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EXISTENCE OF SMOOTH GLOBAL SOLUTIONS FOR A 1-D MODIFIED NAVIER-STOKES-FOURIER MODEL

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Abstract. We prove the existence of strong global solutions of the 1-D modified compressible Navier-Stokes-Fourier equations proposed by Howard Brenner [1, 2].

1. Introduction

We consider the modified Navier-Stokes-Fourier equations proposed by by Brenner [1, 2]:

$$\partial_t\rho + \operatorname{div}(\rho v_m) = 0, \tag{1.1}$$

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v_m) + \nabla p = \operatorname{div}\mathbb{S}, \tag{1.2}$$

$$\partial_t(\rho(\frac{1}{2}v^2 + e)) + \operatorname{div}(\rho(\frac{1}{2}v^2 + e)v_m) + \operatorname{div}(pv) + \operatorname{div} q = \operatorname{div}(\mathbb{S}v), \tag{1.3}$$

$$v|_{\partial\Omega} = 0, v_m \cdot n|_{\partial\Omega} = \nabla\rho \cdot n|_{\partial\Omega} = 0, q \cdot n|_{\partial\Omega} = \nabla\theta \cdot n|_{\partial\Omega} = 0, \tag{1.4}$$

$$(\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0) \quad \text{in } \Omega := (0, 1). \tag{1.5}$$

where ρ is the mass density, v is the fluid-based (Lagrangian) volume velocity, v_m is the mass-based (Eulerian) mass velocity, $p = R\rho\theta$ is the pressure with positive constant $R > 0$, $e = C_V\theta$ the specific internal energy, θ the temperature, $\mathbb{S}$ the viscous stress tensor, we will adopt the Newton's rheological law:

$$\mathbb{S} := \mu\Big(\nabla v + \nabla v^T - \frac{2}{3}\operatorname{div} v\mathbb{I}\Big) + \eta \operatorname{div} v\mathbb{I}, \tag{1.6}$$

where $\mu \geq 0$ and $\eta \geq 0$ stand for the shear and bulk viscosity coefficients, respectively. The relationship between v_m and v is a cornerstone of Brenner's approach. After a careful study [1, 2], Brenner proposes a universal constitutive equation in the form:

$$v - v_m = K\nabla \log\rho, \tag{1.7}$$

with $K \geq 0$ a purely phenomenological coefficient.

Moreover, we suppose the heat flux obeys Fourier's law, specifically,

$$q = -k\nabla\theta, \tag{1.8}$$

where k is the heat conductivity coefficient.

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We will assume $K = 1$, $C_v = 1$, $R = 1$, $\mu > 0$, $\eta = 0$, and

$$k(\theta) := k_0(1 + 4\theta^3), \tag{1.9}$$

with a positive constant $k_0 = 1$. (1.9) is physically relevant as radiation heat conductivity at least for large values of θ (see [8]).

Very recently, Feireisl and Vasseur [4] proved the global-in-time existence of weak solutions to the problem (1.1)-(1.5). Under the conditions that $\rho_0, \theta_0, v_0 \in L^\infty(\Omega)$ and $\rho_0 \geq C > 0, \theta_0 \geq C > 0$ in Ω. Here it should be noted that similar result for the classical Navier-Stokes-Fourier system ((1.1)-(1.3) with $v = v_m$) have not yet been proved. In their proof, they obtained the following global-in-time estimates:

$$\|v\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{1.10}$$

$$\|\theta^{3/2}\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{1.11}$$

$$\|\nabla\theta\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{1.12}$$

where C is a positive constant depending on $\int_\Omega \rho_0 dx$, $\int_\Omega \rho(\frac{1}{2}v_0^2 + C_V\theta_0)dx$, and $\int_\Omega \rho_0 s(\rho_0, \theta_0)dx$, the other norms of ρ_0 and v_0, θ_0.

Our aim in this article is to show the existence of a smooth global solution to the problem (1.1)-(1.5).

