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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH M-POINT BOUNDARY CONDITIONS AND TWO FRACTIONAL ORDERS

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Abstract. We study boundary-value problems of nonlinear fractional differential equations and inclusions with m-point boundary conditions. Several results are obtained by using suitable fixed point theorems when the right hand side has convex or non convex values.

1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order, the fractional calculus may be considered an old and yet novel topic.

Recently, fractional differential equations have been of great interest. This is because of both the intensive development of the theory of fractional calculus itself and its applications in various sciences, such as physics, mechanics, chemistry, engineering, etc. For example, for fractional initial value problems, the existence and multiplicity of solutions were discussed in [3, 13, 43, 44], moreover, fractional derivative arises from many physical processes,such as a charge transport in amorphous semiconductors [42], electrochemistry and material science are also described by differential equations of fractional order [14, 17, 18, 30, 31].

The existence of solutions of initial value problems for fractional order differential equations have been studied in the literature [1, 26, 34, 35, 36, 37, 38, 40, 41] and the references therein. The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim [21]. Also, recently several qualitative results for fractional differential inclusions were obtained in [5, 9, 33, 39] and the references therein.

Bai and Lü [4] considered the boundary-value problem of fractional-order differential equation

$$
\begin{gathered}
D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\
u(0) = u(1) = 0,
\end{gathered}
$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

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Hussein [20] considered the nonlinear m-point boundary-value problem of fractional type

$$\begin{gathered} D_{0+}^{\alpha} x(t) + q(t) f(t, x(t)) = 0, \quad \text{a.e. on } [0,1],\ \alpha \in (n-1, n],\ n \geq 2 \\ x(0) = x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \xi_i x(\eta_i) \end{gathered}$$

where $0 < \eta_1 < \dots < \eta_{m-2} < 1$, $\xi_i > 0$ with $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-1} < 1$, q is a real-valued continuous function and f is a nonlinear Pettis integrable function.

In the past few decades, many important results relative to equation (1.1) with certain boundary value conditions have been obtained. we refer the reader to [10, 22, 28, 29, 45] and the references therein.

Motivated by the mentioned works, our purpose in the first part of this paper is to show the existence and multiplicity of positive solutions for the boundary-value problem of the fractional differential equation

$$\begin{gathered} D_{0+}^{\beta}(D_{0+}^{\alpha} u)(t) = f(t, u(t)), \quad t \in (0,1), \\ D_{0+}^{\alpha} u(0) = D_{0+}^{\alpha} u(1) = 0, \quad u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} a_i\, u(\xi_i) = \lambda, \end{gathered} \tag{1.1}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $m > 2$, $1 < \alpha, \beta \leq 2$, $2 < \alpha + \beta \leq 4$, $0 < \xi_{j1} < \xi_{j2} < \dots < \xi_{jm-2} < 1$, $a_i > 0$ for $i = 1, 2, \dots, m-2$ and $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-1} < 1$, $\lambda > 0$ is a parameter and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

In the second part of this paper, we consider a nonlinear fractional differential inclusion of an arbitrary order with multi-strip boundary conditions

$$\begin{gathered} D_{0+}^{\beta}(D_{0+}^{\alpha} u)(t) \in F(t, u(t)), \quad t \in (0,1), \\ D_{0+}^{\alpha} u(0) = D_{0+}^{\alpha} u(1) = 0, \quad u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} a_i\, u(\xi_i) = \lambda, \end{gathered} \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and $F : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map.

The aim here is to establish existence results for the problem (1.1), when the right-hand side is convex as well as nonconvex valued. In the first result (Theorem 4.11) we consider the case when the right hand side has convex values, and prove an existence result via Nonlinear alternative for Kakutani maps. In the second result (Theorem 4.16), we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result (Theorem 4.20), we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

The rest of the article is organized as follows: in Section 2, we present some preliminaries that will be used in Sections 3 and 4. In Section 3, we give the existence of one and three positive solutions for the problem (1.1) by using the Leray-Schauder Alternative, Leggett-Williams fixed point theorem and nonlinear contractions. The main result and proof for the problem (1.2) will be given in Section 4. Finally, in Section 4, an example is given to demonstrate the application of one our main result.

2. Preliminaries

In this section, we present some notation and preliminary lemmas that will be used in the Sections 3 and 4.

Definition 2.1 ([40]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2 ([25]). The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$ is defined as

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.3 ([25]). *The equality $D_{0^+}^{\gamma} I_{0^+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L^1(0,1)$.*

Lemma 2.4 ([15, 25]). *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then the differential equation*

$$D_{0^+}^{\alpha} u(t) = 0$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \dots, n$, where $n-1 < \alpha < n$.

Lemma 2.5 (([25]). *Let $\alpha > 0$. Then the following equality holds for $u \in L^1(0,1)$, $D_{0^+}^{\alpha} u \in L^1(0,1)$;*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

$c_i \in \mathbb{R}$, $i = 1, \dots, n$, where $n-1 < \alpha \le n$.

In the following, we present the Green function of fractional differential equation boundary value problem. Let

$$y(t) = -(D_{0^+}^{\alpha} u)(t), \tag{2.1}$$

then, the problem

$$\begin{gathered} D_{0^+}^{\beta}(D_{0^+}^{\alpha} u)(t) = h(t), \quad 1 < \beta \le 2,\ t \in (0,1), \\ D_{0^+}^{\alpha} u(0) = D_{0^+}^{\alpha} u(1) = 0, \end{gathered}$$

where $f \in C[0,1]$, is transformed into the problem

$$\begin{gathered} D_{0^+}^{\beta} y(t) + h(t) = 0, \quad 1 < \beta \le 2,\ t \in (0,1), \\ y(0) = y(1) = 0, \end{gathered} \tag{2.2}$$

Lemma 2.6. *Suppose that $h \in C[0,1]$, then the boundary-value problem (2.2) has a unique solution*

$$y(t) = \int_0^1 H(t,s) h(s) ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \frac{t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le t \le 1, \\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

The proof of the above lemma is similar to that of [4, Lemma 2.3], so we omit it here.

Lemma 2.7. *Let $\Delta = \sum_{i=1}^{m-2} a_i\xi_i^{\alpha-1} \neq 1$. Then, for $y \in C[0,1]$, the boundary-value problem*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0,1),\ 1 < \alpha \leq 2, \\ u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} a_i u(\xi_i) = \lambda, \end{gathered} \tag{2.5}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\,ds + \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)y(s)\,ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)}, \tag{2.6}$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.7}$$

Proof. By applying lemma (2.4), equation (2.5) is equivalent to the integral equation

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds - c_1 t^{\alpha-1} - c_2 t^{\alpha-2},$$

for some arbitrary constants $c_1, c_2 \in \mathbb{R}$. By the boundary condition (2.5), one can get $c_2 = 0$ and

$$\begin{aligned} c_1 = &-\frac{1}{\Gamma(\alpha)(1-\Delta_j)} \int_0^1 (1-s)^{\alpha-1} y(s))ds \\ &+ \frac{1}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)ds - \frac{\lambda_j}{1-\Delta}. \end{aligned}$$

