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PROPERTIES OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS IN THE UNIT DISC

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Abstract. In this article we study the growth of solutions of linear differential equations with analytic coefficients in the unit disc. Our investigation is based on the behavior of the coefficients on a neighborhood of a point on the boundary of the unit disc.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ (see [9, 14]). In addition, the order of meromorphic function $f(z)$ in D is defined by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f; and for an analytic function $f(z)$ in D, we have also the definition

$$\sigma_M(f) = \limsup_{r\to 1^-} \frac{\log^+ \log^+ M(r,f)}{\log \frac{1}{1-r}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. Tsuji [13, p. 205] states that

$$\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1.$$

For example, the function $g(z) = \exp\{\frac{1}{(1-z)^\mu}\}$ satisfies $\sigma(g) = \mu - 1$ and $\sigma_M(g) = \mu$. Obviously, we have

$$\sigma(f) < \infty \text{ if and only if } \sigma_M(f) < \infty.$$

Definition 1.1 ([10]). A meromorphic function f in D is called admissible if

$$\limsup_{r\to 1^-} \frac{\log T(r,f)}{\log \frac{1}{1-r}} = \infty;$$

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and f is called nonadmissible if

$$\limsup_{r\to 1^-} \frac{\log T(r,f)}{\log \frac{1}{1-r}} < \infty.$$

There are some similarities between results of linear differential equations in the complex plane and the unit disc. For example, Heittokangas [10] obtained the following results.

Theorem 1.2 ([10]). *Let $A(z)$ and $B(z)$ be analytic functions in the unit disc. If $\sigma(A) < \sigma(B)$ or $A(z)$ is nonadmissible while $B(z)$ is admissible, then all solutions $f(z) \not\equiv 0$ of the linear differential equation*

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

are of infinite order of growth.

Theorem 1.3 ([10]). *Let $A_0(z), \dots, A_{k-1}(z)$ be analytic coefficients in the unit disc of the linear differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_0(z)f = 0. \tag{1.2}$$

Let $A_d(z)$ be the last coefficient not being an $\mathcal{H}$-function while the coefficients $A_{d+1}(z), \dots, A_{k-1}(z)$ are $\mathcal{H}$-functions. Then possesses at most d linearly independent analytic solutions of finite order of growth.

These theorems are analogous of the following results respectively.

Theorem 1.4 ([7]). *Let $A(z)$ and $B(z)$ be entire functions. If $\sigma(A) < \sigma(B)$ or $A(z)$ is a polynomial and $B(z)$ is transcendental, then every solution $f(z) \not\equiv 0$ of* (1.1) *has infinite order.*

Theorem 1.5 ([6]). *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions. Let $A_d(z)$ be the last transcendental coefficient in* (1.2) *while $A_{d+1}(z), \dots, A_{k-1}(z)$ are polynomials. Then* (1.2) *possesses at most d linearly independent entire solutions of finite order of growth.*

In general, the study of growth of solutions of linear differential equations in the complex plane or in the unit disc is based on the dominant of some coefficient by using the order, iterated order, type and the degree; see for example [4, 7, 11, 12]. In this paper, we will get out of these methods by using only the behavior of the coefficients near a point on the boundary of the unit disc. By this concept, we can study certain class of linear differential equations with analytic coefficients in the unit disc having the same order and type. We are motivated by certain results in the complex plane concerning the linear differential equation

$$f'' + A(z)e^{az}f' + B(z)e^{bz}f = 0, \tag{1.3}$$

where $A(z)$ and $B(z)$ are entire functions, see for example [1, 2, 3, 8]. Chen [2] proved that if $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$, then every solution $f(z) \not\equiv 0$ of (1.3) is of infinite order.

We will see that there are similarities and differences between our results and those of the complex plane. In fact, we will prove the following results.

Theorem 1.6. *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ is a real constant, b and z_0 are complex numbers such that*

$b \neq 0$, $|z_0| = 1$. If $A(z)$ and $B(z)$ are analytic on z_0 then every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + A(z)f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0, \tag{1.4}$$

is of infinite order.

Example 1.7. Every solution $f(z) \not\equiv 0$ of the differential equation

$$f'' + e^{\frac{1}{(1+z)^\alpha}} f' + e^{\frac{1}{(1-z)^\beta}} f = 0,$$

is of infinite order, where $\alpha > 1$ and $\beta > 1$ are real constants. We see that, in the case $\alpha = \beta$, the coefficients have the same order and type.