Theorem 1.1. *Let $\rho_0, v_0, \theta_0 \in H^1(\Omega)$ with $\inf \rho_0 > 0, \inf \theta_0 > 0$ in Ω. Then there exists a unique strong solution (ρ, v, θ) to the problem (1.1)-(1.5) satisfying*

$$(\rho, v, \theta) \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), (\partial_t\rho, \partial_t v, \partial_t\theta) \in L^2(0, T; L^2(\Omega))$$

for any given $T > 0$ and

$$\inf \rho(x, t) > 0, \quad \inf \theta(x, t) > 0 \quad \text{in } \Omega \times (0, T). \tag{1.13}$$

Remark 1.2. The methods for the one-dimensional classical Navier-Stokes-Fourier equations [6, 7] do not work here. Because their clever method for proving $0 < \frac{1}{C} \leq \rho \leq C < \infty$ does not work here.

The continuity equation (1.1) can be rewritten as

$$\partial_t\rho + \operatorname{div}(\rho v) = \Delta\rho. \tag{1.14}$$

The energy equation (1.3) can be rewritten as

$$\partial_t(\rho\theta) + \operatorname{div}(\rho v_m\theta) + \operatorname{div} q = \mathbb{S} : \nabla v - p \operatorname{div} v. \tag{1.15}$$

2. Proof of Theorem 1.1

Since it is easy to prove a local existence result for smooth solution, which is very similar as that in [3], we omit the details here. We need to prove only the a priori estimates for smooth solutions and omit the proof of the uniqueness which is standard.

Since we take $x \in \Omega := (0, 1)$ and $\partial\Omega = \{0, 1\}$, it follows that $\operatorname{div} = \nabla = \partial_x$, $\Delta = \partial_x^2$, $\mathbb{S} := (\frac{4}{3}\mu + \eta)\partial_x v$ and (1.4) becomes

$$v|_{\partial\Omega} = 0, \quad \nabla\rho|_{\partial\Omega} = \frac{\partial\rho}{\partial x}\Big|_{\partial\Omega} = 0, \quad \nabla\theta|_{\partial\Omega} = \frac{\partial\theta}{\partial x}\Big|_{\partial\Omega} = 0.$$

First, we note that in 1-D, we have

$$\|\rho\|_{L^\infty} \leq C\|\rho\|_{H^1}, \quad \|\theta\|_{L^\infty} \leq C\|\theta\|_{H^1}, \quad \|v\|_{L^\infty} \leq C\|\nabla v\|_{L^2}. \tag{2.1}$$

Lemma 2.1. *If* (ρ, v, θ) *is a strong solution, then*

$$\|\rho\|_{L^\infty(0,T;H^1)} + \|\rho\|_{L^2(0,T;H^2)} \le C(T),$$
$$\|\partial_t\rho\|_{L^2(0,T;L^2)} \le C(T),$$
$$\frac{1}{C(T)} \le \rho.$$

Proof. Testing (1.14) with ρ, using (1.10) and (2.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \rho^2 dx + \int |\nabla\rho|^2 dx &= \int \rho v \nabla\rho dx\\ &\le \|\rho\|_{L^2}\|v\|_{L^\infty}\|\nabla\rho\|_{L^2} \le C\|\nabla v\|_{L^2}\|\rho\|_{L^2}\|\nabla\rho\|_{L^2}\\ &\le \frac{1}{2}\|\nabla\rho\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|\rho\|_{L^2}^2\end{aligned}$$

which gives

$$\|\rho\|_{L^\infty(0,T;L^2)} + \|\rho\|_{L^2(0,T;H^1)} \le C(T).$$

Similarly, testing (1.14) with $-\Delta\rho$, using (1.10) and (2.1), we see that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int |\nabla\rho|^2 dx + \int |\Delta\rho|^2 dx = \int (\rho \operatorname{div} v + v\nabla\rho)\Delta\rho dx\\ &\le (\|\rho\|_{L^\infty}\|\operatorname{div} v\|_{L^2} + \|v\|_{L^\infty}\|\nabla\rho\|_{L^2})\|\Delta\rho\|_{L^2}\\ &\le C\|\rho\|_{H^1}\|\nabla v\|_{L^2}\|\Delta\rho\|_{L^2}\\ &\le \frac{1}{2}\|\Delta\rho\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|\rho\|_{H^1}^2\end{aligned}$$

which yields (2.1). Here we have $\operatorname{div} v = \nabla v = \frac{\partial v}{\partial x}$. Then (2.1) follows easily from (1.14) and (2.1).

