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BLOW-UP CRITERION FOR STRONG SOLUTIONS TO THE 3D MAGNETO-MICROPOLAR FLUID EQUATIONS IN THE MULTIPLIER SPACE

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Abstract. In this article, we study the blow-up of strong solutions to the magneto-micropolar (MMP) fluid equations in $\mathbb{R}^3$. It is proved that if the gradient field of velocity satisfies

$$\nabla u \in L^{2/(2-r)}(0,T;\dot{X}_r(\mathbb{R}^3)) \quad \text{with } r \in [0,1],$$

then the strong solution (u,w,b) can be extended beyond $t=T$.

1. Introduction

In this article, we consider the 3D magneto-micropolar (MMP) fluid equations

$$\begin{gathered}
\partial_t u + (u\cdot\nabla)u - (\mu+\chi)\Delta u - (b\cdot\nabla)b + \nabla(p+|b|^2) - \chi\nabla\times w = 0,\\
\partial_t w - \gamma\Delta w - \kappa\nabla\nabla\cdot w + 2\chi w + u\cdot\nabla w - \chi\nabla\times u = 0,\\
\partial_t b - \nu\Delta b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0,\\
\nabla\cdot u = \nabla\cdot b = 0,\\
u(x,0)=u_0(x), b(x,0)=b_0(x), w(x,0)=w_0(x).
\end{gathered} \tag{1.1}$$

where $u(x,t)\in\mathbb{R}^3$, $w(x,t)\in\mathbb{R}^3$, $b(x,t)\in\mathbb{R}^3$, $p=p(x,t)\in\mathbb{R}$ denote the velocity of the fluid, the micro-rotational velocity, magnetic field and pressure, respectively. μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities and ν is the magnetic diffusivity. (u_0,w_0,b_0) are the given initial data with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$.

The MMP fluid system (1.1) was first studied by Galdi and Rionero in [3]. Rojas-Medar and Boldrin [11] proved the existence of weak solutions by the Galerkin method, and in 2D case, also proved the uniqueness of the weak solutions. Ortega-Torres and Rojas-Medar [12, 8] established the local in time existence and uniqueness of strong solutions and proved global in time existence of strong solution for small initial data. However, whether the local strong solutions can exist globally or the global weak solution is regular and unique is an outstanding open problem.

If $b=0$, (1.1) reduce to micropolar fluid equations.The micropolar fluid equations were first proposed by Eringen [1]. It is a type of fluids which exhibit the

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micro-rotational effects and micro-rotational inertia, and can be viewed as a non-Newtonian fluid. It can describe some physical phenomena that can't be treated by the classical Navier-Stokes equations for the viscous incompressible fluids, such as the motion of animal blood, liquid crystals and dilute aqueous polymer solutions, etc. The existences of weak and strong solutions were treated by Galdi and Rionero [4], as well as Yamaguchi [14], respectively. Recently, Ferreira and Villamizar-Roa [2] considered the existence and stability of solutions to the micropolar fluids in exterior domains. Villamizar-Roa and Rodríguez-Bellido [13] studied the micropolar system in a bounded domain by using the semigroup approach in L^p-space, showing the global existence of strong solutions for small data and the asymptotic behavior and stability of the solutions.

The purpose of this article is to study the breakdown criteria of smooth solutions to the MMP fluid system (1.1). Some fundamental Serrin's-type regularity criteria was done in [9] and [15] independently. Torres [9] showed the uniqueness of weak solution if $(u, w, b) \in L^p(0, T; L^q(\Omega))$; $\frac{2}{p} + \frac{3}{q} \leq 1$, $q > 3$ in a bounded domain with the no-slip boundary conditions, Yuan [15] proved that if $\nabla u \in L^p(0, T; L^q(\mathbb{R}^3))$; $\frac{2}{p} + \frac{3}{q} = 2$, $\frac{3}{2} < q \leq \infty$, then the weak solution is regular. Zhang [16] consider the regularity criterion for the 3D MMP fluid equations in Triebel-Lizorkin spaces. Torres [10] showed a regularity criterion in term of pressure for the micropolar in a bounded domain.