Theorem 1.8. *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ is a real constant, a, b and z_0 are complex numbers such that $ab \neq 0$, $\arg a \neq \arg b$, $|z_0| = 1$. If $A(z)$ and $B(z)$ are analytic on z_0 then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)e^{\frac{a}{(z_0-z)^\mu}} f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0, \tag{1.5}$$

is of infinite order.

Theorem 1.9. *Let $A(z)$ and $B(z) \not\equiv 0$ be analytic functions in the unit disc. Suppose that $\mu > 1$ is a real constant, a, b and z_0 are complex numbers such that $ab \neq 0$, $a = cb$ $(0 < c < 1)$, $|z_0| = 1$. If $A(z)$ and $B(z)$ are analytic on z_0 then every solution $f(z) \not\equiv 0$ of the differential equation*

$$f'' + A(z)e^{\frac{a}{(z_0-z)^\mu}} f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0, \tag{1.6}$$

is of infinite order

Remark 1.10. In all these our theorems, it may happen that the order of growth of $A(z)$ and $B(z)$ is greater than μ not like of the complex plane case.

We can generalize our previous Theorems to the higher differential equations as follows.

Theorem 1.11. *Consider the linear differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + B(z)e^{\frac{b}{(z_0-z)^\mu}} f = 0, \tag{1.7}$$

where $\mu > 1$ is a real constant, b and z_0 are complex numbers such that $b \neq 0$, $|z_0| = 1$, $B(z) \not\equiv 0$, $A_0(z), \dots, A_{k-1}(z)$ are analytic functions in the unit disc such that either $A_j(z)$ is analytic on z_0 or $A_j(z) = B_j(z)e^{\frac{b_j}{(z_0-z)^\mu}}$ with $B_j(z)$ is analytic on z_0 and $b_j = c_j b$ $(0 < c_j < 1)$ or $\arg b_j \neq \arg b$ for at most one possibility. Then every solution $f(z) \not\equiv 0$ of (1.7) is of infinite order.

Remark 1.12. Throughout this paper, we choose the principal branch of logarithm of the function $e^{\frac{\lambda}{(z_0-z)^\mu}}$ $(\lambda \in \mathbb{C}\backslash\{0\})$.

2. Preliminaries

To prove our results we need the following lemmas.

Lemma 2.1 ([5]). *Let f be a meromorphic function in the unit disc D of finite order σ. Let $\varepsilon > 0$ be a constant; k and j be integers satisfying $k > j \geq 0$. Assume that $f^{(j)} \not\equiv 0$. Then there exists a set $E \subset [0,1)$ which satisfies $\int_E \frac{1}{1-r} dr < \infty$, such that for all $z \in D$ satisfying $|z| \notin E$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq \Big(\frac{1}{1-|z|}\Big)^{(k-j)(\sigma+2+\varepsilon)}.$$

Lemma 2.2. *Let $A(z)$ be an analytic function on a point $z_0 \in \mathbb{C}$. Set $g(z) = A(z)e^{\frac{a}{(z_0-z)^\mu}}$, ($\mu > 0$ is a real constant), $a = \alpha + i\beta \neq 0$, $z_0 - z = Re^{i\varphi}$, $\delta_a(\varphi) = \alpha\cos(\mu\varphi) + \beta\sin(\mu\varphi)$, and $H = \{\varphi \in [0, 2\pi) : \delta_a(\varphi) = 0\}$, (obviously, H is of linear measure zero). Then for any given $\varepsilon > 0$ and for any $\varphi \in [0, 2\pi)\backslash H$, there exists $R_0 > 0$ such that for $0 < R < R_0$, we have:*

(i) if $\delta_a(\varphi) > 0$, then

$$\exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\} \leq |g(z)| \leq \exp\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\}, \tag{2.1}$$

(ii) if $\delta_a(\varphi) < 0$, then

$$\exp\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\} \leq |g(z)| \leq \exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\}. \tag{2.2}$$

Proof. We have

$$\big|e^{\frac{a}{(z_0-z)^\mu}}\big| = \exp\{\delta_a(\varphi)\frac{1}{R^\mu}\}. \tag{2.3}$$

If z_0 is a zero of order m of $A(z)$, then there exist $c_1 > 0,\ c_2 > 0$ such that

$$c_1R^m \leq |A(z)| \leq c_2R^m, \quad \text{for } z \text{ near enough } z_0. \tag{2.4}$$

Using (2.3) and (2.4) , we obtain

$$c_1R^m\exp\{\delta_a(\varphi)\frac{1}{R^\mu}\} \leq |g(z)| \leq c_2R^m\exp\{\delta_a(\varphi)\frac{1}{R^\mu}\}, \tag{2.5}$$

for z near enough z_0.