To prove (2.1), we multiply (1.14) by $\frac{1}{\rho}$ to obtain

$$\begin{aligned}\partial_t \log\rho - \Delta\log\rho &= |\nabla\log\rho|^2 - v\cdot\nabla\log\rho - \operatorname{div} v\\ &= \Big(\nabla\log\rho - \frac{1}{2}v\Big)^2 - \frac{1}{4}v^2 - \operatorname{div} v\\ &\ge -\frac{1}{4}v^2 - \operatorname{div} v.\end{aligned}$$

By the classical comparison principle, it is easy to infer that $\log\rho \ge w$, with w a solution to the problem

$$\partial_t w - \Delta w = -\frac{1}{4}v^2 - \operatorname{div} v, \quad \nabla w|_{\partial\Omega} = \frac{\partial w}{\partial x}\big|_{\partial\Omega} = 0, \quad w|_{t=0} = \log\rho_0, \tag{2.2}$$

with fixed v satisfying (1.10).

Testing (2.2) with w, using (1.10), we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w^2 dx + \int |\nabla w|^2 dx &\le (\frac{1}{4}\|v\|_{L^\infty}\|v\|_{L^2} + \|\operatorname{div} v\|_{L^2})\|w\|_{L^2}\\ &\le C(\|\nabla v\|_{L^2} + \|\nabla v\|_{L^2}^2)\|w\|_{L^2}\end{aligned}$$

which gives

$$\|w\|_{L^\infty(0,T;L^2)} + \|w\|_{L^2(0,T;H^1)} \le C(T).$$

Similarly, testing (2.2) with $-\Delta w$, using (1.10), we infer that

$$\frac{1}{2}\frac{d}{dt}\int |\nabla w|^2 dx + \int |\Delta w|^2 dx \le |\int \frac{1}{4}\nabla v^2\cdot\nabla w dx| + |\int \operatorname{div} v\cdot\Delta w dx|$$

$$
\begin{aligned}
&\le \frac{1}{2}\|v\|_{L^\infty}\|\operatorname{div} v\|_{L^2}\|\nabla w\|_{L^2} + \|\operatorname{div} v\|_{L^2}\|\Delta w\|_{L^2} \\
&\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + C\|\operatorname{div} v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|\nabla w\|_{L^2}
\end{aligned}
$$

which yields

$$
\|w\|_{L^\infty(0,T;H^1)} \le C(T).
$$

This yields

$$
\log\rho \ge w \ge -C(T) > -\infty
$$

and thus (2.1) holds. The proof is complete. □

Using (1.1), (1.2), (2.1), (2.1), $p := R\rho\theta$, (1.11), (1.12) and the method in [4], it is easy to verify the following lemma.

Lemma 2.2 ([4])**.** *If (ρ, v, θ) is a weak solution, then*

$$
\|v\|_{L^\infty(0,T;L^m(\Omega))} \le C(T) \quad \text{for some } m > 2. \tag{2.3}
$$

It follows from (1.11) and (2.1) that

$$
\|\theta\|_{L^3(0,T;L^\infty(\Omega))} \le C(T). \tag{2.4}
$$

Lemma 2.3. *If (ρ, v, θ) is a strong solution, then*

$$
\|v\|_{L^\infty(0,T;H^1)} + \|v_t\|_{L^2(0,T;L^2)} \le C(T), \tag{2.5}
$$

$$
\|v\|_{L^2(0,T;H^2)} \le C(T). \tag{2.6}
$$

Proof. We start rewriting the momentum equation (1.2) in the form

$$
\rho(\partial_t v + v_m \cdot \nabla v) + R\nabla(\rho\theta) = \mu\Delta v + \frac{1}{3}\mu\nabla \operatorname{div} v. \tag{2.7}
$$