Then, the unique solution of (2.5) is given by the formula

$$\begin{aligned} &u(t) \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + \frac{1}{\Gamma(\alpha)(1-\Delta)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds \\ &\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\,ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds + \frac{\Delta}{\Gamma(\alpha)(1-\Delta)} \int_0^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds \\ &\quad - \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\,ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)} \\ &= \frac{1}{\Gamma(\alpha)} \Big[\int_0^t \Big(t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \Big)\, y(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds \Big] \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 \xi_i^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&- \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1} y(s)\,ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta_j)} \\
&= \frac{1}{\Gamma(\alpha)} \Big[\int_0^t \Big(t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} \Big) y(s)\,ds + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-1} y(s)\,ds \Big] \\
&+ \frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\Delta)} \sum_{i=1}^{m-2} a_{ji} \Big[\int_0^{\xi_i} \Big({\xi_i}^{\alpha-1}(1-s)^{\alpha-1} - (\xi_i - s)^{\alpha-1} \Big) y(s) ds \\
&+ \int_{\xi_i}^1 {\xi_i}^{\alpha-1}(1-s)^{\alpha-1} y(s) ds \Big] + \frac{\lambda t^{\alpha-1}}{(1-\Delta)} \\
&= \int_0^1 G(t,s) y(s) ds + \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s) v(s) ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)}.
\end{aligned}
$$

Thus, the proof is complete. □

Lemma 2.8. *The functions $H(t,s)$ and $G(t,s)$ defined by (2.4) and (2.7), respectively satisfies the following conditions:*

(i) *$G(t,s) \ge 0$, $G(t,s) \le G(s,s) \le \frac{1}{\Gamma(\alpha)}$ for any $t,s \in [0,1]$;*
(ii) *$H(t,s) \ge 0$, $H(t,s) \le H(s,s) \le \frac{1}{\Gamma(\alpha)}$ for any $t,s \in [0,1]$;*
(iii) *there exists a positive function $g \in C(0,1)$ such that $\min_{\gamma \le t \le \delta} G(t,s) \ge g(s)G(s,s), s \in (0,1)$, where $0 < \gamma < \delta < 1$ and*

$$
g(s) = \begin{cases} \frac{\delta^{\alpha-1}(1-s)^{\alpha-1} - (\delta - s)^{\alpha-1}}{s^{\alpha-1}(1-s)^{\alpha-1}} & s \in (0, m_1] \\ (\frac{\gamma}{s})^{\alpha-1} & s \in [m_1, 1) \end{cases} \tag{2.8}
$$

with $\gamma < m_1 < \delta$.

Proof. First, we define

$$
g_1(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \le s \le t \le 1,
$$
$$
g_2(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 \le t \le s \le 1.
$$

One can obtain

$$
\begin{aligned}
g_1(t,s) &\ge \frac{1}{\Gamma(\alpha)} \Big[t^{\alpha-1}(1-s)^{\alpha-1} - (t-ts)^{\alpha-1} \Big] \\
&= \frac{1}{\Gamma(\alpha)} \Big[t^{\alpha-1} \Big((1-s)^{\alpha-1} - (1-s)^{\alpha-1} \Big) \Big] = 0,
\end{aligned}
$$

On the other hand, it is obvious that $g_2(t,s) \ge 0$, $0 \le t \le s \le 1$. Thus, $G(t,s) \ge 0$ for any $t,s \in [0,1]$.

Now, we show that $G(t,s) \le G(s,s)$ for any $t,s \in [0,1]$. Also, we have

$$
\begin{aligned}
\frac{\partial g_1(t,s)}{\partial t} &= \frac{1}{\Gamma(\alpha)} \Big[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1} - (\alpha-1)(t-s)^{\alpha-2} \Big] \\
&= \frac{1}{\Gamma(\alpha)} (\alpha-1) t^{\alpha-2} \Big[(1-s)^{\alpha-1} - (1 - \frac{s}{t})^{\alpha-2} \Big] \\
&\le \frac{1}{\Gamma(\alpha)} (\alpha-1) t^{\alpha-2} \Big[(1-s)^{\alpha-1} - (1-s)^{\alpha-2} \Big] \le 0,
\end{aligned}
$$

then, $g_1(t,s)$ is non-increasing with respect to t on $[s,1]$, hence, we obtain

$$g_1(t,s) \le g_1(s,s) \quad \forall 0 \le s \le t \le 1.$$

On the other hand, we obtain

$$\frac{\partial g_2(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha)}\Big[(\alpha-1)t^{\alpha-2}(1-s)^{\alpha-1}\Big] \ge 0,$$

then $g_2(t,s)$ is increasing with respect to t on $[0,s]$, therefore,

$$g_2(t,s) \le g_2(s,s) \quad \forall 0 \le t \le s \le 1.$$

Then, we have

$$G(t,s) \le G(s,s) \quad \forall t,s \in [0,1]. \tag{2.9}$$

(ii) Since $g_1(.,s)$ is non-increasing and $g_2(.,s)$ is non-decreasing, for all $s,t \in [0,1]$, we have

$$\begin{aligned}
\min_{\gamma\le t\le\delta} G(t,s) &= \begin{cases} \min_{\gamma\le t\le\delta} g_1(t,s) & s\in[0,\gamma] \\ \min_{\gamma\le t\le\delta}\{g_1(t,s),g_2(t,s)\} & s\in[\gamma,\delta] \\ \min_{\gamma\le t\le\delta} g_2(t,s) & s\in[\delta,1] \end{cases} \\
&= \begin{cases} \min_{\gamma\le t\le\delta} g_1(t,s) & s\in[0,m_1] \\ \min_{\gamma\le t\le\delta} g_2(t,s) & s\in[m_1,1] \end{cases} \\
&\ge \begin{cases} g_1(\delta,s) & s\in[0,m_1] \\ g_2(\gamma,s) & s\in[m_1,1] \end{cases} \\
&= \begin{cases} \delta^{\alpha-1}(1-s)^{\alpha-1} - (\delta-s)^{\alpha-1} & s\in[0,m_1] \\ \gamma^{\alpha-1}(1-s)^{\alpha-1} & s\in[m_1,1] \end{cases}
\end{aligned}$$

where $\gamma < m_1 < \delta$ is the solution of

$$\delta^{\alpha-1}(1-m_1)^{\alpha-1} - (\delta-m_1)^{\alpha-1} = \gamma^{\alpha-1}(1-m_1)^{\alpha-1}.$$

It follows from the monotonicity of g_1 and g_2 that

$$\max_{0\le t\le 1} G(t,s) = G(s,s) = \frac{(s(1-s))^{\alpha-1}}{\Gamma(\alpha)} \quad s\in(0,1),$$

The proof is complete. □

Remark 2.9. If $\gamma \in (0,1/4)$ and $\delta = 1-\gamma$, then Lemma 2.8 holds.

In this article, we assume that $\gamma \in (0,1/4)$ and $\delta = 1-\gamma$. Now, we consider the system (1.1). By applying lemmas 2.6 and 2.7, $u \in C(0,1)$ is a solution of (1.1) if and only if $u \in C[0,1]$ is a solution of the nonlinear integral equation

$$\begin{aligned}
u(t) &= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\
&\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds,
\end{aligned} \tag{2.10}$$

where $\lambda t^{\alpha-1}/(1-\Delta)$. Let $z(t) = \frac{\lambda t^{\alpha-1}}{1-\Delta}$ and $C := \sup_{t\in I}|z(t)| = \|z\|$.

3. Main result for the single-valued case

Now we are able to present the existence results for problem (1.1).