Now, if z_0 is not a zero of $A(z)$, then there exist $c_1' > 0,\ c_2' > 0$ such that

$$c_1' \leq |A(z)| \leq c_2', \quad \text{for } z \text{ near enough } z_0,$$

and so

$$c_1'\exp\{\delta_a(\varphi)\frac{1}{R^\mu}\} \leq |g(z)| \leq c_2'\exp\{\delta_a(\varphi)\frac{1}{R^\mu}\}, \tag{2.6}$$

for z near enough z_0. From (2.5) and (2.6), we can easily obtain (2.1) and (2.2). $\square$

Remark 2.3. In general, we can write $\delta_a(\varphi) = c\cos(\mu\varphi + \varphi_0)$, where $c > 0$, $\varphi_0 \in [0, 2\pi)$. By this formula, it is easy to prove that if $\mu > 1$, $\delta_a(\varphi)$ changes its sign on each interval (φ_1, φ_2) of linear measure equal to π.

3. Proof of theorems

Proof of Theorem 1.6. Suppose that $f \not\equiv 0$ is a solution of (1.4) of finite order $\sigma(f) = \sigma < \infty$. Since $\mu > 1$, By Remark 2.3, there exist $(\varphi_1, \varphi_2) \subset [0, 2\pi)$ such that for $z \in D$ and $\arg(z_0 - z) = \varphi \in (\varphi_1, \varphi_2)$ we have $\delta_b(\varphi) > 0$.

From (1.4), we obtain

$$\big|B(z)e^{\frac{b}{(z_0-z)^\mu}}\big| \leq |\frac{f''}{f}| + |A(z)||\frac{f'}{f}|. \tag{3.1}$$

From Lemma 2.1, for a given $\varepsilon > 0$ there exists a set $E \subset [0,1)$ which satisfies $\int_E \frac{1}{1-r} dr < \infty$, such that for all $z \in D$ satisfying $|z| \notin E$, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \le \Big(\frac{1}{1-|z|}\Big)^{k(\sigma+2+\varepsilon)}, \quad (k=1,2). \tag{3.2}$$

From Lemma 2.2, for any given $0 < \varepsilon < 1$ and for for $z \in D$ and $\arg(z_0 - z) = \varphi \in (\varphi_1, \varphi_2)$ with $|z_0 - z| = R$, there exists $R_0 > 0$ such that for $0 < R < R_0$, we have

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le |B(z)e^{\frac{b}{(z_0-z)^\mu}}|. \tag{3.3}$$

Since $A(z)$ is analytic on z_0, for z near enough z_0, we have

$$|A(z)| \le M, \quad M > 0. \tag{3.4}$$

Using (3.2), (3.3) and (3.4) in (3.1), we obtain

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le (\frac{1}{1-|z|})^{2(\sigma+2+\varepsilon)} + M(\frac{1}{1-|z|})^{(\sigma+2+\varepsilon)}, \tag{3.5}$$

where $z \in D$, $|z| \notin E$, $\arg(z_0 - z) = \varphi \in (\varphi_1, \varphi_2)$ with $|z_0 - z| = R$ and $0 < R < R_0$. By the metric relations in the triangle (oz_0z), we have $|z|^2 = 1 + R^2 - 2R\cos\varphi^*$ and then

$$1 - |z| = R(\frac{2\cos\varphi^* - R}{1+|z|}). \tag{3.6}$$

For z near enough z_0 and by considering that φ is fixed, there exists certain $\varepsilon_0 > 0$ such that

$$\frac{2\cos\varphi^* - R}{1+|z|} > \varepsilon_0; \tag{3.7}$$

we signal here that $0 \le \varphi^* < \frac{\pi}{2}$. By combining (3.6) and (3.7), we obtain

