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SUBHARMONIC SOLUTIONS FOR NONAUTONOMOUS SECOND-ORDER HAMILTONIAN SYSTEMS

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Abstract. In this article, we prove the existence of subharmonic solutions for the non-autonomous second-order Hamiltonian system $\ddot{u}(t) + V'(t, u(t)) = 0$. Also we study the minimality of their periods, when the nonlinearity $V'(t, x)$ grows faster than $|x|^\alpha$, $\alpha \in [0, 1[$ at infinity. The proof is based on the Least Action Principle and the Saddle Point Theorem.

1. Introduction

Consider the non-autonomous second-order Hamiltonian system

$$\ddot{u}(t) + V'(t, u(t)) = 0, \tag{1.1}$$

where $V : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(t, x) \to V(t, x)$ is a continuous function, T-periodic ($T > 0$) in the first variable and differentiable with respect to the second variable such that the gradient $V'(t, x) = \frac{\partial V}{\partial x}(t, x)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$. In this work, we are interested in the existence of subharmonic solutions of (1.1). Assuming that $T > 0$ is the minimal period of the time dependence of $V(t, x)$, by subharmonic solution of (1.1) we mean a kT-periodic solution, where k is any integer; when moreover the solution is not T-periodic we call it a true subharmonic.

Using variational methods, there have been various types of results concerning the existence of subharmonic solutions to system (1.1). Many solvability conditions are given, such as a convexity condition [4, 12], a super-quadratic condition [7, 11], a subquadratic condition [4, 6], a periodic condition [8], a bounded nonlinearity condition [1,2,5], and a sublinear condition [10]. In particular, under the assumptions that there exists a constant $M > 0$ such that

$$|V'(x)| \le M, \quad \forall x \in \mathbb{R}^N, \tag{1.2}$$

$$\lim_{|x|\to\infty} (V'(x) - \bar{e})x = +\infty, \tag{1.3}$$

where $e : \mathbb{R} \to \mathbb{R}^N$ is a continuous periodic function having minimal period $T > 0$, and $\bar{e}$ is the mean value of e, A. Fonda and Lazer in [2] showed that the system

$$\ddot{u}(t) + V'(u(t)) = e(t) \tag{1.4}$$

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admitted periodic solutions with minimal period kT, for any sufficiently large prime number k. After that, Tang and Wu in [10] generalized these results without studying the minimality of periods. Precisely, it was assumed that the nonlinearity satisfied the following restrictions:

$$|V'(t,x)| \leq f(t)|x|^{\alpha} + g(t),\ \forall x \in \mathbb{R}^N, \quad \text{a. e. } t \in [0,T], \tag{1.5}$$

$$\frac{1}{|x|^{2\alpha}} \int_0^T V(t,x)dt \to +\infty \quad \text{as } |x| \to +\infty, \tag{1.6}$$

here $f, g \in L^1(0,T;\mathbb{R}^+)$ are T-periodic and $\alpha \in [0,1[$.

In [2, 10], the nonlinearity is required to grow at infinity at most like $|x|^{\alpha}$ with $\alpha \in [0,1[$. In this article, we will firstly, establish the existence of subharmonic solutions for the system (1.1) when the nonlinearity $V'(t,x)$ is required to have a sublinear growth at infinity faster than $|x|^{\alpha}$, $\alpha \in [0,1[$. Our first main result is as follows.

Theorem 1.1. *Let $\omega \in C([0,\infty[,\mathbb{R}^+)$ be a nonincreasing positive function with the properties:*

$$\liminf_{s\to\infty} \frac{\omega(s)}{\omega(s^{1/2})} > 0, \tag{1.7}$$

$$\omega(s) \to 0, \quad \omega(s)s \to \infty \quad \text{as } s \to \infty. \tag{1.8}$$

Assume that V satisfies: There exist two T-periodic functions $f \in L^2(0,T;\mathbb{R}^+)$ and $g \in L^1(0,T;\mathbb{R}^+)$ such that

$$|V'(t,x)| \leq f(t)\omega(|x|)|x| + g(t), \quad \forall x \in \mathbb{R}^N, \text{a.e. } t \in [0,T]; \tag{1.9}$$

$$\frac{1}{[\omega(|x|)|x|]^2} \int_0^T V(t,x)dt \to +\infty \quad \text{as } |x| \to \infty; \tag{1.10}$$

There is a subset C of $[0,T]$ with $\operatorname{meas}(C) > 0$ and $h \in L^1(0,T;\mathbb{R})$ such that

$$\lim_{|x|\to\infty} V(t,x) = +\infty, \quad \text{a.e. } t \in C, \tag{1.11}$$

$$V(t,x) \geq h(t) \quad \text{for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T]. \tag{1.12}$$

Then for all positive integer k, the system (1.1) has at least one kT-periodic solution u_k satisfying

$$\lim_{k\to\infty} \|u_k\|_\infty = +\infty,$$

where $\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$.

