nq} U(nq,\rho)\widetilde{f}_{x,\mu}(\rho)d\rho &= \sum_{k=0}^{n-1} \int_{kq}^{(k+1)q} e^{i\mu\rho} U(nq,\rho) g_x(\rho)d\rho \\
&= \sum_{k=0}^{n-1} \int_0^q e^{i\mu(kq+\tau)} U((n-k)q,\tau) g_x(\tau)d\tau \\
&= \sum_{k=0}^{n-1} \Big(\int_0^q e^{i\mu\tau} g(\tau)d\tau\Big) e^{i\mu kq} U(q,0)^{n-k} x \\
&= m(\mu) \sum_{k=0}^{n-1} e^{i\mu kq} U(q,0)^{n-k} x.
\end{aligned}$$

Passing to the norms, we obtain

$$\begin{aligned}
\|\sum_{k=0}^{n-1} e^{i\mu kq} U(q,0)^{n-k} x\| &= \frac{1}{|m(\mu)|}\|\int_0^{nq} U(nq,\rho)\widetilde{f}_{x,\mu}(\rho)d\rho\| \\
&\le \frac{k}{|m(\mu)|}\|f_{x,\mu}\|_\infty \le \frac{kk_1K}{|m(\mu)|}\|x\|,
\end{aligned} \tag{4.1}$$

for every x in D, where $K := \sup_{t\in[0,q]} \|U(t,0)\| = Me^{\omega q}$. Taking into account that D is a dense set in X yields that (4.1) holds for every x in X. The assertion follows applying by Proposition 2.1. □

Remark 4.3. (i) Let q be a positive number and let h be a $\mathbb{C}$-valued continuous function that is not differentiable at any point in $[0,q]$. In addition, we suppose that $h(t) \neq 0$, for any $t \in [0,q]$, and $h(0) = h(q)$. Denote by $\widetilde{h}$ the extension by periodicity of the function h to $\mathbb{R}_+$. An example of such a function could be found in [15]. We can easily verify that

$$\Big\{U(t,s) := \frac{\widetilde{h}(t)}{\widetilde{h}(s)},\ t \ge s \ge 0\Big\},$$

is a strongly continuous and q-periodic evolution family on $\mathbb{C}$. As is well known, every complex-valued function, defined and satisfying a Lipschitz condition on the interval $[0,q]$, is almost everywhere differentiable on that interval. Then, the map $t \mapsto U(t,0) = \frac{1}{\widetilde{h}(0)}\widetilde{h}(t)$ does not satisfy a Lipschitz condition on $(0,q)$.

(ii) First recall the Weis-Wrobel Theorem, [16].

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on a complex Banach space. For each λ in the resolvent set, let us denote by $R(\lambda, A)$ the resolvent operator of the infinitesimal generator A of $\mathbf{T}$. If the resolvent set, $\rho(A)$, contains the closed half plane $\overline{\mathbb{C}_+} := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\}$ and the map $R(\cdot, A)$ is bounded on $\overline{\mathbb{C}_+}$, then, the semigroup $\mathbf{T}$ is exponentially stable; i. e., the growth bound

$$\omega_1(\mathbf{T}) = \limsup_{t\to\infty} \frac{\ln \|T(t)R(z,A)\|}{t}$$

is negative, for some (and then for all) $z \in \rho(A)$.

Note that, if $f \in \widetilde{E}(\mathbb{R}, X)$, then, for each $\mu \in \mathbb{R}$, $e^{i\mu\cdot} f$ belongs to $\widetilde{E}(\mathbb{R}, X)$. Indeed, if $f \in \widetilde{E}(\mathbb{R}, X)$, then, there exists a sequence (f_n), with $f_n \in E(\mathbb{R}, X)$, such that f_n converge uniformly to f. Thus, for every $\mu \in \mathbb{R}$,

$$\sup_{s\geq 0} \|e^{i\mu s} f_n(s) - e^{i\mu s} f(s)\| = \sup_{s\geq 0} \|f_n(s) - f(s)\| \to 0.$$

Now, assume that the family $\mathcal{U}$ is uniformly bounded and that the statement (5), from the previous theorem, holds. Then

$$\sup_{t\geq 0} \| \int_0^t e^{i\mu s} U(t,s) g(s) ds\| \leq M\|g\|,$$

where $g(s) := e^{-i\mu s} f(s)$, for all $f \in \widetilde{E}(\mathbb{R}, X)$ and $\mu \in \mathbb{R}$. Applying first [5, Lemma 3.3] (which remains true when replacing $E(\mathbb{R}, X)$) with $\widetilde{E}(\mathbb{R}, X)$ and then [14, Lemma 3], it follows that $i\mu \in \rho(G)$ and $\sup_{\mu\in\mathbb{R}} \|R(i\mu, G)\| < \infty$. Therefore, by Hille-Yoshida theorem and Phragmen-Lindelöf theorem follows that the map $R(\cdot, A)$ is bounded on $\overline{\mathbb{C}_+}$.