$$\frac{1}{1-|z|} < \frac{1}{\varepsilon_0 R}. \tag{3.8}$$

Now by (3.5) and (3.8), we obtain

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le M'(\frac{1}{\varepsilon_0 R})^{2(\sigma+2+\varepsilon)}, \quad M' > 1,$$

which gives a contradiction as $R \to 0$. □

Proof of Theorem 1.8. Suppose that $f \not\equiv 0$ is a solution of (1.5) of finite order $\sigma(f) = \sigma < \infty$. Since $\arg a \ne \arg b$ and $\mu > 1$, then there exist $(\varphi_1, \varphi_2) \subset [0, 2\pi)$ such that for $z \in D$ and $\arg(z_0 - z) = \varphi \in (\varphi_1, \varphi_2)$ we have $\delta_b(\varphi) > 0$ and $\delta_a(\varphi) < 0$.

From (1.5), we obtain

$$\big|B(z)e^{\frac{b}{(z_0-z)^\mu}}\big| \le |\frac{f''}{f}| + |A(z)||A(z)e^{\frac{a}{(z_0-z)^\mu}}||\frac{f'}{f}|. \tag{3.9}$$

From Lemma 2.1, for a given $\varepsilon > 0$ there exists a set $E \subset [0,1)$ which satisfies $\int_E \frac{1}{1-r} dr < \infty$, such that for all $z \in D$ satisfying $|z| \notin E$, we have

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \le \Big(\frac{1}{1-|z|}\Big)^{k(\sigma+2+\varepsilon)}, \quad (k=1,2). \tag{3.10}$$

From Lemma 2.2, for any given $0 < \varepsilon < 1$ and for for $z \in D$ and $\arg(z_0 - z) = \varphi \in (\varphi_1, \varphi_2)$ with $|z_0 - z| = R$, there exists $R_0 > 0$ such that for $0 < R < R_0$, we have

$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le \big|B(z)e^{\frac{b}{(z_0-z)^\mu}}\big|, \tag{3.11}$$

and
$$\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right| \le \exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\}. \tag{3.12}$$
Using (3.9) -(3.12) and (3.8), we obtain
$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le (\frac{1}{\varepsilon_0 R})^{2(\sigma+2+\varepsilon)} + (\frac{1}{\varepsilon_0 R})^{(\sigma+2+\varepsilon)}\exp\{(1-\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\},$$
where $z \in D$, $|z| \notin E$, $\arg(z_0-z) = \varphi \in (\varphi_1,\varphi_2)$ with $|z_0-z| = R$ and $0 < R < R_0$. A contradiction follows as $R \to 0$. $\square$

Proof of Theorem 1.9. Suppose that $f \not\equiv 0$ is a solution of (1.6) of finite order $\sigma(f) = \sigma < \infty$. Since $\mu > 1$, then there exist $(\varphi_1,\varphi_2) \subset [0,2\pi)$ such that for $z \in D$ and $\arg(z_0-z) = \varphi \in (\varphi_1,\varphi_2)$ we have $\delta_a(\varphi) > 0$.

From (1.6), we obtain
$$\left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right| \le |\frac{f''}{f}| + \left|A(z)\right|\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right||\frac{f'}{f}|. \tag{3.13}$$
From Lemma 2.1, for a given $\varepsilon > 0$ there exists a set $E \subset [0,1)$ which satisfies $\int_E \frac{1}{1-r}dr < \infty$, such that for all $z \in D$ satisfying $|z| \notin E$, we have
$$|\frac{f^{(k)}(z)}{f(z)}| \le \Big(\frac{1}{1-|z|}\Big)^{k(\sigma+2+\varepsilon)}, \quad (k = 1,2). \tag{3.14}$$
From Lemma 2.2, for any $\varepsilon > 0$ and for for $z \in D$ and $\arg(z_0-z) = \varphi \in (\varphi_1,\varphi_2)$ with $|z_0-z| = R$, there exists $R_0 > 0$ such that for $0 < R < R_0$, we have
$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \le \left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right|, \tag{3.15}$$
and
$$\left|A(z)e^{\frac{a}{(z_0-z)^\mu}}\right| \le \exp\{(1+\varepsilon)\delta_a(\varphi)\frac{1}{R^\mu}\}. \tag{3.16}$$
By using (3.14) – (3.16) in (3.13) and taking into account that $\delta_a(\varphi) = c\delta_b(\varphi)$ $(0 < c < 1)$, we obtain
$$\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \quad \le \big(\frac{1}{\varepsilon_0 R}\big)^{2(\sigma+2+\varepsilon)} + \big(\frac{1}{\varepsilon_0 R}\big)^{(\sigma+2+\varepsilon)}\exp\{(1+\varepsilon)c\delta_b(\varphi)\frac{1}{R^\mu}\}.$$
By taking $0 < \varepsilon < \frac{1-c}{1+c}$, we obtain a contradiction to $R \to 0$. $\square$