3.1. **Existence result via Leray-Schauder alternative.**

Theorem 3.1 (Nonlinear alternative for single valued maps [19])**.** *Let X be a Banach space, C a closed convex subset of X, U an open subset of D and $0 \in U$. Suppose that $T : \overline{U} \to D$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subsets of D) map. Then either*

- (i) *T has a fixed point in U, or*
- (ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda T(u)$.*

Theorem 3.2. *If the continuous function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies:*

- (B1) *There exist a function $\varphi \in L^1(I, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,+\infty) \to (0,+\infty)$ such that*
$$|f(t,x)| \le \varphi(t)\psi(\|x\|), \quad \text{for all } (t,x) \in I \times \mathbb{R}.$$
- (B2) *There exists a constant $M > 0$ such that*
$$\begin{aligned} &M\Big[C + \psi(M)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &+ \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]\Big]^{-1} > 1. \end{aligned} \tag{3.1}$$

Then (1.1) has at least one solution on I.

Proof. Consider $F : E \to E$ with $u = F(u)$, where
$$\begin{aligned} (Tu)(t) &= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds, \end{aligned}$$
for all $t \in I$. We show that T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{u \in C([0,1],\mathbb{R}) : \|u\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then
$$\begin{aligned} |(Tu)(t)| &\le |z(t)| + \psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]. \end{aligned}$$
Consequently,
$$\begin{aligned} \|Tu\| &\le C + \psi(r)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]. \end{aligned}$$

Next we show that T maps bounded sets into equi-continuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain
$$\begin{aligned} &|(Tu)(t_1) - (Tu)(t_2)| \\ &\le \int_0^1 |G(t_1,s) - G(t_2,s)|\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2}a_i}{1-\Delta}|t_1^{\alpha-1}-t_2^{\alpha-1}|\int_0^1 G_1(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds\\
&\le \int_0^1 |G(t_1,s)-G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(r)\varphi(r)dr\Big)ds\\
&+\frac{\sum_{i=1}^{m-2}a_i}{1-\Delta}|t_1^{\alpha-1}-t_2^{\alpha-1}|\int_0^1 G_1(\xi_i,s)\Big(\int_0^1 H(s,r)\psi(r)\varphi(r)dr\Big)ds
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^1 |G(t_1,s)-G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(r)\varphi(r)dr\Big)ds\\
&\le \Big(\int_0^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^1\Big)|G(t_1,s)-G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(r)\varphi(r)dr\Big)ds\\
&\le \psi(r)\int_0^{t_1}[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}+((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})]\\
&\quad\times\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\quad+\psi(r)\int_{t_1}^{t_2}[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}+(t_2-s)^{\alpha-1}]\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\quad+\psi(m)\int_{t_2}^1[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}]\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\to 0\quad\text{uniformly as } t_1\to t_2.
\end{aligned}
$$

Therefore,

$$
|(Tu)(t_1)-(Tu)(t_2)|\to 0\quad\text{uniformly as } t_1\to t_2,\text{ for } u\in B_r.
$$

As F satisfies the above assumptions, it follows by the Arzelá-Ascoli theorem that $T: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Theorem 3.1) once we have proved the boundendness of the set of all solutions to equations $u=\lambda Fu$ for $\lambda\in[0,1]$.

Let u be a solution. Then, for $t\in[0,1]$, and using the computations in proving that T is bounded, we have

$$
\begin{aligned}
|u(t)|=|\lambda(Tu)(t)|\le |z(t)|+\psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
+\frac{\sum_{i=1}^{m-2}a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|u\|\Big[C+\psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&+\frac{\sum_{i=1}^{m-2}a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]\Big]^{-1}\le 1.
\end{aligned}
$$

In view of (B2), there exists M such that $\|u\|\ne M$. Let us set

$$
U:=\{u\in C(I,\mathbb{R}):\|u\|<M\}.
$$

Note that the operator $T : \overline{U} \to C([0,1], \mathbb{R}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda T(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.1), we deduce that has a fixed point $u \in U$ which is a solution of the problem (1.1). This completes the proof. □

3.2. **Existence result via Leggett-Williams fixed point theorem.** In this section, we assume that $\gamma \in (0, 1/4)$ and $\delta = 1 - \gamma$. To establish the existence of three positive solutions to system (1.1), we use the following Leggett-Williams fixed point theorem. For the convenience of the reader, we present here the Leggett-Williams fixed point theorem [27].

Given a cone K in a real Banach space E, a map α is said to be a nonnegative continuous concave (resp. convex) functional on K provided that $\alpha : K \to [0. +\infty)$ is continuous and

$$\begin{gathered}\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),\\ (\text{resp. } \alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)),\end{gathered}$$

for all $x, y \in K$ and $t \in [0,1]$. Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$\begin{gathered}P_r = \{x \in K | \|x\| < r\},\\ P(\alpha, a, b) = \{x \in K | a \leq \alpha(x), \|x\| \leq b\}.\end{gathered}$$

Theorem 3.3 (Leggett-Williams fixed point theorem)**.** *Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \leq c$ such that*

- (A1) *$\{x \in P(\alpha, b, d) | \alpha(x) > b\} \neq \emptyset$, and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$,*
- (A2) *$\|Ax\| < a$ for $\|x\| \leq a$, and*
- (A3) *$\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.*

Then A has at least three fixed points x_1, x_2, and x_3 and such that $\|x_1\| < a$, $b < \alpha(x_2)$ and $\|x_3\| > a$, with $\alpha(x_3) < b$.

Also, we introduce the following notation. Define

$$\eta = \min_{\gamma \leq t \leq \delta}\{g(t)\}, \quad \sigma = \min\{\eta, \gamma^{\alpha-1}\},$$

and

$$\begin{aligned}M &= \int_0^1 G(s,s)\Big(\int_0^1 H(s,r)dr\Big)\, ds\\ &\quad + \frac{1}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r)dr\Big)ds,\\ m &= \min_{\gamma \leq t \leq \delta}\Big\{\int_\gamma^\delta G(t,s)\Big(\int_\gamma^\delta H(s,r)dr\Big)\, ds\\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_\gamma^\delta G(\xi_i, s)\Big(\int_\gamma^\delta H(s,r)dr\Big)\, ds\Big\}.\end{aligned}$$

Note that $0 < m < M$. The basic space used in this paper is a real Banach space $E = C([0,1], \mathbb{R})$ with the norm $\|u\| := \max_{t\in[0,1]} |u(t)|$. Then, we define a cone $K \subset E$, by

$$K = \{u \in E : u(t) \geq 0 \ \min_{\gamma\leq t\leq\delta}(u(t)) \geq \frac{\sigma}{3}\|u\|\},$$

and an operator $T : E \to E$ by

$$\begin{aligned} T(u)(t) = &\int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds + \frac{\lambda t^{\alpha-1}}{1-\Delta}. \end{aligned} \tag{3.2}$$

Lemma 3.4. *The operator T is defined from K to K; i.e., $T(K) \subseteq K$.*

Proof. For any $u \in K$, by Lemma 2.8, $T(u)(t) \geq 0$, $t \in [0,1]$, and it follows from (3.2) that

$$\begin{aligned} &\|T(u)\| \\ &\leq \int_0^1 G(s,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\ &\quad + \frac{1}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds + \frac{\lambda}{(1-\Delta)} \\ &= \Big(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\Big)\Big(G(s,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds\Big) \\ &\quad + \frac{1}{(1-\Delta)}\sum_{i=1}^{m-2} a_i\Big(\int_0^\gamma + \int_\gamma^\delta + \int_\delta^1\Big)\Big(G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds\Big) \\ &\quad + \frac{\lambda}{(1-\Delta)} \\ &\leq 3\Big[\int_\gamma^\delta G(s,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\ &\quad + \frac{1}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_\gamma^\delta G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds + \frac{\lambda}{3(1-\Delta)}\Big]. \end{aligned} \tag{3.3}$$

