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HÖLDER REGULARITY FOR SIGNED SOLUTIONS TO SINGULAR POROUS MEDIUM TYPE EQUATIONS

SIMONA PUGLISI

Abstract. We prove Hölder regularity for bounded signed solution to singular porous medium type equations, whose prototype is

$$u_t - \operatorname{div} m|u|^{m-1}Du = 0 \quad \text{weakly in } E_T,$$

with $m \in (0, 1)$.

1. Introduction and statement of main result

Let E be an open set in $\mathbb{R}^N$, for $T > 0$ denote the cylindrical domain

$$E_T = E \times (0, T]$$

and let $\Gamma = \partial E_T \setminus \bar{E} \times \{T\}$ be its parabolic boundary. We consider quasi-linear homogeneous singular parabolic partial differential equation

$$u_t - \operatorname{div} \mathcal{A}(x, t, u, Du) = 0 \quad \text{weakly in } E_T, \tag{1.1}$$

where $\mathcal{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is measurable and subject to the structure conditions

$$\begin{cases} \mathcal{A}(x, t, z, \xi) \cdot \xi \geq C_0 m|z|^{m-1}|\xi|^2 \\ |\mathcal{A}(x, t, z, \xi)| \leq C_1 m|z|^{m-1}|\xi| \end{cases} \tag{1.2}$$

for a.e. $(x, t) \in E_T$, for every $z \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where C_0, C_1 are given positive constants and $0 < m < 1$.

The prototype of this class of parabolic equations is the porous medium equation

$$u_t - \operatorname{div} m|u|^{m-1}Du = 0 \quad \text{weakly in } E_T.$$

The modulus of ellipticity of this class of parabolic equations is $m|u|^{m-1}$. Whenever $m > 1$, such a modulus vanishes when u vanishes, and for this reason we say that the equation (1.1)-(1.2) is *degenerate*. Whenever $0 < m < 1$, such a modulus approaches infinity as $u \to 0$, and for this reason we say that the equation (1.1)-(1.2) is *singular*. One also speaks about *slow*, when $m > 1$, or *fast diffusion*, when $0 < m < 1$ (see the monograph [8]).

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We are interested only in *local* solutions to *singular* porous medium type equation. The parameters $\{N, m, C_0, C_1\}$ are the data, and we say that a generic constant $\gamma = \gamma(N, m, C_0, C_1)$ depends upon the data, if it can be quantitatively determined a priori only in terms of the indicated parameters. As usual, in the following the constant γ may change from line to line.

Let us give the notion of weak solution for this kind of equations as follows. A function $u \in C_{\rm loc}(0,T; L^2_{\rm loc}(E))$ with $|u|^m \in L^2_{\rm loc}(0,T; H^1_{\rm loc}(E))$ is a local weak sub(super)-solution to (1.1) if for every compact set $\mathcal{K} \subset E$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_{\mathcal{K}} u\varphi\, dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}} [-u\varphi_t + \mathcal{A}(x,t,u,Du)\cdot D\varphi]\, dx\, dt \le (\ge)\, 0,$$

for all non-negative test functions $\varphi \in H^1_{\rm loc}(0,T; L^2(\mathcal{K})) \cap L^2_{\rm loc}(0,T; H^1_0(\mathcal{K}))$.

Our aim is to show that locally bounded, local, weak solutions of variable sign to our problem (1.1)-(1.2), with $0 < m < 1$, are locally Hölder continuous.

Let us introduce the parabolic m-distance of a compact set $\mathcal{K} \subset E_T$ from the parabolic boundary Γ in the following way

$$\text{m-dist}(\mathcal{K},\Gamma) = \inf_{(x,t)\in\mathcal{K},(y,s)\in\Gamma} \Big(\|u\|_{\infty,E_T}^{\frac{1-m}{2}} |x-y| + |t-s|^{1/2} \Big).$$

We can state the main result of this paper as follows.

Theorem 1.1. *Let u be a bounded, local, weak solution to* (1.1)-(1.2). *Then u is locally Hölder continuous in E_T and there exist constants $c > 1$ and $\alpha \in (0,1)$ such that for every compact set $\mathcal{K} \subset E_T$*

$$|u(x_1,t_1) - u(x_2,t_2)| \le c\, \|u\|_{\infty,E_T} \Big(\frac{\|u\|_{\infty,E_T}^{\frac{1-m}{2}} |x_1 - x_2| + |t_1 - t_2|^{1/2}}{\text{m-dist}(\mathcal{K},\Gamma)} \Big)^{\alpha},$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \mathcal{K}$.