Remark 1.2. Let

$$V(t,x) = \gamma(t)\frac{|x|^2}{ln(2+|x|^2)}, \quad \forall x \in \mathbb{R}^N,\ \forall t \in \mathbb{R},$$

where

$$\gamma(t) = \begin{cases} \sin(2\pi t/T), & t \in [0,T/2] \\ 0, & t \in [T/2,T]. \end{cases}$$

Taking $\omega(s) = \frac{1}{ln(2+s^2)}$, $C = [0, \frac{T}{2}]$. By a simple computation, we prove that $V(t,x)$ satisfies (1.9)–(V3) and does not satisfy the conditions (1.2), (1.3) nor (1.5), (1.6).

Corollary 1.3. *Assume that* (1.9) *holds and there exists a subset C of* $[0,T]$ *with* $\operatorname{meas}(C)>0$ *and* $h\in L^1(0,T;\mathbb{R})$ *such that*

$$\lim_{|x|\to\infty}\frac{V(t,x)}{[\omega(|x|)|x|]^2}=+\infty,\quad a.e.\ t\in C, \tag{1.13}$$

$$V(t,x)\geq h(t),\quad \text{for all} x\in\mathbb{R}^N,\ a.e.\ t\in[0,T]. \tag{1.14}$$

Then the conclusion of Theorem 1.1 holds.

There are a few results studying the minimality of periods of the subharmonics, see [12] for the case of convexity, and [2] for the case of bounded gradient. We study this problem and obtain the following result.

Theorem 1.4. *Assume that V satisfies* (1.9) *and*

$$\frac{V'(t,x)x}{[\omega(|x|)|x|]^2}\to+\infty\quad \text{as } |x|\to\infty,\ \text{uniformly for } t\in[0,T]. \tag{1.15}$$

Then, for all integer $k\geq 1$, Equation (1.1) *possesses a kT-periodic solution u_k such that* $\lim_{k\to\infty}\|u_k\|_\infty=+\infty$. *If moreover V satisfies the assumption:*

If $u(t)$ is a periodic function with minimal period rT with r rational, and $V'(t,u(t))$ is a periodic function with minimal period rT, then r is necessarily an integer. (1.16)

Then, for any sufficiently large prime number k, kT is the minimal period of u_k.

As an example of a function V we have

$$V(t,x)=(2+\cos(\frac{2\pi}{T}t))\frac{|x|^2}{\ln(2+|x|^2)},\quad \omega(s)=\frac{1}{\ln(2+s^2)}.$$

Then $V(t,x)$ satisfies (1.9) (1.15) and (1.16), Our main tools, for proving our results, are the Least Action Principle and the Saddle Point Theorem.

2. Proof of theorems

Let us fix a positive integer k and consider the continuously differentiable function

$$\varphi_k(u)=\int_0^{kT}[\frac{1}{2}|\dot u(t)|^2-V(t,u(t))]dt$$

defined on the space H^1_{kT} of kT-periodic absolutely continuous vector functions whose derivatives have square-integrable norm. This set is a Hilbert space with the norm

$$\|u\|_k=\Big[\int_0^{kT}|u(t)|^2dt+\int_0^{kT}|\dot u(t)|^2dt\Big]^{1/2},\quad u\in H^1_{kT}$$

and the associated inner product

$$\langle u,v\rangle_k=\int_0^{kT}[u(t)v(t)+\dot u(t)\dot v(t)]dt,\quad u,v\in H^1_{kT}.$$

For $u\in H^1_{kT}$, let $\bar u=\frac{1}{kT}\int_0^{kT}u(t)dt$ and $\tilde u(t)=u(t)-\bar u$, then we have Sobolev's inequality,

$$\|\tilde u\|_\infty^2\leq\frac{kT}{12}\int_0^{kT}|\dot u(t)|^2dt, \tag{2.1}$$

and Wirtinger's inequality,

$$\int_0^{kT} |\tilde{u}(t)|^2 dt \leq \frac{k^2T^2}{4\pi^2} \int_0^{kT} |\dot{u}(t)|^2 dt\,. \tag{2.2}$$

It is easy to see that the norm $\|u\|_k$ is equivalently to the norm

$$\|u\| = \Big[\int_0^{kT} |\dot{u}(t)|^2 dt + |\bar{u}|^2\Big]^{1/2}.$$

In the following, we will use this last norm. It is well known that φ_k is continuously differentiable with

$$\varphi_k'(u)v = \int_0^{kT} [\dot{u}(t)\dot{v}(t) - V'(t, u(t))v(t)]dt, \quad \forall u, v \in H^1_{kT}$$

and its critical points correspond to the kT-periodic solutions of the system (1.1).