Thus, for any $u \in K$, it follows from Lemma 2.8 and (3.2) that

$$\begin{aligned} &\min_{\gamma\leq t\leq\delta} T(u)(t) \\ &= \min_{\gamma\leq t\leq\delta}\Big\{\int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta_1)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)}\Big\} \\ &\geq \int_0^1 g(s)G(s,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\gamma^{\alpha-1}}{(1-\Delta_1)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big)\, ds + \frac{\lambda\gamma^{\alpha-1}}{3(1-\Delta)} \\
&\geq \eta \int_\gamma^\delta G(s,s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big) ds \\
&\quad + \frac{\gamma^{\alpha-1}}{(1-\Delta_1)} \sum_{i=1}^{m-2} a_i \int_\gamma^\delta G(\xi_i, s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big)\, ds + \frac{\lambda\gamma^{\alpha-1}}{3(1-\Delta)} \\
&\geq \sigma \Big[\int_\gamma^\delta G(s,s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big) ds \\
&\quad + \frac{1}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_\gamma^\delta G(\xi_i, s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big)\, ds + \frac{\lambda}{3(1-\Delta)}\Big] \\
&\geq \frac{\sigma}{3}\|T(u)\|.
\end{aligned}
$$

Therefore, from the above, we conclude that $T(u)(t) \in K$; that is, $T(K) \subset K$. This completes the proof. $\square$

It is clear that the existence of a positive solution for (1.1) is equivalent to the existence of a nontrivial fixed point of T in K.

Next, we define the nonnegative continuous concave functional on K by

$$\alpha(u) = \min_{\gamma \leq t \leq \delta}(u(t)).$$

It is obvious that, for each $u \in K$, $\alpha(u) \leq \|u\|$.

Throughout this section, we assume that p, q, are two positive numbers satisfying $\frac{1}{p} + \frac{1}{q} \leq 1$. Now, we can state our main result in this section.

Theorem 3.5. *Assume that there exist nonnegative numbers a, b, c such that $0 < a < b \leq \frac{\sigma}{3}c$, and $f(t,u)$, $j = 1, 2$, satisfy the following conditions:*

- (H1) $f(t,u) \leq \frac{1}{p} \cdot \frac{c}{M}$, *for all* $t \in [0,1], u \in [0,c]$;
- (H2) $f(t,u) \leq \frac{1}{p} \cdot \frac{a}{M}$, *for all* $t \in [0,1], u \in [0,a]$;
- (H3) *(i)* $f(t,u) > \frac{b}{m}$ *for all* $t \in [\gamma,\delta]$, $u \in [b, \frac{3b}{\sigma}]$.

Then, for

$$0 < \lambda < \frac{c(1-\Delta)}{q}, \tag{3.4}$$

problem (1.1) *has at least three positive solutions u_1, u_2, u_3 such that $\|u_1\| < a$, $b < \min_{\gamma \leq t \leq \delta}(u_2(t))$, and $\|u_3\| > a$, with $\min_{\gamma \leq t \leq \delta}(u_3(t)) < b$.*

Proof. First, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator. If $u \in \overline{P_c}$, then $\|u\| \leq c$. Then, by applying condition (H1), we have

$$
\begin{aligned}
\|T(u)\| &= \max_{0 \leq t \leq 1} |T(u)(t)| \\
&= \max_{0 \leq t \leq 1} \Big\{ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big) ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big)\, ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)}\Big\} \\
&\le \frac{1}{p} \cdot \frac{c}{M}\Big\{\int_0^1 G(s,s)\Big(\int_0^1 H(s,r) dr\Big)\, ds \\
&\quad + \frac{1}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r) dr\Big)\, ds\Big\} + \frac{\lambda}{(1-\Delta)} \\
&\le \frac{1}{p} \cdot c + \frac{1}{q} \cdot c \le c.
\end{aligned}
$$

Therefore, $\|T(u)\| \le c$, that is, $T : \overline{P_c} \to \overline{P_c}$. The operator T is completely continuous by an application of the Ascoli-Arzela theorem.

In the same way, condition (H2) implies that condition (A2) of Theorem 3.3 is satisfied. We now show that condition (A1) of Theorem 3.3 is satisfied. Clearly $\{u \in P(\alpha, b, \frac{3b}{\sigma}) | \alpha(u) > b\} \ne \emptyset$. If $u \in P(\alpha, b, \frac{3b}{\sigma})$, then $b \le u(s) \le \frac{3b}{\sigma}$, $s \in [\gamma, \delta]$.

By condition (H3), we obtain

$$
\begin{aligned}
&\alpha(T(u)(t)) \\
&= \min_{\gamma \le t \le \delta} (T(u)(t)) \\
&= \min_{\gamma \le t \le \delta} \Big\{\int_0^1 G(t,s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big) ds \\
&\quad + \frac{t^{\alpha-1}}{(1-\Delta_1)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r) f(r,u(r)) dr\Big)\, ds + \frac{\lambda t^{\alpha-1}}{(1-\Delta)}\Big\} \\
&= \frac{b}{m} m = b.
\end{aligned}
$$

Therefore, condition (A1) of Theorem 3.3 is satisfied.

Finally, we show that the condition (A3) of Theorem 3.3 is also satisfied. If $u \in P(\alpha, b, c)$, and $\|T(u)\| > 3b/\sigma$, then

$$
\alpha(T(u)(t)) = \min_{\gamma \le t \le \delta} (T(u)(t)) \ge \frac{\sigma}{3} \|T(u)\| > b.
$$

Therefore, condition (A3) of Theorem 3.3 is also satisfied. By Theorem 3.3, there exist three positive solutions u_1, u_2, u_3 such that $\|u_1\| < a$, $b < \min_{\gamma \le t \le \delta}(u_2(t))$, and $\|u_3\| > a$, with $\min_{\gamma \le t \le \delta}(u_3(t)) < b$. we have the conclusion. $\square$

3.3. **Existence result via nonlinear contractions.** Now we present the existence results for problem (1.1).

Definition 3.6. Let X be a Banach space and let $T : X \to X$ be a mapping. T is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(0) = 0$ and $\psi(\xi) < \xi$ for all $\xi > 0$ with the property:

$$
\|T(u) - T(v)\| \le \psi(\|u - v\|), \quad \forall u, v \in X.
$$

Lemma 3.7 (Boyd and Wong [6]). *Let X be a Banach space and let $T : X \to X$ be a nonlinear contraction. Then T has a unique fixed point in X.*

Theorem 3.8. *Assume that f satisfy the following conditions:*

(H4) $|f(t,u)-f(t,v)| \le h(t)\frac{|u-v|}{\theta+|u-v|}$ *for all* $t\in[0,1]$, $u,v\ge 0$, *where* $h:[0,1]\to\mathbb{R}^+$ *is continuous and*

$$\begin{aligned}\theta &= \int_0^1 G(s,s)\Big(\int_0^1 H(s,r)h(r)dr\Big)\,ds\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)h(r)dr\Big)\,ds,\quad t\in[0,1].\end{aligned}\tag{3.5}$$

AUTHORS: IT SEEMS THAT PART OF THE THEOREM IS MISSING ???