Proof of Theorem 1.11. Let
$$A_{j_1}(z) = B_{j_1}(z)e^{\frac{b_{j_1}}{(z_0-z)^\mu}}$$
such that $B_{j_1}(z)$ is analytic on z_0 and $\arg b_{j_1} \neq \arg b$;
$$A_{j_m}(z) = B_{j_m}(z)e^{\frac{b_{j_m}}{(z_0-z)^\mu}} \quad (m = 2,\dots,s)$$
such that $B_{j_m}(z)$ are analytic on z_0 and $b_{j_m} = c_{j_m}b$ $(0 < c_{j_m} < 1)$, and the remaining coefficients $A_{j_m}(z)$ are analytic on z_0 $(m = s+1,\dots,k-1)$. Suppose that $f \not\equiv 0$ is a solution of (1.7) of finite order $\sigma(f) = \sigma < \infty$. Since $\arg b_{j_1} \neq \arg b$ then there exist $(\varphi_1,\varphi_2) \subset [0,2\pi)$ such that for $z \in D$ and $\arg(z_0-z) = \varphi \in (\varphi_1,\varphi_2)$

we have $\delta_b(\varphi) > 0$ and $\delta_{b_{j_1}}(\varphi) < 0$. Set $c = \max\{c_{j_m} : m = 2, \dots, s\}$. We have $\delta_{b_{j_m}}(\varphi) = c_{j_m}\delta_b(\varphi) \le c\delta_b(\varphi)$. From (1.7), we can write

$$\begin{aligned} \left|B(z)e^{\frac{b}{(z_0-z)^\mu}}\right| &\le \left|\frac{f^{(k)}}{f}\right| + \sum_{m=s+1}^{k-1} |A_{j_m}(z)|\left|\frac{f^{(j_m)}}{f}\right| \\ &\quad + \sum_{m=2}^{s} |A_{j_m}(z)|\left|\frac{f^{(j_m)}}{f}\right| + |A_{j_1}(z)|\left|\frac{f^{(j_1)}}{f}\right|. \end{aligned} \tag{3.17}$$

Using the same reasoning as above, from (3.17), we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta_b(\varphi)\frac{1}{R^\mu}\} \\ &\le \Big(\frac{1}{\varepsilon_0 R}\Big)^{k(\sigma+2+\varepsilon)}\Big(M + (s-1)\exp\{(1+\varepsilon)c\delta_b(\varphi)\frac{1}{R^\mu}\} + \exp\{(1-\varepsilon)\delta_{b_{j_1}}(\varphi)\frac{1}{R^\mu}\}\Big). \end{aligned}$$

By taking $0 < \varepsilon < \frac{1-c}{1+c}$, we obtain a contradiction to $R \to 0$. $\square$

Remark 3.1. Concerning the case when $0 < \mu \le 1$, our method is not valid in general. For example, for the differential equation

$$f'' + A(z)f' + B(z)e^{\frac{-1}{(1-z)}} f = 0,$$

where $A(z)$ and $B(z)$ are analytic on $z_0 = 1$. However, we cannot apply our method because for all $z \in D$ we have $\delta_{-1}(\varphi) < 0$, where $\varphi = \arg(1-z)$. Furthermore, for the differential equation

$$f'' + A(z)f' + B(z)e^{\frac{1}{(1-z)}} f = 0,$$

our method is valid. So we can deduce that every solution $f \not\equiv 0$ of this differential equation is of infinite order. In general, our method is valid for $0 < \mu \le 1$ except the case when we have $\delta_b(\varphi) < 0$ for $\arg z_0 - \frac{\pi}{2} < \varphi < \arg z_0 + \frac{\pi}{2}$.

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