Proof of Theorem 1.1. Here, we will show that, for every positive integer k, one can find a kT-periodic solution u_k of (1.1) in such a way that the sequence (u_k) satisfies

$$\lim_{k\to\infty} \frac{1}{k}\varphi_k(u_k) = -\infty. \tag{2.3}$$

This will be done by using some estimates on the critical levels of φ_k given by the Saddle Point Theorem. The following lemma will be needed for the study of the geometry of the functionals φ_k.

Lemma 2.1. *Assume that* (1.9), (1.10) *hold. Then there exist a nonincreasing positive function* $\theta \in C(]0, \infty[, \mathbb{R}^+)$ *and a positive constant* c_0 *satisfying the following conditions:*

(i) $\theta(s) \to 0$, $\theta(s)s \to +\infty$ *as* $s \to \infty$,
(ii) $\|V'(t, u)\|_{L^1} \leq c_0[\theta(\|u\|)\|u\| + 1]$, *for all* $u \in H^1_{kT}$,
(iii)

$$\frac{1}{[\theta(|x|)|x|]^2} \int_0^{kT} V(t, x)dt \to +\infty \quad \text{as } |x| \to +\infty.$$

Proof. For $u \in E$, let $A = \{t \in [0, kT]/|u(t)| \geq \|u\|^{1/2}\}$. By (1.9), we have

$$\begin{aligned}
&\|V'(t, u)\|_{L^1}\\
&\leq \int_0^T \big[f(t)\omega(|u(t)|)|u(t)| + g(t)\big]dt\\
&\leq \|f\|_{L^2}\big(\int_0^T [\omega(|u(t)|)|u(t)|]^2 dt\big)^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\Big(\int_A [\omega^2(\|u\|^{1/2})|u(t)|^2 dt + \int_{[0,kT]-A} \sup_{s\geq 0} \omega^2(s)\|u\|dt\Big)^{1/2} + \|g\|_{L^1}\\
&\leq \|f\|_{L^2}\big[\omega^2(\|u\|^{1/2})\|u\|^2_{L^2} + kT\sup_{s\geq 0}\omega^2(s)\|u\|\big]^{1/2} + \|g\|_{L^1}.
\end{aligned}$$

So there exists a positive constant c_0 such that

$$\|V'(t, u)\|_{L^1} \leq c_0\big(\big[\omega^2(\|u\|^{1/2})\|u\|^2 + \|u\|\big]^{1/2} + 1\big).$$

Take

$$\theta(s) = \big[\omega^2(s^{1/2}) + \frac{1}{s}\big]^{1/2},\ s > 0, \tag{2.4}$$

then θ satisfies (ii) and it is easy to see that θ satisfies (i).

Now, by (a), we have $\rho = \liminf_{s\to\infty} \frac{\omega^2(s)}{\omega^2(s^{1/2})} > 0$. By (1.10), for any $\gamma > 0$, there exists a positive constant c_1 such that

$$\int_0^{kT} V(t,x)dt \geq \gamma[\omega(|x|)|x|]^2 - c_1. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\frac{\int_0^{kT} V(t,x)dt}{[\theta(|x|)|x|]^2} \geq \frac{\gamma[\omega(|x|)|x|]^2 - c_1}{\omega^2|x|^{1/2})|x|^2 + |x|}. \tag{2.6}$$

By the definition of ρ, there exists $R > 0$ such that for all $s \geq R$

$$\frac{\omega^2(s)s^2}{\omega^2(s^{1/2})s^2 + s} \geq \frac{\rho}{2}, \tag{2.7}$$

and

$$\frac{c_1}{\omega^2(s)s^2 + s} \leq \frac{\gamma\rho}{4}, \tag{2.8}$$

Therefore, by (2.6)-(2.8), we have

$$\frac{\int_0^{kT} V(t,x)dt}{[\theta(|x|)|x|]^2} \geq \frac{\gamma\rho}{4}, \quad \forall |x| \geq R.$$

Since γ is arbitrary chosen, condition (iii) holds. The proof of Lemma 2.1 is complete. □

Now, we need to show that, for every positive integer k, one can find a critical point u_k of the functional φ_k in such a way that (2.3) holds. To this aim, we will apply the Saddle Point Theorem to each of the φ_k' s. Let us fix k and write $H^1_{kT} = \mathbb{R}^N \oplus \tilde{H}^1_{kT}$ where $\mathbb{R}^N$ is identified with the set of constant functions and $\tilde{H}^1_{kT}$ consists of functions u in H^1_{kT} such that $\int_0^{kT} u(t)dt = 0$. First, we prove the Palais-Smale condition.