Motivated by these works, we establish a similar regularity criterion to the MMP fluid system (1.1) in multiplier space $\dot{X}_r(R^3)$. More precisely, we prove the following result.

Theorem 1.1. *Let $T > 0$, (u, w, b) be a strong solution of 3D MMP equation (1.1) on $(0, T)$ with the initial data $(u_0, w_0, b_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. If the gradient field of velocity satisfies*

$$\nabla u \in L^{2/(2-r)}(0, T; \dot{X}_r(\mathbb{R}^3)), \quad 0 \leq r \leq 1 \tag{1.2}$$

then the solution (u, w, b) can be extended smoothly beyond $t = T$.

Definition 1.2. Let $T > 0$, $(u_0, w_0, b_0) \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. A measurable R^3-valued triple (u, w, b) is said to be a weak solution of the MMP equation on $(0, T]$ if the following conditions hold:

(1) $(u, w, b) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$,
(2) (u, w, b) verifies (1.1) in the sense of distribution,
(3) the following energy inequality is satisfied,

$$\begin{aligned} &\|(u, w, b)\|_2^2 + 2(\mu + \chi) \int_0^t \|\nabla u\|_2^2 ds \\ &+ 2\gamma \int_0^t \|\nabla w\|_2^2 ds + 2\nu \int_0^t \|\nabla b\|_2^2 ds + 2\chi \int_0^t \|w\|_2^2 ds \leq \|(u_0, b_0, w_0)\|_2^2. \end{aligned} \tag{1.3}$$

For the convenience, we set $\mu = \chi = 1/2$, $\kappa = \gamma = \nu = 1$, throughout this article, the L^p-norm of a function denoted by $\| \cdot \|_p$, and the H^s-norm by $\| \cdot \|_{H^s}$.

2. The multiplier space

In this section, we describe the multiplier space $\dot{X}_r$ introduced by Lemarie-Rieusset [6] .

Definition 2.1. For $0 \le r < 3/2$, the space $\dot{X}_r$ is defined as the space of $f(x) \in L^2_{loc}(\mathbb{R}^3)$ such that

$$\|f\|_{\dot{X}_r} = \sup_t \|g\|_{\dot{H}^r \le 1} \|fg\|_2 < \infty,$$

where we denote by $\dot{H}^r(R^d)$ the completion of the space $D(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \|(-\Delta)^{\frac{r}{2}} u\|_2 = \||\xi|^r \hat{u}(\xi)\|_2$, where $\hat{u}(\xi)$ denotes the Fourier transform of u.

For any function $f(x,t)$ defined for both spatial and time variables, we have

$$\|f_\lambda\|_{L^{\frac{2}{1-r}}(0,\frac{T}{\lambda^2},\dot{X}_r)} = \|f_\lambda\|_{L^{\frac{2}{1-r}}(0,T,\dot{X}_r)}$$

for any $\lambda > 0$, with $f_\lambda(x,t) = \lambda f(\lambda x, \lambda^2 t)$. So, if (u,w,b) solves the MMP equation, then so does $(u_\lambda, w_\lambda, b_\lambda)$ for all $\lambda > 0$. This is so called scaling dimension zero property. For more details, we refer the reader to [6, 7, 17]. In particular, we have the imbedding

$$L^{3/r}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3), \quad 0 \le r < \frac{3}{2}$$

holds, $r = 0$, it is clear that, [5],

$$\dot{X}_0 \cong BMO.$$

3. Proof of main theorem

We take gradient of the both sides of (1.1) and take the L^2 inner product of the resulting equation with $(\nabla u, \nabla w, \nabla b)$. With help of integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_2^2 + \|\nabla w\|_2^2 + \|\nabla b\|_2^2) + (\|\nabla^2 u\|_2^2 + \|\nabla^2 w\|_2^2 \\
&+ \|\nabla^2 b\|_2^2) + \|\nabla\nabla\cdot w\|_2^2 + \|\nabla w\|_2^2 dx \\
&= -\int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)u]\nabla u dx + \int_{\mathbb{R}^3} \nabla[(b\cdot\nabla)b]\nabla u dx - \int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)w]\nabla w dx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^3} \nabla(\nabla\times w)\nabla u dx + \frac{1}{2}\int_{\mathbb{R}^3} \nabla(\nabla\times u)\nabla w dx \\
&\quad - \int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)b]\nabla b dx + \int_{\mathbb{R}^3} \nabla[(b\cdot\nabla)u]\nabla b dx \\
&= \sum_{i=1}^{7} I_i.
\end{aligned} \tag{3.1}$$