Proof. We define the operator $T:(E=C([0,1],\mathbb{R}))\to E$ by

$$\begin{aligned}T(u)(t)&=\int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)ds\\ &\quad+\frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r,u(r))dr\Big)\,ds+\frac{\lambda t^{\alpha-1}}{1-\Delta},\end{aligned}$$

for $t\in[0,1]$. Let $\psi:\mathbb{R}^+\to\mathbb{R}^+$ a continuous nondecreasing function such that $\psi(0)=0$ and $\psi(\xi)<\xi$ for all $\xi>0$ be defined by

$$\psi(\xi)=\frac{\theta\xi}{\theta+\xi},\quad \forall\xi\ge 0.$$

Let $u,v\in E$. Then

$$|f(s,u(s))-f(s,v(s))|\le\frac{h(s)}{\theta}\psi(\|u-v\|).$$

Thus

$$\begin{aligned}|T(u)(t)-T(v)(t)|&\le\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)h(r)\frac{|u(r)-v(r)|}{\theta+|u(r)-v(r)|}dr\Big)ds\\ &\quad+\frac{\sum_{i=1}^{m-2}a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\frac{|u(r)-v(r)|}{\theta+|u(r)-v(r)|}dr\Big)\,ds,\end{aligned}$$

for all $t\in[0,1]$. In view of (3.5), it follows that $\|T(u)-T(v)\|\le\psi(\|u-v\|)$ and hence T is a nonlinear contraction. Thus, by Lemma 3.7, the operator T has a unique fixed point in E, which in turn is a unique solution of problem (1.1). $\square$

4. Existence results of the multi-valued case

4.1. **The upper semi-continuous case.** To obtain the complete continuity of existence solutions operator, the following lemmas and definitions are needed.

Let (X,d) be a metric space with the corresponding norm $\|\cdot\|$ and let $I=[0,1]$. Denoted by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{B}(X)$ the family of all nonempty subsets of X and by $\mathcal{P}(X)$ the family of all Borel subsets of X. If $A\subset I$ then $\chi_A:I\to\{0,1\}$ denotes the characteristic function of A. For any subset $A\subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A,B\subset X$ is defined by

$$d_H(A,B)=\max\{d^*(A,B),d^*(B,A)\},\quad d^*(A,B)=\sup\{d(a,B),a\in A\},$$

where $d(x,B)=\inf_{y\in B}d(x,y)$. Define

$$\mathcal{P}(X)=\{Y\subset X:\ Y\ne\emptyset\},$$

$$\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\},$$
$$\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\},$$
$$\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\},$$
$$\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\}.$$

Also, we denote $C(I, X)$ the Banach space of all continuous functions $x : I \to X$ endowed with the norm $|x|_c = \sup_{t\in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x : [0, 1] \to X$ endowed with the norm $|x|_1 = \int_I |x(t)|dt$.

A subset $D \subset L^1(I, X)$ is said to be decomposable if for any $x, y \in D$ and any subset $A \in \mathcal{L}(I)$ one has $x\chi_A + y\chi_B \in D$, where $B = I \setminus A$.

Let (X, d_1) and (Y, d_2) be two metric spaces. If $T : X \to \mathcal{P}(X)$ a set-valued map, then a point $x \in X$ is called a fixed point for T if $x \in T(x)$. T is said to be bounded on bounded sets if $T(B) := \cup_{x\in B}T(x)$ is a bounded subset of X for all bounded sets B in X. T is said to be compact if $T(B)$ is relatively compact for any bounded sets B in X. T is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. T is said to be upper semi-continuous if for any open set $D \subset X$, the set $\{x \in X : T(x) \subset D\}$ is open in X. T is called completely continuous if it is upper semi-continuous and and, for every bounded subset $A \subset X$, $T(A)$ is relatively compact. It is well known that a compact set-valued map T with nonempty compact values is upper semi-continuous if and only if T has a closed graph.

We define the graph of T to be the set $Gr(T) = \{(x, y) \in X \times Y, y = T(x)\}$ and recall a useful result regarding connection between closed graphs and upper semi-continuity.

Lemma 4.1 ([12, Proposition 1.2]). *If $T : X \to \mathcal{P}_{cl}(Y)$ is upper semi-continuous, then $Gr(T)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in T(x_n)$, then $y_* \in T(x_*)$. Conversely, if T is completely continuous and has a closed graph, then it is upper semi-continuous.*

For convenience of the reader, we present here the following nonlinear alternative of Leray-Schauder type and its consequences.

Theorem 4.2 (Nonlinear alternative for Kakutani maps [19]). *Let X be a Banach space, C a closed convex subset of X, U an open subset of C and $0 \in U$. Suppose that $T : \overline{U} \to \mathcal{P}_{cl,cv}(C)$ is a upper semi-continuous compact map; here $\mathcal{P}_{cl,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either*

(i) *T has a fixed point in U, or*
(ii) *there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda T(u)$.*

Definition 4.3. The multifunction $T : X \to \mathcal{P}(X)$ is said to be lower semi-continuous if for any closed subset $C \subset X$, the subset $\{s \in X : T(s) \subset C\}$ is closed.

If $F : I\times\mathbb{R}\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map with compact values and $x \in C(I, \mathbb{R})$ we define
$$S_F(x) := \{f \in L^1(I, \mathbb{R}) : f(t) \in F(t, x(t)) \text{ a.e. } I\}.$$
Then F is of lower semi-continuous type if $S_F(\cdot)$ is lower semi-continuous with closed and decomposable values.

Theorem 4.4 ([7]). *Let S be a separable metric space and $G : S \to \mathcal{P}(L^1(I,\mathbb{R}))$ be a lower semi-continuous set-valued map with closed decomposable values. Then G has a continuous selection (i.e., there exists a continuous mapping $g : S \to L^1(I,\mathbb{R})$ such that $g(s) \in G(s)$ for all $s \in S$).*

Definition 4.5. (i) A set-valued map $G : I \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for any $x \in \mathbb{R}$ the function $t \to d(x, G(t))$ is measurable.

(ii) A set-valued map $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if $t \to F(t, x)$ is measurable for all $x \in \mathbb{R}$ and $x \to F(t, x)$ is upper semi-continuous for almost all $t \in I$.

(iii) F is said to be L^1-Carathéodory if for any $l > 0$ there exists $h_l \in L^1(I,\mathbb{R})$ such that $\sup\{|v| : v \in F(t, x)\} \le h_l(t)$ a.e. I for all $x \in \mathbb{R}$.

The following results are easily deduced from the theoretical limit set properties.

Lemma 4.6 ([2, Lemma 1.1.9]). *Let $\{K_n\}_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is a compact subset of a separable Banach space X. Then*

$$\overline{\mathrm{co}}(\limsup_{n\to\infty} K_n) = \cap_{N>0}\overline{\mathrm{co}}\big(\cup_{n\ge N} K_n\big),$$

where $\overline{\mathrm{co}}(A)$ refers to the closure of the convex hull of A.

Lemma 4.7 ([2, Lemma 1.4.13]). *Let X and Y be two metric spaces. If $G : X \to \mathcal{P}_{cp}(Y)$ is upper semi-continuous, then for each $x_0 \in X$,*

$$\limsup_{x\to x_0} G(x) = G(x_0).$$

Definition 4.8. Let X be a Banach space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset L^1([a, b], X)$ is said to be semi-compact if

(a) it is integrably bounded; i.e., there exists $q \in L^1([a, b],\mathbb{R}^+)$ such that

$$|x_n(t)|_E \le q(t), \quad \text{for a.e. } t \in [a, b] \text{ and every } n \in \mathbb{N},$$

(b) the image sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in E for a.e. $t \in [a, b]$.

The following important result follows from the Dunford-Pettis theorem (see [23, Proposition 4.2.1]).