Lemma 2.2. *Assume that* (1.9) *and* (1.10) *hold. Then* φ_k *satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a sequence in H^1_{kT} such that $(\varphi_k(u_n))$ is bounded and $\varphi_k'(u_n) \to 0$ as $n \to \infty$. In particular, for a positive constant c_2 we will have

$$\varphi_k'(u_n)\tilde{u}_n = \int_0^{kT} [|\dot{u_n}(t)|^2 - V'(t,u_n(t))\tilde{u}_n(t)]dt \leq c_2\|\tilde{u}_n\|. \tag{2.9}$$

Since θ is non increasing and $\|u\| \geq max(|\bar{u}|, \|\tilde{u}\|)$, we obtain

$$\theta(\|u\|) \leq \min(\theta(|\bar{u}|), \theta(\|\tilde{u}\|)). \tag{2.10}$$

Combining Sobolev's inequality, Lemma 2.1 (ii) and (2.10), we can find a positive constant c_3 such that

$$\begin{aligned} |\int_0^{kT} V'(t,u_n)\tilde{u}_n dt| &\leq c_0\|\tilde{u}_n\|_\infty[\theta(\|u_n\|)\|u_n\| + 1] \\ &\leq c_0\|\tilde{u}_n\|_\infty\big[\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \theta(|\bar{u}_n|)|\bar{u}_n| + 1\big] \\ &\leq c_3\|\tilde{u}_n\|\big[\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \theta(|\bar{u}_n|)|\bar{u}_n| + 1\big] \end{aligned} \tag{2.11}$$

for all $n \in \mathbb{N}$.

From Wirtinger's inequality, there exists a constant $c_4 > 0$ such that

$$\|\dot{u}_n\|_{L^2} \le \|\tilde{u}_n\| \le c_4^{-1/2}\|\dot{u}_n\|_{L^2}. \tag{2.12}$$

Therefore, by (2.9) and (2.11), we obtain

$$\begin{aligned} c_2\|\tilde{u}_n\| \ge \varphi_k'(u_n).\tilde{u}_n &= \int_0^{kT} |\dot{u_n}|^2 dt - \int_0^{kT} V'(t,u_n)\tilde{u}_n dt \\ &\ge c_4\|\tilde{u}_n\|^2 - c_3\|\tilde{u}_n\|[\theta(\|\tilde{u}_n\|)\|\tilde{u}_n\| + \theta(|\bar{u}_n|)|\bar{u}_n| + 1]. \end{aligned} \tag{2.13}$$

Assume that $(\|\tilde{u}_n\|)$ is unbounded, by going to a subsequence if necessary, we can assume that $\|\tilde{u}_n\| \to \infty$ as $n \to \infty$. Since $\theta(s) \to 0$ as $s \to \infty$, we deduce from (2.13) that there exists a positive constant c_5 such that

$$\|\tilde{u}_n\| \le c_5\theta(|\bar{u}_n|)|\bar{u}_n| = c_5\big[\omega^2(|\bar{u}_n|^{1/2})|\bar{u}_n|^2 + |\bar{u}_n|\big]^{1/2} \tag{2.14}$$

for n large enough. Since ω is bounded, it follows that $|\bar{u}_n| \to \infty$ as $n \to \infty$.

Now, by the Mean Value Theorem and Lemma 2.1 (ii), we obtain

$$\begin{aligned} |\int_0^{kT}(V(t,u_n) - V(t,\bar{u}_n))dt| &= |\int_0^{kT}\int_0^1 V'(t,\bar{u}_n) + s\tilde{u}_n)\tilde{u}_n\, ds\, dt| \\ &\le \|\tilde{u}_n\|_\infty \int_0^1\int_0^{kT} |V'(t,\bar{u}_n + s\tilde{u}_n)|\, ds\, dt \\ &\le c_0\|\tilde{u}_n\|_\infty \int_0^1 \big[\theta(\|\bar{u}_n + s\tilde{u}_n\|)\|\bar{u}_n + s\tilde{u}_n\| + 1\big] ds. \end{aligned} \tag{2.15}$$

Since $\|\bar{u}_n + s\tilde{u}_n\| \ge \|\bar{u}_n\|$ for all $s \in [0,1]$, we deduce from (2.1), (2.14) and (2.15) that there exists a positive constant c_6 such that

$$\begin{aligned} &|\int_0^{kT}(V(t,u_n) - V(t,\bar{u}_n))dt| \\ &\le c_6\Big([\theta(|\bar{u}_n|)|\bar{u}_n|]^2 + \theta(|\bar{u}_n|)[\theta(|\bar{u}_n|)|\bar{u}_n|]^2 + \theta(|\bar{u}_n|)|\bar{u}_n|\Big). \end{aligned} \tag{2.16}$$