Testing (2.7) with v_t, using (2.1), (2.1), (1.12), (2.3) and (2.4), we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \mu|\nabla v|^2 + \frac{1}{3}\mu(\operatorname{div} v)^2 dx + \int \rho v_t^2 dx \\
&= -\int \rho v_m \cdot \nabla v \cdot v_t dx - R\int \nabla(\rho\theta)\cdot v_t dx \\
&= -\int \rho v \cdot \nabla v \cdot v_t dx + \int \nabla\rho \cdot \nabla v \cdot v_t dx - R\int (\rho\nabla\theta + \theta\nabla\rho) v_t dx \\
&\le \|\rho\|_{L^\infty}\|v\|_{L^\infty}\|\nabla v\|_{L^2}\|v_t\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\nabla v\|_{L^2}\|v_t\|_{L^2} \\
&\quad + R(\|\rho\|_{L^\infty}\|\nabla\theta\|_{L^2} + \|\theta\|_{L^\infty}\|\nabla\rho\|_{L^2})\|v_t\|_{L^2} \\
&\le C\|\nabla v\|_{L^2}^2\|v_t\|_{L^2} + C\|\Delta\rho\|_{L^2}\|\nabla v\|_{L^2}\|v_t\|_{L^2} \\
&\quad + C(\|\nabla\theta\|_{L^2} + \|\theta\|_{L^\infty})\|v_t\|_{L^2}.
\end{aligned} \tag{2.8}
$$

On the other hand, using (2.7) and the H^2-theory of second order elliptic equations, we have

$$
\begin{aligned}
\|v\|_{H^2} &\le C\|\rho\partial_t v + \rho v_m \cdot \nabla v + R\nabla(\rho\theta)\|_{L^2} \\
&\le C(\|v_t\|_{L^2} + \|v\cdot\nabla v\|_{L^2} + \|\nabla\rho\|_{L^\infty}\|\nabla v\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\theta\|_{L^\infty}) \\
&\le C(\|v_t\|_{L^2} + \|\nabla v\|_{L^2}^2 + \|\Delta\rho\|_{L^2}\|\nabla v\|_{L^2} + \|\nabla\theta\|_{L^2} + \|\theta\|_{L^\infty}).
\end{aligned} \tag{2.9}
$$

Now using (2.3), Young's inequality and the Gagliardo-Nirenberg inequality [5],

$$
\|\nabla v\|_{L^2}^2 \le C\|v\|_{L^m}^{2\alpha}\|v\|_{H^2}^{2(1-\alpha)} \le C\|v\|_{H^2}^{2(1-\alpha)} \le \frac{1}{2C}\|v\|_{H^2} + C,
$$

with $1-\alpha=\frac{m+2}{3m+2}<\frac{1}{2}$), we obtain

$$\|v\|_{H^2}\le C(\|v_t\|_{L^2}+\|\Delta\rho\|_{L^2}\|\nabla v\|_{L^2}+\|\nabla\theta\|_{L^2}+\|\theta\|_{L^\infty}+C). \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) and using Gronwall's inequality, we obtain (2.5) and (2.6). This completes the proof. □

Lemma 2.4. *Let $K(\theta):=\theta+\theta^4$. If (ρ,v,θ) is a strong solution, then*

$$\|K(\theta)\|_{L^\infty(0,T;L^2)}+\|K(\theta)\|_{L^2(0,T;H^1)}\le C(T). \tag{2.11}$$

Proof. We start by rewriting the energy equation (1.15) in the form:

$$\rho\partial_t K(\theta)+\rho v_m\cdot\nabla K(\theta)-\Delta K(\theta)=(\mathbb{S}:\nabla v-p\operatorname{div}v)K'(\theta). \tag{2.12}$$

Testing (2.12) with $K(\theta)$, using (1.1), (2.5), (2.1) and (2.1), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho K^2(\theta)dx+\int|\nabla K(\theta)|^2dx\\
&=\int(\mathbb{S}:\nabla v-p\operatorname{div}v)K'(\theta)K(\theta)dx\\
&\le\|\mathbb{S}\|_{L^2}\|\nabla v\|_{L^2}\|K'(\theta)K(\theta)\|_{L^\infty}+C\|\rho\|_{L^\infty}\|\operatorname{div}v\|_{L^2}\|K(\theta)\|_{L^4}^2\\
&\le C\|K(\theta)\|_{L^\infty}^{7/4}+C\|K(\theta)\|_{L^4}^2\\
&\le C\|K(\theta)\|_{L^2}^{7/8}\|K(\theta)\|_{H^1}^{7/8}+\frac{1}{8}\|\nabla K(\theta)\|_{L^2}^2+C\|K(\theta)\|_{L^2}^2\\
&\le\frac{1}{4}\|\nabla K(\theta)\|_{L^2}^2+C\|K(\theta)\|_{L^2}^2+C
\end{aligned}$$

which yields (2.11). Here we have used the Gagliardo-Nirenberg inequalities:

$$\begin{gathered}
\|K(\theta)\|_{L^\infty}\le C\|K(\theta)\|_{L^2}^{1/2}\|K(\theta)\|_{H^1}^{1/2},\\
\|K(\theta)\|_{L^4}\le C\|K(\theta)\|_{L^2}^{3/4}\|K(\theta)\|_{H^1}^{1/4}.
\end{gathered}$$