Lemma 4.9. *Every semi-compact sequence $L^1([a, b], X)$ is weakly compact in the space $L^1([a, b], X)$.*

When the nonlinearity takes convex values, Mazur's Lemma may be useful:

Lemma 4.10 ([32, Theorem 21.4]). *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^m \alpha_{mk}x_k$ with $\alpha_{mk} > 0$ for $k = 1, 2, \dots, m$ and $\sum_{k=1}^m \alpha_{mk} = 1$ which converges strongly to x.*

Theorem 4.11. *The Carathéodory multivalued map $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty, compact, convex values and satisfies:*

(H5) *There exist a continuous nondecreasing function $\psi : [0,+\infty) \to (0,+\infty)$ and $\varphi \in L^1(I,\mathbb{R}^+)$ such that*

$$\|F(t, x)\|_{\mathcal{P}} := \sup\Big\{|v(t)| : \ v \in F(t, x)\Big\} \le \varphi(t)\psi(\|x\|),$$

for a.e. $t \in I$ and each $x \in \mathbb{R}$.

(H6) *There exists a constant* $M > 0$ *such that*

$$M\Big[C + \psi(M)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]\Big]^{-1} > 1. \tag{4.1}$$

Then problem (1.2) *has at least one solution.*

Proof. Let $X = E$ and consider $M > 0$ as in (3.1). It is obvious that the existence of solutions to problem (1.2) reduces to the existence of the solutions of the integral inclusion

$$u(t) = z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)F(r,u(r))dr\Big)ds + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)F(r,u(r))dr\Big)\,ds.$$

$$u(t) \in z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)F(r,u(r))dr\Big)ds + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)F(r,u(r))dr\Big)\,ds, \quad t \in I, \tag{4.2}$$

where $G(t,s)$ and $H(t,s)$ defined by (2.4) and (2.7), respectively. Consider the set-valued map $T : E \to \mathcal{P}(X)$ defined by

$$T(u) := \Big\{v \in X;\ v(t) = z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)\,ds, \quad f \in \overline{S_F(u)}\Big\}. \tag{4.3}$$

We show that T satisfies the hypotheses of the Theorem 4.2.

Claim 1. We show that $T(u) \subset X$ is convex for any $u \in X$. If $v_1, v_2 \in T(u)$ then there exist $f_1, f_2 \in S_F(u)$ such that for any $t \in I$ one has

$$v_i(t) = z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f_i(r)dr\Big)ds + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f_i(r)dr\Big)\,ds, \quad i = 1,2. \tag{4.4}$$

Let $0 \le \lambda \le 1$. Then for any $t \in I$ we have

$$\begin{aligned}
&(\lambda v_1 + (1-\lambda)v_2)(t)\\
&= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)[\lambda f_1(r) + (1-\lambda)f_2(r)]dr\Big)ds\\
&\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)[\lambda f_1(r) + (1-\lambda)f_2(r)]dr\Big)\,ds.
\end{aligned}$$

The values of F are convex, thus $S_F(u)$ is a convex set and hence $\lambda v_1 + (1-\lambda)v_2 \in T(u)$.

Claim 2. we show that T is bounded on bounded sets of X. Let B be any bounded subset of X. Then there exist $m > 0$ such that $\|u\| \le m$ for all $u \in B$. If $v \in T(u)$ there exists $f \in S_F(u)$ such that

$$\begin{aligned} v(t) &= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)\,ds. \end{aligned} \tag{4.5}$$

One may write for any $t \in I$,

$$\begin{aligned} |v(t)| &\le |z(t)| + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)|f(r)|dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)|f(r)|dr\Big)\,ds \\ &\le |z(t)| + \int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)\psi(\|u\|)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)\psi(\|u\|)dr\Big)\,ds \\ &\le |z(t)| + \psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|v\| &= \max_{t\in I} |v(t)| \\ &\le C + \psi(m)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big], \end{aligned}$$

for all $v \in T(u)$; i.e., $T(B)$ is bounded.

Claim 3. We show that T maps bounded sets into equi-continuous sets. Let B be any bounded subset of X as before and $v \in T(u)$ for some $u \in B$. Then, there exists $f \in S_F(u)$ such that $v(t)$ is defined as (4.5). So, for any $t_1, t_2 \in [0,1]$, without loss of generality we may assume that $t_2 > t_1$ and one can get

$$\begin{aligned} |v(t_1) - v(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)|\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\Delta}|t_1^{\alpha-1} - t_2^{\alpha-1}| \int_0^1 G_1(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &\le \int_0^1 |G(t_1,s) - G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(m)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{1-\Delta}|t_1^{\alpha-1} - t_2^{\alpha-1}| \int_0^1 G_1(\xi_i,s)\Big(\int_0^1 H(s,r)\psi(m)\varphi(r)dr\Big)ds \end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\int_0^1 |G(t_1,s)-G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(m)\varphi(r)dr\Big)ds\\
&\le \Big(\int_0^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^1\Big)|G(t_1,s)-G(t_2,s)|\Big(\int_0^1 H(s,r)\psi(m)\varphi(r)dr\Big)ds\\
&\le \psi(m)\int_0^{t_1}[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}+((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})]\\
&\quad\times\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\quad+\psi(m)\int_{t_1}^{t_2}[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}+(t_2-s)^{\alpha-1}]\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\quad+\psi(m)\int_{t_2}^1[(t_2^{\alpha-1}-t_1^{\alpha-1})(1-s)^{\alpha-1}]\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds\\
&\to 0\quad\text{uniformly as } t_1\to t_2.
\end{aligned}
\tag{4.6}
$$

Thus,

$$
|v(t_1)-v(t_2)|\to 0\quad\text{uniformly as } t_1\to t_2. \tag{4.7}
$$

Therefore, $T(B)$ is an equi-continuous set in X. As satisfies the above Claims 2 and 3, therefore it follows by the Arzelá-Ascoli theorem that $T:C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Claim 5. T is upper semi-continuous. To this end, it is sufficient to show that T has a closed graph. Let $v_n\in T(u_n)$ be such that $v_n\to v$ and $u_n\to u$, as $n\to+\infty$. Then there exists $m>0$ such that $\|u_n\|\le m$. We shall prove that $v\in T(u)$ means that there exists $f_n\in S_F(u_n)$ such that, for a.e. $t\in I$, we have

$$
\begin{aligned}
v_n(t)&=z(t)+\int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f_n(r)dr\Big)ds\\
&\quad+\frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f_n(r)dr\Big)\,ds.
\end{aligned}
$$

The condition (H5) implies that $f_n(t)\in\varphi(t)\psi(m)B_1(0)$. Then $\{f_n\}_{n\in\mathbb{N}}$ is integrable bounded in $L^1(I,\mathbb{R})$. Since F has compact values, we deduce that $\{f_n\}_n$ is semi-compact. By Lemma 4.9, there exists a subsequence, still denoted $\{f_n\}_{n\in\mathbb{N}}$, which converges weakly to some limit $f\in L^1(I,\mathbb{R})$. Moreover, the mapping $\Gamma:L^1(I,\mathbb{R})\to X=E$ defined by

$$
\begin{aligned}
\Gamma(g)(t)&=\int_0^1 G(t,s)\Big(\int_0^1 H(s,r)g(r)dr\Big)ds\\
&\quad+\frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2}a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)g(r)dr\Big)\,ds
\end{aligned}
$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [24, 32]. Moreover for a.e. $t\in I$, $u_n(t)$ converges to $u(t)$. Then we have

$$\begin{aligned}v(t) &= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)\,ds.\end{aligned}$$