Thus, by (2.14) and (2.16), we obtain for a positive constant c_7,

$$\begin{aligned} &\varphi_k(u_n) \\ &= \frac{1}{2}\|\dot{u}_n\|_{L^2}^2 - \int_0^{kT}(V(t,u_n) - V(t,\bar{u}_n))dt - \int_0^{kT} V(t,\bar{u}_n)dt \\ &\le c_7\Big([\theta(|\bar{u}_n|)|\bar{u}_n|]^2 + \theta(|\bar{u}_n|)[\theta(|\bar{u}_n|)|\bar{u}_n|]^2 + \theta(|\bar{u}_n|)|\bar{u}_n|\Big) - \int_0^{kT} V(t,\bar{u}_n)dt \\ &= c_7[\theta(|\bar{u}_n|)|\bar{u}_n|]^2\Big[1 + \theta(|\bar{u}_n|) + \frac{1}{\theta(|\bar{u}_n|)|\bar{u}_n|} - \frac{1}{c_7[\theta(|\bar{u}_n|)|\bar{u}_n|]^2}\int_0^{kT} V(t,\bar{u}_n)dt\Big] \end{aligned}$$

so $\varphi_k(u_n) \to -\infty$ *as* $n \to \infty$. This contradicts the boundedness of $(\varphi_k(u_n))$. Therefore $(\|\tilde{u}_n\|)$ is bounded.

It remains to prove that $(|\bar{u}_n|)$ is bounded. Assume the contrary. By taking a subsequence, if necessary, we can assume that $\|\bar{u}_n\| \to \infty$ as $n \to \infty$. By the preceding calculus, we obtain for some positive constants c_8, c_9 such that

$$\varphi_k(u_n)$$

$$
\begin{aligned}
&\le c_8\Big[\|\tilde{u}_n\|^2+\|\tilde{u}_n\|\theta(|\bar{u}_n|)|\bar{u}_n|+\theta(|\bar{u}_n|)\|\tilde{u}_n\|+1\Big]-\int_0^{kT}V(t,\bar{u}_n)dt\\
&\le c_9\big[1+\theta(|\bar{u}_n|)|\bar{u}_n|+\theta(|\bar{u}_n|)\big]-\int_0^{kT}V(t,\bar{u}_n)dt\\
&\le c_9[\theta(|\bar{u}_n|)|\bar{u}_n|]^2\Big[\frac{1+\theta(|\bar{u}_n|)}{[\theta(|\bar{u}_n|)|\bar{u}_n|]^2}+\frac{1}{\theta(|\bar{u}_n|)|\bar{u}_n|}-\frac{1}{c_9[\theta(|\bar{u}_n|)|\bar{u}_n|]^2}\int_0^{kT}V(t,\bar{u}_n)dt\Big]
\end{aligned}
$$

so $\varphi_k(u_n)\to-\infty$ as $n\to\infty$, which also contradicts the boundedness of $(\varphi_k(u_n))$. So $(|\bar{u}_n|)$ is bounded and then $(\|u_n\|)$ is also bounded. By a standard argument, we conclude that (u_n) possesses a convergent subsequence and the proof is complete. □

Now, it is easy to show that (1.10) yields

$$
\varphi_k(u)=-\int_0^{kT}V(t,u)dt\to-\infty\quad\text{as } |u|\to\infty \text{ in } \mathbb{R}^N. \tag{2.17}
$$

On the other hand, by the Mean Value Theorem, (1.9) and Hölder's inequality, for all $u\in\tilde{H}^1_{kT}$ and $a\in\mathbb{R}^N$ $|a|>0$, we have

$$
\begin{aligned}
&|\int_0^{kT}(V(t,u)-V(t,a))dt|\\
&=|\int_0^{kT}\int_0^1 V'(t,a+s(u-a))(u-a)\,ds\,dt|\\
&\le\|u-a\|_\infty\int_0^1\int_0^{kT}|V'(t,a+s(u-a))|\,dt\,ds\\
&\le\|u-a\|_\infty\int_0^1\int_0^{kT}\big[f(t)\omega(|a+s(u-a)|)|a+s(u-a)|+g(t)\big]dt\\
&\le\|u-a\|_\infty\Big(\|f\|_{L^2}\int_0^1\big[\int_0^{kT}(\omega(|a+s(u-a)|)|a+s(u-a)|)^2dt\big]^{\frac12}ds+\|g\|_{L^1}\Big).
\end{aligned}
$$

For $s\in[0,1]$, take

$$
A(s)=\{t\in[0,kT]/|a+s(u(t)-a)|\ge|a|\}.
$$

By a classical calculation as in the proof of Lemma 2.1, we obtain some positive constants c_{10} and $c(a)$ such that

$$
|\int_0^{kT}(V(t,u)-V(t,a))dt|\le c_{10}\omega(a)\|u\|^2+c(a)(\|u\|+1).
$$

Since $\omega(|a|)\to0$ as $|a|\to\infty$, there exists $|a|>0$ such that $c_{10}\omega(|a|)\le\frac14c_4^2$ and then we obtain

$$
|\int_0^{kT}(V(t,u)-V(t,a))dt|\le\frac14c_4^2\|u\|^2+c(a)(\|u\|+1)
$$

which implies that

$$
\varphi_k(u)\ge\frac14c_4^2\|u\|^2-c(a)(\|u\|+1)-\int_0^{kT}V(t,a)dt\to\infty\quad\text{as }\|u\|\to\infty. \tag{2.18}
$$