To estimate I_1, we integrate by parts and apply Holder's inequality:

$$\begin{aligned}
I_1 &= -\int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)u]\nabla u dx \\
&= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k(u_i\partial_i u_j)\partial_k u_j dx \\
&= -\sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} \partial_k u_i \partial_i u_j \partial_k u_j dx - \sum_{i,j,k=1}^{3} \int_{\mathbb{R}^3} u_i \partial_i \partial_k u_j \partial_k u_j dx \\
&= I_{11} + I_{12}.
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
|I_{11}| &\le \int_{\mathbb{R}^3} |\nabla u|^3 dx \\
&\le \|\nabla u \cdot \nabla u\|_2 \|\nabla u\|_2 \\
&\le \|\nabla u\|_{\dot{X}_r} \|\nabla u\|_2^{2-r} \|\nabla^2 u\|_2^r \\
&\le C\|\nabla u\|_{\dot{X}_r}^{2/(2-r)} \|\nabla u\|_2^2 + \frac{1}{4}\|\nabla^2 u\|_2^2.
\end{aligned} \tag{3.3}
$$

Here we have used the inequality $\|f\|_{\dot{H}^r} \le \|f\|_2^{1-r}\|\nabla f\|_2^r$. Using the incompressible condition $\nabla \cdot u = 0$, we obtain

$$
|I_{12}| = 0. \tag{3.4}
$$

Similarly, for I_2 one can deduce

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^3} \nabla[(b\cdot\nabla)b]\nabla u dx \\
&= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k(b_i\partial_i b_j)\partial_k u_j dx \\
&= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k b_i \partial_i b_j \partial_k u_j dx + \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} b_i \partial_i \partial_k b_j \partial_k u_j dx \\
&= I_{21} + I_{22}.
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
|I_{21}| &\le \int_{\mathbb{R}^3} |\nabla b|^2 |\nabla u| dx \\
&\le \|\nabla u \cdot \nabla b\|_2 \|\nabla b\|_2 \\
&\le \|\nabla u\|_{\dot{X}_r} \|\nabla b\|_2^{2-r} \|\nabla^2 b\|_2^\gamma \\
&\le C\|\nabla u\|_{\dot{X}_r}^{2/(2-r)} \|\nabla b\|_2^2 + \frac{1}{6}\|\nabla^2 b\|_2^2.
\end{aligned} \tag{3.6}
$$

For I_{22}, we will give a result of $I_{22} + I_{72} = 0$ later. In the same way, for I_3, we have

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}^3} \nabla[(u\cdot\nabla)w]\nabla w dx \\
&= -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k(u_i\partial_i w_j)\partial_k w_j dx \\
&= -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_i w_j \partial_k w_j dx - \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i \partial_i \partial_k w_j \partial_k w_j dx \\
&= I_{31} + I_{32}.
\end{aligned} \tag{3.7}
$$

$$
|I_{31}| \le C\|\nabla u\|_{\dot{X}_r}^{2/(2-r)} \|\nabla w\|_2^2 + \frac{1}{2}\|\nabla^2 w\|_2^2. \tag{3.8}
$$

Using the incompressible condition $\nabla \cdot u = 0$, we obtain

$$
|I_{32}| = 0. \tag{3.9}
$$

Integrating by parts and Holder's inequality, we find

$$\begin{aligned} |I_4+I_5| &= |\frac{1}{2}\int_{\mathbb{R}^3}\nabla(\nabla\times w)\nabla u+\nabla(\nabla\times u)\nabla w dx| \\ &\le \int_{\mathbb{R}^3}|\nabla^2 u||\nabla w|dx \\ &\le \frac{1}{4}\|\nabla^2 u\|_2^2+\|\nabla w\|_2^2. \end{aligned} \tag{3.10}$$