Now, in view of Weis-Wrobel's theorem, the evolution semigroup $\mathcal{T}$, associated to $\mathcal{U}$ on $\widetilde{E}(\mathbb{R}, X)$, is exponential stable. At least for us, remains as open problem whether the exponential stability of the evolution semigroup $\mathcal{T}$ implies the uniform exponential stability of $\mathcal{U}$.

Thus, building an example on arbitrary Banach spaces, where the implication (5) $\Rightarrow$ (1) does not occur, seems to be a difficult matter that could be the subject for an another paper.

Corollary 4.4. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup acting on X and let $\mathcal{T}$ be its associated evolution semigroup on $\widetilde{E}(\mathbb{R}, X)$. Denote by G its infinitesimal generator. The following five statements are equivalent.*

(1) *$\mathbf{T}$ is uniformly exponentially stable.*
(2) *$\mathcal{T}$ is uniformly exponentially stable.*
(3) *G is invertible.*
(4) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map*

$$t \mapsto g_{\widetilde{f}}(t) := \int_0^t T(t-s)\widetilde{f}(s) ds$$

belongs to $\widetilde{E}(\mathbb{R}, X)$.

(5) *For each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$ the map $g_{\widetilde{f}}$ is bounded on $\mathbb{R}_+$.*

Proof. In the proof of (5) $\Rightarrow$ (1) we no longer need additional conditions. Indeed, for $x \in D(A)$, the map $t \mapsto T(t)x$, is Lipschitz on $\mathbb{R}_+$. To see this, let $t, s \in [0, q]$. Then

$$\begin{aligned}\|T(t)x - T(s)x\| &= \|\int_s^t \frac{d}{d\tau} T(\tau)x d\tau\| \\ &= \|\int_s^t T(\tau)Ax d\tau\| \\ &\leq \sup_{\tau \in [0,q]} \|T(\tau)\| \|Ax\| \, |t - s|.\end{aligned}$$

When $t = t^* + k_1 q$ and $s = s^* + k_2 q$, with $t^*, s^* \in [0, q]$ and k_1, k_2 non-negative integer numbers, have that

$$\|T(t)x - T(s)x\| = \|T(t^*)x - T(s^*)x\| \leq \sup_{\tau \in [0,q]} \|T(\tau)\| \|Ax\| \, |t - s|.$$

□

In terms of well-posed abstract Cauchy problems, the result contained in Theorem 4.1 may be read as follows.

Corollary 4.5. *Let $(A(t), D(A(t)))_{t \geq 0}$ be a family of linear operators acting on a Banach space X. Assume that the evolution family $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\}$, generated by the family $\{A(t)\}_{t \geq 0}$, is strongly continuous, q-periodic and there is a dense subset D of X such that the map $U(\cdot, 0)x$ satisfy a Lipschitz condition on $\mathbb{R}_+$ for every $x \in D$. Then, $\mathcal{U}$ is uniformly exponentially stable if and only if for each $\widetilde{f} \in \widetilde{E}(\mathbb{R}, X)$, the solution of the abstract Cauchy Problem*

$$\begin{gathered}\dot{u}(t) = A(t)u(t) + \widetilde{f}(t), \quad t > 0 \\ u(0) = 0,\end{gathered}$$

is bounded on $\mathbb{R}_+$.

5. Applications

An immediate consequence of Theorem 4.1 is the spectral mapping theorem for the evolution semigroup $\mathcal{T}$ on $\widetilde{E}(\mathbb{R}, X)$. Similar results can be found in [3, Theorem 2.5], [6, Theorem 3.5], [8, Theorem 3.6], [7, Theorem 3.1], [13, Corollary 2.4] for evolution semigroups acting on other spaces.

Theorem 5.1. *Let $\mathcal{U}$ be a strongly continuous and q-periodic evolution family acting on X and let $\mathcal{T}$ be its associated evolution semigroup on $\widetilde{E}(\mathbb{R}, X)$. Let denote by G the infinitesimal generator of $\mathcal{T}$. Suppose that there is a dense subset D of X such that for each $x \in D$ the map $s \mapsto U(s, 0)x : \mathbb{R}_+ \to X$ satisfy a Lipschitz condition on $\mathbb{R}_+$. Then*

$$\sigma(G) = \{z \in \mathbb{C} : Re(z) \leq s(G)\}.$$

Proof. It is well-known that $\rho(G) \supseteq \{z \in \mathbb{C} : Re(z) > s(G)\}$. To establish the converse inclusion, let $\lambda \in \rho(G)$ and $\mu \in \mathbb{C}$ with $Re(\mu) \geq Re(\lambda)$. We prove that $\mu \in \rho(G)$. Consider the evolution family $U_\lambda(t, s) := e^{-\lambda(t-s)} U(t, s)$, $t \geq s \geq 0$ whose associated evolution semigroup is $\mathcal{T}_\lambda(t) := e^{-\lambda t} \mathcal{T}(t)$. Obviously, $\lambda I - G$ is the infinitesimal generator of $\mathcal{T}_\lambda$. Because $\lambda I - G$ is invertible and applying

Theorem 4.1, $\mathcal{T}_\lambda$ (and then $\mathcal{T}_\mu$)is uniformly exponentially stable. Therefore, by applying again Theorem 4.1, $\mu \in \rho(G)$. □

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