We deduce from Lemma 2.2 and (2.17), (2.18) that all the Saddle Point Theorem's assumptions are satisfied. Therefore the functional φ_k possesses at least a critical point u_k satisfying

$$-\infty < \inf_{\tilde{H}^1_{kT}} \varphi_k \le \varphi_k(u_k) \le \sup_{\mathbb{R}^N + e_k} \varphi_k \tag{2.19}$$

where $e_k(t) = k\cos(\frac{\sigma t}{k})x_0$ for $t \in \mathbb{R}$, $\sigma = \frac{2\pi}{T}$ and some $x_0 \in \mathbb{R}^N$ with $|x_0| = 1$. The first part of Theorem 1.1 is proved.

Next, we will prove that the sequence $(u_k)_{k\ge1}$ obtained above satisfies (2.3). For this aim, the following two lemmas will be needed.

Lemma 2.3 ([9])**.** *If* (1.11) *holds, then for every $\delta > 0$ there exists a measurable subset C_δ of C with* $\operatorname{meas}(C - C_\delta) < \delta$ *such that*

$$V(t,x) \to +\infty \quad \text{as } |x| \to \infty, \text{ uniformly in } t \in C_\delta.$$

Lemma 2.4. *Suppose that V satisfies* (1.11) - (1.12)*, then*

$$\limsup_{k\to\infty} \sup_{x\in\mathbb{R}^N} \frac{1}{k}\varphi_k(x + e_k) = -\infty. \tag{2.20}$$

Proof. Let $x \in \mathbb{R}^N$, we have

$$\varphi_k(x + e_k) = \frac{1}{4}kT\sigma^2 - \int_0^{kT} V(t, x + e_k(t))dt.$$

By (1.11) and Lemma 2.3, for $\delta = \frac{1}{2}\operatorname{meas}(C)$ and all $\gamma > 0$, there exist a measurable subset $C_\delta \subset C$ with $\operatorname{meas}(C - C_\delta) < \delta$ and $r > 0$ such that

$$V(t,x) \ge \gamma, \quad \forall |x| \ge r \ \forall t \in C_\delta. \tag{2.21}$$

Let

$$B_k = \{t \in [0,kT] : |x + e_k(t)| \le r\}.$$

Then we have

$$\operatorname{meas}(B_k) \le \frac{k\delta}{2}. \tag{2.22}$$

In fact, if $\operatorname{meas}(B_k) > k\delta/2$, there exists $t_0 \in B_k$ such that

$$\frac{k\delta}{8} \le t_0 \le \frac{kT}{2} - \frac{k\delta}{8}, \tag{2.23}$$

$$\frac{kT}{2} + \frac{k\delta}{8} \le t_0 \le kT - \frac{k\delta}{8}, \tag{2.24}$$

and there exists $t_1 \in B_k$ such that

$$|t_1 - t_0| \ge \frac{k\delta}{8}, \tag{2.25}$$

$$|t_1 - (kT - t_0)| \ge \frac{k\delta}{8}. \tag{2.26}$$

It follows from (2.26) that

$$|\frac{t_0 + t_1}{2k} - \frac{T}{2}| \ge \frac{\delta}{16}. \tag{2.27}$$

By (2.23) and (2.24), one has

$$\frac{\delta}{16} \le \frac{t_0 + t_1}{2k} \le T - \frac{\delta}{16}. \tag{2.28}$$

Combining (2.27) and (2.28), yields

$$|\sin(\frac{t_0+t_1}{2k}\sigma)| \ge \sin(\frac{\sigma\delta}{16}). \tag{2.29}$$

On the other hand, by (2.25) we have

$$|\cos(\frac{\sigma t_0}{k}) - \cos(\frac{\sigma t_1}{k})| = 2|\sin(\frac{t_0+t_1}{2k}\sigma)||\sin(\frac{t_0-t_1}{2k}\sigma)| \ge 2\sin^2(\frac{\sigma\delta}{16}),$$

and

$$|\cos(\frac{\sigma t_0}{2k}) - \cos(\frac{\sigma t_1}{2k})| = \frac{1}{k}|(x+e_k(t_1)) - (x+e_k(t_0))| \le \frac{2r}{k},$$

which is impossible for large k. Hence (2.22) holds.