With the similar derivation as of I_1, one has

$$\begin{aligned} I_6 &= -\int_{\mathbb{R}^3}\nabla[(u\cdot\nabla)b]\nabla b dx \\ &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k(u_i\partial_i b_j)\partial_k b_j dx \\ &= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_i b_j\partial_k b_j dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}u_i\partial_i\partial_k b_j\partial_k b_j dx \\ &= I_{61}+I_{62}. \end{aligned} \tag{3.11}$$

$$\begin{aligned} |I_{61}| &\le \int_{\mathbb{R}^3}|\nabla b|^2|\nabla u|dx \\ &\le \|\nabla u\cdot\nabla b\|_2\|\nabla b\|_2 \\ &\le \|\nabla u\|_{\dot{X}_r}\|\nabla b\|_2^{2-r}\|\nabla^2 b\|_2^{r} \\ &\le C\|\nabla u\|_{\dot{X}_r}^{2/(2-r)}\|\nabla b\|_2^2+\frac{1}{6}\|\nabla^2 b\|_2^2. \end{aligned} \tag{3.12}$$

Using the incompressible condition $\nabla\cdot u=0$, we have

$$|I_{62}|=0. \tag{3.13}$$

Similarly,

$$\begin{aligned} I_7 &= \int_{\mathbb{R}^3}\nabla[(b\cdot\nabla)u]\nabla b dx \\ &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k(b_i\partial_i u_j)\partial_k b_j dx \\ &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k b_i\partial_i u_j\partial_k b_j dx+\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}b_i\partial_i\partial_k u_j\partial_k b_j dx \\ &= I_{71}+I_{72}. \end{aligned} \tag{3.14}$$

$$|I_{71}|\le C\|\nabla u\|_{\dot{X}_r}^{2/(2-r)}\|\nabla b\|_2^2+\frac{1}{6}\|\nabla^2 b\|_2^2. \tag{3.15}$$

Now, we give a simple result

$$\begin{aligned} I_{22}+I_{72} &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}b_i\partial_i(\partial_k b_j\partial_k u_j)dx \\ &= \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i[b_i(\partial_k b_j\partial_k u_j)]dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_i(\partial_k b_j\partial_k u_j)=0. \end{aligned} \tag{3.16}$$

Here we have used the incompressible condition and the $|(u,w,b)| \to 0$ as $|x| \to \infty$. Now combing the estimates of (3.1)–(3.16), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_2^2 + \|\nabla w\|_2^2 + \|\nabla b\|_2^2) + (\|\nabla^2 u\|_2^2 + \|\nabla^2 w\|_2^2 + \|\nabla^2 b\|_2^2) + \|\nabla\nabla\cdot w\|_2^2 \\
&\le C(\|\nabla u\|_2^2 + \|\nabla w\|_2^2 + \|\nabla b\|_2^2)\|\nabla u\|_{\dot{X}_r}^{2/(2-r)}.
\end{aligned} \tag{3.17}$$

Applying Gronwall's inequality, we have

$$(\|\nabla u\|_2^2 + \|\nabla w\|_2^2 + \|\nabla b\|_2^2) \le C\exp\{\int_0^T \|\nabla u\|_{\dot{X}_r}^{2/(2-r)} dt\}. \tag{3.18}$$

Combining the a priori estimate (3.18) with the energy inequality (1.3) and by standard arguments of continuation of local solutions, we conclude that the solutions (u,w,b) can be extended beyond $t = T$ provided that $\nabla u \in L^{2/(2-r)}(0,T;\dot{X}_r(R^3))$, $r \in [0,1]$. This completes the proof of Theorem 1.1.

Remark. We point out that the above methods do not seem to work for a bounded domains. For bounded domains, the main difficulty lies in controlling the pressure. If we removed the contribution of the pressure p, which can be recovered with the help of (u,b), there is no difficulty in considering a bounded domain.

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