It remains to prove that $f \in F(t,u(t))$, a.e. $t \in I$. Mazur's Lemma 4.10 yields the existence of $\alpha_i^n \geq 0$, $i = n,\dots,k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n f_i(\cdot)$ converges strongly to f in L^1. Using Lemma 4.6, we obtain that

$$\begin{aligned}v(t) &\in \cap_{n\geq 1}\overline{\{g_n(t)\}}, \quad \text{a.e. } t \in I \\ &\subset \cap_{n\geq 1}\overline{\text{co}}\{f_k(t),\ k \geq n\} \\ &\subset \cap_{n\geq 1}\overline{\text{co}}\Big\{\cup_{n\geq 1} F(t,u_k(t))\Big\} \\ &= \overline{\text{co}}\Big(\limsup_{k\to+\infty} F(t,u_k(t))\Big).\end{aligned} \tag{4.8}$$

The fact that the multivalued function $x \to F(.,x)$ is upper semi-continuous and has compact values, together with Lemma 4.7, implies that

$$\limsup_{n\to+\infty} F(t,u_n(t)) = F(t,u(t)), \quad \text{a.e. } t \in I.$$

This with (4.8) yields that $f(t) \in \overline{\text{co}}F(t,u(t))$. Finally $F(\cdot,\cdot)$ has closed, convex values; hence $f(t) \in F(t,u(t))$, a.e. $t \in I$. Thus $v \in T(u)$, proving that T has a closed graph. Finally, with Lemma 4.1 and the compactness of T, we conclude that T is upper semi-continuous.

Claim 6. A priori bounds of solutions. Let u be a solution of (1.2). Then there exists $f \in L^1(I,\mathbb{R})$ with $f \in S_F(u)$ such that

$$\begin{aligned}u(t) &= z(t) + \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &\quad + \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)\,ds.\end{aligned}$$

In view of (H5), and using the computations in the Clime 2 above, for each $t \in I$, we obtain

$$\begin{aligned}|u(t)| &\leq |z(t)| + \psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big].\end{aligned}$$

Consequently,

$$\begin{aligned}&\|u\|\Big[C + \psi(\|u\|)\Big[\int_0^1 G(s,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)ds \\ &+ \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)\varphi(r)dr\Big)\,ds\Big]\Big]^{-1} \leq 1.\end{aligned}$$

In view of (H6), there exists M such that $\|u\| \neq M$. Let us set

$$U := \{u \in C(I, \mathbb{R}) : \|u\| < M\}.$$

Note that the operator $T : \overline{U} \to \mathcal{P}(C([0,1], \mathbb{R})$ is upper semi-continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda T(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 4.2), we deduce that has a fixed point $u \in U$ which is a solution of the problem (1.2). This completes the proof. □

4.2. **The lower semi-continuous case.** Here, we study the case when T is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [7] for lower semi-continuous maps with decomposable values. Consider a Banach space Y and $I = [a, b]$ an interval of the real line.

Definition 4.12. A subset $A \subset L^1(I, Y)$ is decomposable if for all $u, v \in A$ and for every Lebesgue measurable subset $I' \subset I$, we have $u\chi_{I'} + v\chi_{I \setminus I'} \in A$, where χ_A stands for the characteristic function of the set A.

Let $F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0, +\infty), \mathbb{R}) \to \mathcal{P}(L^1([0, +\infty), \mathbb{R}))$ associated with F as

$$\mathcal{F}(u) := \Big\{ w \in L^1(I, \mathbb{R})) : w(t) \in F(t, u(t)), \text{ a.e. } t \in I \Big\}, \tag{4.9}$$

which is called the Nemytskii operator associated with F.

Definition 4.13. Let $F : I \times Y \to \mathcal{P}(Y)$ be a multivalued map with nonempty compact values. We say that F is of lower semi-continuous type if its associated Nemytskii operator $\mathcal{F} : C(I, Y) \to \mathcal{P}(L^1(I, Y))$ defined by $\mathcal{F}(y) = S_F(y)$ is lower semi-continuous and has nonempty, closed, and decomposable values.

Definition 4.14. Let A be a subset of $I \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in I and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.

Next, we state the celebrated selection theorem of Bressan and Colombo [7].

Lemma 4.15. *Let X be a separable metric space and let Y be a Banach space. Then every lower semi-continuous multivalued operator $T : X \to \mathcal{P}_{cl}(L^1(I, Y))$ with nonempty closed decomposable values has a continuous selection; i.e., there exists a continuous single-valued function $f : X \to L^1(I, Y)$ such that $f(x) \in T(x)$ for every $x \in X$.*

Theorem 4.16. *Assume that (H5) and the following condition holds:*

- (H7) *$F : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that:*
  - (a) *$(t, x) \to F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
  - (b) *$x \to F(t, x)$ is lower semi-continuous for each $t \in I$.*

Then boundary value problem (1.2) *has at least one solution on I.*

Proof. We note first that if (H5) and (H7) are satisfied then F is of lower semi-continuous type (e.g., [16]). Then, by Lemma 4.15, there exists a continuous function $f : C(I,\mathbb{R}) \to L^1(I,\mathbb{R})$ such that $f(u) \in \mathcal{F}(u)$ for all $u \in C(I,\mathbb{R})$. Consider the problem

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = f(u(t)), \quad 0 < t < +\infty, \\ D_{0^+}^{\alpha} u(0) = D_{0^+}^{\alpha} u(1) = 0, \quad u(0) = 0, \quad u(1) - \sum_{i=1}^{m-2} a_i \, u(\xi_i) = \lambda, \end{gathered} \tag{4.10}$$

in the space $C([0,1),\mathbb{R})$. It is clear that if u is a solution of the problem (4.10), then u is a solution to the problem (1.2). In order to transform the problem (4.10) into a fixed point problem, we define the operator as

$$\begin{aligned} \Theta u(t) = \Big\{ z(t) &+ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r) f(u(r))dr\Big)ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r) f(u(r))dr\Big)\, ds, \ t \in I\Big\}. \end{aligned}$$

It can easily be shown that is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.11. So we omit it. This completes the proof. □

4.3. **The Lipschitz case.** Now we prove the existence of solutions for the problem (1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [11].

Definition 4.17. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called:

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that $d_H(N(x), N(y)) \leq d(x,y)$ for each $x, y \in X$;
(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 4.18 (Covitz-Nadler [11]). *Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

Definition 4.19. A measurable multi-valued function $F : [0,+\infty) \to \mathcal{P}(X)$ is said to be integrably bounded if there exists a function $f \in L^1(I,X)$ such that for all $v \in F(t)$, $\|v\| \leq f(t)$ for a.e. $t \in [0,+\infty)$.

Theorem 4.20. *Assume that the following conditions hold:*

(H8) *$F : I \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : I \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*
(H9) *There exists $l : I \to [0,+\infty)$ are not identical zero on any closed subinterval of I, and*

$$\int_0^1 H(r,r) l(r) dr < +\infty, \quad i = 1, 2,$$

such that for almost all $t \in [0,+\infty)$,

$$d_H(F(t,x_1), F(t,x_2)) \leq l(t)|x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}$ with $d(0, F(t,0,0)) \leq l(t)$ for almost all $t \in I$.

Then boundary-value problem (1.2) *has at least one solution on* $I = [0,1]$ *if*

$$\int_0^1 G(s,s)\Big(\int_0^1 H(r,r)l(r)dr\Big)ds + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)}\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(r,r)l(r)dr\Big)\,ds < 1.$$

Proof. We transform problem (1.2) into a fixed point problem. Consider the set-valued map $T : C[0,1] \to \mathcal{P}(C[0,1],\mathbb{R}))$ defined at the beginning of the proof of Theorem 4.11. It is clear that the fixed points of T are solutions of (1.2).