Now, let $C_k = \cup_{j=0}^{k-1}(jT + C_\delta)$. It follows from (2.22) that for all k,

$$\operatorname{meas}(C_k - B_k) \ge \frac{k\delta}{2}.$$

By (2.21), we have

$$\begin{aligned} k^{-1}\varphi_k(x+e_k) &= \frac{1}{4}T\sigma^2 - k^{-1}\int_0^{kT} V(t, x+e_k(t))dt \\ &\le \frac{1}{4}T\sigma^2 - k^{-1}\int_{[0,kT]-(C_k-B_k)} h(t)dt - k^{-1}\gamma\operatorname{meas}(C_k - B_k) \\ &\le \frac{1}{4}T\sigma^2 - \int_0^T |h(t)|dt - \frac{\delta\gamma}{2} \end{aligned}$$

for all large k, which implies

$$\limsup_{k\to\infty}\sup_{x\in\mathbb{R}^N} k^{-1}\varphi_k(x+e_k) \le \frac{1}{4}T\sigma^2 + \int_0^T |h(t)|dt - \frac{\delta\gamma}{2}.$$

By the arbitrariness of γ, we obtain

$$\limsup_{k\to\infty}\sup_{x\in\mathbb{R}^N} k^{-1}\varphi_k(x+e_k) = -\infty.$$

The proof of Lemma 2.4 is complete. □

It remains to prove that the sequence $(\|u_k\|_\infty)$ of solutions of (1.1) obtained above, is unbounded. Arguing by contradiction, assume that $(\|u_k\|_\infty)$ is bounded, then there exists $R > 0$ such that $(\|u_k\|_\infty) \le R$ for all $k \ge 1$. We have

$$k^{-1}\varphi_k(u_k) \ge -k^{-1}\int_0^{kT} V(t, u_k)dt. \tag{2.30}$$

Since V is T-periodic in t and continuous, then there exists a constant $\rho > 0$ such that

$$|V(t,x)| \le \rho, \quad \forall x \in \mathbb{R}^N,\ |x| \le R,\ \text{a.e. } t \in \mathbb{R}.$$

Therefore,

$$k^{-1}\varphi_k(u_k) \ge -\rho T \tag{2.31}$$

which contradicts (2.19) with (2.20). The proof of Theorem 1.1 is complete.

Proof of Theorem 1.4. The following lemma will be needed.

Lemma 2.5. *Let (1.9), (1.15) hold. Then for all $\rho > 0$, there exists a constant $c_\rho \geq 0$ such that for all $x \in \mathbb{R}^N$, $|x| > 1$ and for a.e. $t \in [0, T]$,*

$$V(t,x) \geq V(t,0) + \frac{\rho}{2}[\omega(|x|)|x|]^2(1 - \frac{1}{|x|^2}) - c_\rho ln(|x|) - \frac{1}{2}a \sup_{r\geq 0}\omega(r) - g(t). \quad (2.32)$$

Proof. For $x \in \mathbb{R}^N$, $|x| > 1$, we have

$$\begin{aligned} V(t,x) &= V(t,0) + \int_0^1 V'(t,sx)x\,ds \\ &= V(t,0) + \int_0^{\frac{1}{|x|}} V'(t,sx)x\,ds + \int_{\frac{1}{|x|}}^1 V'(t,sx)x\,ds. \end{aligned} \quad (2.33)$$

By (1.9), we have

$$\begin{aligned} |\int_0^{\frac{1}{|x|}} V'(t,sx)x\,ds| &\leq |x| \int_0^{\frac{1}{|x|}} [f(t)\omega(|sx|)|sx| + g(t)]dt \\ &\leq |x|[f(t)\sup_{r\geq 0}\omega(r)|x| \int_0^{\frac{1}{|x|}} s\,ds + g(t)\frac{1}{|x|}] \\ &\leq \frac{1}{2}f(t)\sup_{r\geq 0}\omega(r) + g(t). \end{aligned} \quad (2.34)$$

Let $\rho > 0$, then by (1.15), there exists a positive constant c_ρ such that

$$V'(t,x)x \geq \rho[\omega(|x|)|x|]^2 - c_\rho. \quad (2.35)$$

Therefore,

$$\begin{aligned} \int_{\frac{1}{|x|}}^1 V'(t,sx)x\,ds &= \int_{\frac{1}{|x|}}^1 V'(t,sx)sx\frac{ds}{s} \\ &\geq \int_{\frac{1}{|x|}}^1 (\rho[\omega(|sx|)|sx|]^2 - c_\rho)\frac{ds}{s} \\ &\geq \frac{\rho}{2}[\omega(|x|)|x|]^2(1 - \frac{1}{|x|^2}) - c_\rho b(t) ln(|x|). \end{aligned} \quad (2.36)$$

Combining (2.33), (2.34) and (2.36), we obtain (2.32) and Lemma 2.5 is proved. □

Now, since $a_0 = \liminf_{s\to\infty} \frac{\omega(s)}{\omega(s^{\frac{1}{2}})} > 0$, for s large enough, we have

$$\frac{1}{\omega(s)} \leq \frac{1}{a_0\omega(s^{1/2})} \quad (2.37)$$

which implies that for $|x|$ large enough

$$\frac{ln(|x|)}{[\omega(|x|)|x|]^2} \leq \frac{ln(|x|)}{|x|} \frac{1}{a_0^2[\omega(|x|^{1/2})|x|^{1/2}]^2} \to 0 \quad \text{as } |x| \to \infty. \quad (2.38)$$

Combining (2.32), (2.38) we obtain (V_4). By applying Corollary 1.3, we obtain a sequence (u_k) of kT-periodic solutions of (1.1) such that $\lim_{k\to\infty} \|u_k\|_\infty = +\infty$.