The constant c depends only upon the data, the norm $\|u\|_{\infty,\mathcal{K}}$ and m-dist$(\mathcal{K},\Gamma)$; the constant α depends only upon the data and the norm $\|u\|_{\infty,\mathcal{K}}$.

In some physical applications it is natural to consider positive solutions to quasilinear parabolic equations of the form (1.1), and it is also a very useful simplification from the mathematical point of view. Therefore, most of the papers directly deal with positive solutions.

A Hölder regularity result for signed solutions was obtained first by DiBenedetto in [3] for degenerate ($p > 2$) p-laplacian type equations and then by Chen and DiBenedetto in [1] for singular ($1 < p < 2$) p-laplacian type ones (see also [4]). Later on, in 1993 Porzio and Vespri [7] considered the case of a degenerate doubly non-linear equation, whose prototype is

$$u_t - \operatorname{div}\big(|u|^{m-1}|Du|^{p-2}Du\big) = 0,$$

for $p \ge 2$ and $m \ge 1$. Notice that this kind of equations admits as a particular case both the degenerate p-laplacian type equations (for $m = 1$ and $p > 2$) and the degenerate porous medium type equations (for $p = 2$ and $m > 1$). As a consequence, it only remained open the case of the singular porous medium type equations.

We want to point out that the difficulty in our case is due to the presence of the term $|u|^{m-1}$ in the modulus of continuity; indeed, the fact that u changes sign plays a crucial role here. In the p-laplacian case, the modulus of continuity is $|Du|^{p-2}$, thus the proof does not change if u is positive or if it changes sign. One could think to follow the lines of [1] with minor modification, but at some point it will appear $|u|^{m-1}$ that one cannot control from above in a sublevel of the modulus of u, being $0 < m < 1$.

An important point of our strategy is to work with a different equation, apparently more complicated, but instead easier to handle, to which we can reduce, thanks to a change of variables introduced by Vespri in [9]. We will apply a technique due to DiBenedetto [3, 4] via an alternative argument; we will write energy estimates for super(sub)-solutions and logarithmic estimates. We notice that, due to the change of variables, our logarithmic function has to be different by the usual one (see for instance [4]). Then we will use the so-called reduction of oscillation procedure: the Hölder continuity of a solution u to the transformed equation (2.2) will be heuristically a consequence of the following fact: for every $(x_0, t_0) \in E_T$ there exists a family of nested and shrinking cylinders in which the essential oscillation of u goes to zero in a way that can be quantitatively determined in terms of the data. Since this result is well known for non-negative solutions (see [4, 5]), it will suffice to consider the case in which the infimum of our solution is negative and the supremum is positive.

2. Change of variables

To justify some of the following calculations, we assume u to be smooth. In no way this is a restrictive assumption: indeed the modulus of continuity of u will play no role in the forthcoming calculations.

Let us consider $n \in \mathbb{N}$ such that

$$n > \frac{1}{m}\,,$$

and define

$$|v|^{n-1}v = u,$$

which is equivalent to

$$v = |u|^{\frac{1}{n}-1}u.$$

Notice that

$$Du = n|v|^{n-1}Dv, \quad Dv = \frac{1}{n}|u|^{\frac{1}{n}-1}Du.$$

With this substitution equation (1.1) becomes

$$\left(|v|^{n-1}v\right)_t - \operatorname{div}\widetilde{\mathcal{A}}(x,t,v,Dv) = 0 \quad \text{weakly in } E_T,$$

where

$$\widetilde{\mathcal{A}}(x,t,v,Dv) = \mathcal{A}(x,t,u,Du)\big|_{u=|v|^{n-1}v}.$$

Now, let us see what the structure conditions become. We have

$$\begin{aligned}
\widetilde{\mathcal{A}}(x,t,v,Dv)\cdot Dv &= \frac{1}{n}|u|^{\frac{1}{n}-1}\mathcal{A}(x,t,u,Du)\cdot Du \\
&\geq \frac{m}{n}\,C_0|u|^{\frac{1}{n}+m-2}|Du|^2 \\
&= nmC_0|v|^{1+nm-2n}|v|^{2(n-1)}|Dv|^2 \\
&= nmC_0|v|^{nm-1}|Dv|^2;
\end{aligned}$$

since the exponent is $nm-1>0$, the equation is "degenerate".