This completes the proof. □

Lemma 2.5. *If (ρ,v,θ) is a strong solution, then*

$$\|\theta\|_{L^\infty(0,T;H^1)}+\|\theta\|_{L^2(0,T;H^2)}\le C(T), \tag{2.13}$$

$$\|\theta_t\|_{L^2(0,T;L^2)}\le C(T). \tag{2.14}$$

Proof. We start by rewriting the energy equation (2.12) in the form:

$$\partial_t K(\theta)+v_m\cdot\nabla K(\theta)-\frac{1}{\rho}\Delta K(\theta)=\frac{\mathbb{S}:\nabla v-p\operatorname{div}v}{\rho}K'(\theta).$$

Testing the above equation with $-\Delta K(\theta)$, using (2.5), (2.6), (2.1), (2.1) and (2.11), we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla K(\theta)|^2dx+\int\frac{1}{\rho}|\Delta K(\theta)|^2dx\\
&=\int\big[(v-\nabla\log\rho)\nabla K(\theta)-\frac{\mathbb{S}:\nabla v-p\operatorname{div}v}{\rho}K'(\theta)\big]\Delta K(\theta)dx\\
&\le\Big(\|v\|_{L^\infty}\|\nabla K(\theta)\|_{L^2}+\Big\|\frac{1}{\rho}\Big\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla K(\theta)\|_{L^2}\\
&\quad+\|\frac{1}{\rho}\|_{L^\infty}\|\mathbb{S}:\nabla v\|_{L^2}\|K'(\theta)\|_{L^\infty}+C\|\operatorname{div}v\|_{L^2}\|K(\theta)\|_{L^\infty}\Big)\|\Delta K(\theta)\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C(\|K(\theta)\|_{H^1} + \|\rho\|_{H^2}\|K(\theta)\|_{H^1} + \|\nabla v\|_{L^4}^2 \|K(\theta)\|_{L^\infty}^{3/4})\|\Delta K(\theta)\|_{L^2} \\
&\leq C(\|K(\theta)\|_{H^1} + \|\rho\|_{H^2}\|K(\theta)\|_{H^1} + \|v\|_{H^2}^{1/2} \|K(\theta)\|_{H^1}^{3/8})\|\Delta K(\theta)\|_{L^2} \\
&\leq \frac{1}{2}\|\Delta K(\theta)\|_{L^2}^2 + C\|K(\theta)\|_{H^1}^2 + C\|\rho\|_{H^2}^2\|K(\theta)\|_{H^1}^2 + C\|v\|_{H^2}\|K(\theta)\|_{H^1}^{3/4}
\end{aligned}
$$

which yields (2.13). Here we have used the Gagliardo-Nirenberg inequalities:

$$
\begin{gathered}
\|\nabla v\|_{L^4}^2 \leq C\|\nabla v\|_{L^2}^{3/2}\|v\|_{H^2}^{1/2}, \|K(\theta)\|_{L^\infty} \leq C\|K(\theta)\|_{L^2}^{1/2}\|K(\theta)\|_{H^1}^{1/2}, \\
\|\theta\|_{L^\infty(0,T;L^\infty)} \leq C\|\theta\|_{L^\infty(0,T;H^1)}, \\
\|\nabla\theta\|_{L^\infty(0,T;L^2)} \leq C\|\nabla K(\theta)\|_{L^\infty(0,T;L^2)}, \\
\|\Delta\theta\|_{L^2(0,T;L^2)} \leq C\|\Delta K(\theta)\|_{L^2(0,T;L^2)}.
\end{gathered}
$$

Equation (2.14) follows easily from (2.12), (2.13), (2.5), (2.6), and (2.1). This completes the proof. □

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