Note that since the set-valued map $F(\cdot,u(\cdot))$ is measurable with the measurable selection theorem (e.g., [8, Theorem III.6]) it admits a measurable selection $f : I \to \mathbb{R}$. Moreover, since F is integrably bounded, $f \in L^1([0,1],\mathbb{R})$. Therefore, $S_F(u) \neq \emptyset$.

We shall prove that T fulfills the assumptions of Covitz-Nadler contraction principle (Lemma 4.18).

First, we note that since $S_F(u) \neq \emptyset$, $T(u) \neq \emptyset$ for any $u \in C([0,+\infty))$. Second, we prove that $T(u)$ is closed for any $u \in AC^1([0,+\infty),\mathbb{R})$. Let $\{u_n\}_{n\geq 0} \in T(u)$ such that $u_n \to u_0$ in $AC^1([0,+\infty),\mathbb{R})$. Then $u_0 \in AC^1([0,+\infty),\mathbb{R})$ and there exists $f_n \in S_F(u)$ such that

$$\begin{aligned} u_n(t) = z(t) &+ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f_n(r)dr\Big)ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f_n(r)dr\Big)\,ds. \end{aligned}$$

Since F has compact values, we may pass onto a subsequence (if necessary) to obtain that f_n converges to $f \in L^1(([0,1],\mathbb{R}))$ in $L^1(([0,1],\mathbb{R}))$. In particular, $f \in S_F(u)$ and for any $t \in [0,1]$ we have

$$\begin{aligned} u_n(t) \to u_0(t) = z(t) &+ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f(r)dr\Big)\,ds; \end{aligned}$$

i.e., $u_0 \in T(u)$ and $T(u)$ is closed.

Next we show that T is a contraction on $C([0,1],\mathbb{R})$. Let $u_1,u_2 \in C([0,1],\mathbb{R})$ and $v_1 \in T(u_1)$. Then there exist $f_1 \in S_F(u_1)$ such that

$$\begin{aligned} v_1(t) = z(t) &+ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r)f_1(r)dr\Big)ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)}\sum_{i=1}^{m-2} a_i\int_0^1 G(\xi_i,s)\Big(\int_0^1 H(s,r)f_1(r)dr\Big)\,ds, \quad t \in I. \end{aligned}$$

Consider the set-valued map

$$H(t) := F(t,u_2(t)) \cap \{u \in \mathbb{R} : |f_1(t) - u| \leq l(t)|x_1 - x_2|\}, \quad t \in [0,+\infty).$$

By (H5), we have

$$d_H(F(t,x_1),F(t,x_2)) \leq l(t)|x_1 - x_2|,$$

hence H has nonempty closed values. Moreover, since H is measurable (e.g., [8, Proposition III.4]), there exists f_2 a measurable selection of H. It follows that

$f_2 \in S_F(u_2)$ and for any $t \in [0,1]$,

$$|f_1(t) - f_2(t)| \le l(t)|x_1 - x_2|.$$

Define

$$\begin{aligned} v_2(t) = z(t) &+ \int_0^1 G(t,s)\Big(\int_0^1 H(s,r) f_2(r) dr\Big) ds \\ &+ \frac{t^{\alpha-1}}{(1-\Delta)} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(s,r) f_2(r) dr\Big)\, ds, \quad t \in I, \end{aligned}$$

and one can obtain

$$\begin{aligned} |v_1(t) - v_2(t)| &\le \int_0^1 G(s,s)\Big(\int_0^1 H(r,r)|f_1(r) - f_1(r)| dr\Big) ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(r,r)|f_1(r) - f_1(r)| dr\Big)\, ds \\ &\le \int_0^1 G(s,s)\Big(\int_0^1 H(r,r) l(r)|x_1(r) - x_2(r)| dr\Big) ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(r,r) l(r)|x_1(r) - x_2(r)| dr\Big)\, ds \\ &\le \|x_1 - x_2\| \Big[\int_0^1 G(s,s)\Big(\int_0^1 H(r,r) l(r) dr\Big) ds \\ &\quad + \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(r,r) l(r) dr\Big)\, ds\Big]. \end{aligned}$$

Therefore,

$$\begin{aligned} \|v_1 - v_2\| \le \|x_1 - x_2\| \Big[&\int_0^1 G(s,s)\Big(\int_0^1 H(r,r) l(r) dr\Big) ds \\ &+ \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(r,r) l(r) dr\Big)\, ds\Big]. \end{aligned}$$

From an analogous reasoning by interchanging the roles of u_1 and u_2 it follows

$$\begin{aligned} d_H(T(u_1), T(u_2)) \le \|x_1 - x_2\| \Big[&\int_0^1 G(s,s)\Big(\int_0^1 H(r,r) l(r) dr\Big) ds \\ &+ \frac{\sum_{i=1}^{m-2} a_i}{(1-\Delta)} \int_0^1 G(\xi_i, s)\Big(\int_0^1 H(r,r) l(r) dr\Big)\, ds\Big]. \end{aligned}$$

Since T is a contraction, it follows by the Lemma 4.18 that T admits a fixed point which is a solution to problem (1.2). $\square$

5. An application

Consider the singular boundary-value problem

$$\begin{gathered} D_{0^+}^{3/2}(D_{0^+}^{3/2} u)(t) = f(t, u(t)), \quad t \in (0,1), \\ D_{0^+}^{3/2} u(0) = D_{0^+}^{3/2} u(1) = 0, \quad u(0) = 0, \quad u(1) - \frac{7}{4} u(\frac{1}{16}) - u(\frac{1}{4}) = \lambda_1. \end{gathered} \tag{5.1}$$

Here, $\alpha=\beta=3/2$, $p=2$, $m=4$, $a_1=7/4$, $a_2=1$, $\xi_1=1/16$ and $\xi_2=1/4$. Let

$$f(t,u)=\begin{cases}\frac{\sqrt{1-t^2}}{1000}+\frac{1}{400}u^2, & t\in[0,1],\ 0\le u\le 1,\\ \frac{\sqrt{1-t^2}}{1000}+5[u^2-u]+\frac{1}{400}, & t\in[0,1],\ 1<u<2,\\ \frac{\sqrt{1-t^2}}{1000}+2[\log_2 u+2u]+\frac{1}{400}, & t\in[0,1],\ 2\le u\le 4\\ \frac{\sqrt{1-t^2}}{1000}+\frac{\sqrt{u}}{2}+19+\frac{1}{400}, & t\in[0,1],\ 4<u<+\infty,\end{cases}$$

Choose $\gamma=1/4$ and $\delta=3/4$. Then, by direct calculations, we can obtain that $\eta=0.3780,\sigma=0.3536$ and

$$\Delta=0.9375,\quad m=1.0765,\quad M=2.9786.$$

Then, by Choosing $a=1$, $b=2$, $c=1000$, one obtains

$$\frac{1}{p}\cdot\frac{a}{M}=0.0671,\quad \frac{1}{p}\cdot\frac{c}{M}=67.1456,\quad \frac{b}{m}=1.8579.$$

It is easy to check that f satisfy the conditions (H1)–(H3). Then, all conditions of theorem 3.5 hold. Hence, for $0<\lambda<3.1250$, the system (5.1) has at least three positive solutions u_1,u_2,u_3 such that $\|u_1\|<1, 2<\min_{\frac14\le t\le\frac34}(u_2(t))$, and $\|u_3\|>1$, with $\min_{\frac14\le t\le\frac34}(u_3(t))<2$.

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