It remains to analyst the minimal periods of the subharmonic solutions found with the previous results. For this, we will split the problem into two parts. Firstly, we claim that for a sufficiently large integer k, the subharmonic solution u_k is not

T-periodic. In fact, let S_T be the set of T-periodic solutions of (1.1), we will show that S_T is bounded in H_T^1. Assume by contradiction that there exists a sequence (u_n) in S_T such that $\|u_n\|_1 \to \infty$ as $n \to \infty$. Let us write $u_n(t) = \bar{u}_n + \tilde{u}_n(t)$, where $\bar{u}_n$ is the mean value of u_n. Multiplying both sides of the identity

$$\ddot{u}_n(t) + V'(t, u_n(t)) = 0 \tag{2.39}$$

by $\tilde{u}_n(t)$ and integrating, we obtain by (1.9) and Hölder's inequality

$$\begin{aligned} \int_0^T |\dot{u}_n|^2 dt &= -\int_0^T \ddot{u}_n \tilde{u}_n dt \\ &= \int_0^T V'(t, u_n)\tilde{u}_n dt \\ &\leq \|\tilde{u}_n\|_\infty \int_0^T \big[f(t)\omega(|u_n(t)|)|u_n(t)| + g(t)\big] dt \\ &\leq \|\tilde{u}_n\|_\infty \Big[\|f\|_{L^2}\big(\int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt\big)^{1/2} + \|g\|_{L^1}\Big]. \end{aligned} \tag{2.40}$$

By (2.1), (2.2) and (2.40), there exists a positive constant c_{11} such that

$$\|\tilde{u}_n\|_1 \leq c_{11}\big[\big(\int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt\big)^{1/2} + 1\big]. \tag{2.41}$$

Let $\rho > 0$ and let c_ρ be a constant satisfying (2.35). Multiplying both sides of the identity (2.39) by $u_n(t)$ and integrating

$$\begin{aligned} \int_0^T |\dot{u}_n|^2 dt &= -\int_0^T \ddot{u}_n u_n dt \\ &= \int_0^T V'(t, u_n) u_n dt \\ &\geq \rho \int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt - c_\rho T. \end{aligned} \tag{2.42}$$

We deduce from (2.42) and Wirtinger inequality that there exists a positive constant c_{12} such that

$$\|\tilde{u}_n\|_1^2 \geq c_{12}\big[\rho \int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt - c_\rho T\big]. \tag{2.43}$$

Combining (2.41) with (2.43), we can find a positive constant c_{13} such that

$$\rho \int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt - c_\rho T \leq c_{13}\big[\int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt + 1\big]. \tag{2.44}$$

Since ρ is arbitrary chosen,

$$\big(\int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt\big) \quad \text{is bounded.} \tag{2.45}$$

Combining (2.41) and (2.45) yields $(\tilde{u}_n)$ is bounded in H_T^1 and then $|\bar{u}_n| \to \infty$ as $n \to \infty$. Since the embedding $H_T^1 \to L^2(0, T; \mathbb{R}^N)$, $u \to u$ is compact, then we can assume, by going to a subsequence if necessary, that $\tilde{u}_n(t) \to \tilde{u}(t)$ as $n \to \infty$, a.e. $t \in [0, T]$. We deduce that

$$|u_n(t)| \to \infty \quad \text{as } n \to \infty, \text{ a.e. } t \in [0, T]. \tag{2.46}$$

Fatou's lemma and (2.46) imply

$$\int_0^T [\omega(|u_n(t)|)|u_n(t)|]^2 dt \to \infty \quad \text{as } n \to \infty \tag{2.47}$$

which contradicts (2.45). Therefore S_T is bounded in H_T^1. As a consequence, $\varphi_1(S_T)$ is bounded, and since for any $u \in S_T$ one has $\varphi_k(u) = k\varphi_1(u)$, then there exists a positive constant c_{14} such that

$$\frac{1}{k}|\varphi_k(u)| \le c_{14}, \quad \forall u \in S_T,\ \forall k \ge 1. \tag{2.48}$$

Consequently by (2.3), for k large enough, $u_k \notin S_T$. Finally, assumption (1.16) requires that the minimal period of each solution u_k of (1.1) is an integer multiple of T. So if k is chosen to be a prime number, the minimal period of u_k has to be kT. The proof of Theorem 1.4 is complete.

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