In the same way

$$\begin{aligned}|\widetilde{\mathcal{A}}(x,t,v,Dv)| &= |\mathcal{A}(x,t,u,Du)| \le mC_1|u|^{m-1}|Du| \\ &= mC_1|v|^{n(m-1)}n|v|^{n-1}|Dv| = nmC_1|v|^{nm-1}|Dv|.\end{aligned}$$

If we denote our variable with u again, we are then led to consider equations of the type

$$(|u|^{n-1}u)_t - \operatorname{div}\widetilde{\mathcal{A}}(x,t,u,Du) = 0 \quad \text{weakly in } E_T,$$

with structure conditions

$$\begin{cases} \widetilde{\mathcal{A}}(x,t,z,\xi)\cdot\xi \ge nmC_0|z|^{nm-1}|\xi|^2 \\ |\widetilde{\mathcal{A}}(x,t,z,\xi)| \le nmC_1|z|^{nm-1}|\xi|, \end{cases} \tag{2.1}$$

for a.e. $(x,t)\in E_T$ and for every $z\in\mathbb{R}$, $\xi\in\mathbb{R}^N$.

Without loss of generality, we can assume n to be odd; in this case

$$|u|^{n-1}u = u^n,$$

and we can rewrite the equation as

$$(u^n)_t - \operatorname{div}\widetilde{\mathcal{A}}(x,t,u,Du) = 0 \quad \text{weakly in } E_T. \tag{2.2}$$

Hence we have reduced problem (1.1)-(1.2) to (2.2) with structure conditions (2.1).

Let us now see what the notion of weak solution becomes in this new setting. A function u such that $u^n\in C_{\rm loc}(0,T;L^2_{\rm loc}(E))$ with $|u|^{nm}\in L^2_{\rm loc}(0,T;H^1_{\rm loc}(E))$ is a local weak sub(super)-solution to (2.2) if for every compact set $\mathcal{K}\subset E$ and every subinterval $[t_1,t_2]\subset(0,T]$

$$\int_{\mathcal{K}} u^n\varphi\,dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}}[-u^n\varphi_t + \widetilde{\mathcal{A}}(x,t,u,Du)\cdot D\varphi]\,dx\,dt \le (\ge)\, 0,$$

for all non-negative test functions $\varphi\in H^1_{\rm loc}(0,T;L^2(\mathcal{K}))\cap L^2_{\rm loc}(0,T;H^1_0(\mathcal{K}))$.

3. Preliminaries

Let $r,s\ge1$ and let us consider the Banach spaces

$$\begin{aligned}V^{r,s}(E_T) &= L^\infty\big(0,T;L^r(E)\big)\cap L^s\big(0,T;W^{1,s}(E)\big),\\ V_0^{r,s}(E_T) &= L^\infty\big(0,T;L^r(E)\big)\cap L^s\big(0,T;W_0^{1,s}(E)\big),\end{aligned}$$

both equipped with the norm

$$\|v\|_{V^{r,s}(E_T)} = \operatorname*{ess\,sup}_{0<t<T}\|v(\cdot,t)\|_{r,E} + \|Dv\|_{s,E_T};$$

when $r=s$, let $V^{r,r}(E_T)=V^r(E_T)$ and $V_0^{r,r}(E_T)=V_0^r(E_T)$. Both spaces are embedded in $L^q(E_T)$, for some $q>s$ (for a proof one can see [4]).

Proposition 3.1. *If $v\in V_0^{r,s}(E_T)$, then there exists a positive constant γ, depending only upon N,r, and s, such that*

$$\iint_{E_T}|v|^q\,dx\,dt \le \gamma^q\Big(\iint_{E_T}|Dv|^s\,dx\,dt\Big)\Big(\operatorname*{ess\,sup}_{0<t<T}\int_E|v|^r dx\Big)^{s/N}$$

with $q=s\frac{N+r}{N}$. In particular

$$\|v\|_{q,E_T}\le\gamma\|v\|_{V^{r,s}(E_T)}.$$

Note that, taking $r=s$ in the previous proposition, and applying Hölder inequality, one obtains the following result.

Proposition 3.2. *If $v \in V_0^r(E_T)$, then there exists a positive constant γ depending only upon N and r, such that*

$$\|v\|_{r,E_T}^r \le \gamma \left|\{|v|>0\}\right|^{\frac{r}{N+r}} \|v\|_{V^r(E_T)}^r.$$

Given $(y,s) \in E_T$, and $\lambda, R > 0$, we will denote by $K_R(y)$ the cube centered at y with edge $2R$; i.e.,

$$K_R(y) = \Big\{x \in \mathbb{R}^N : \max_{1\le i\le N} |x_i - y_i| < R\Big\},$$

and let $\partial K_R(y)$ be its boundary. Let $(y,s) + Q_R(\lambda)$ be the generic cylinder

$$(y,s) + Q_\rho(\lambda) = K_\rho(y) \times [s-\lambda, s].$$

If $k \in \mathbb{R}$, introduce the truncated functions

$$(u-k)_\pm = \max\{\pm(u-k), 0\}.$$

The following lemma, proved in [2], will be very useful in the sequel.

Lemma 3.3. *Let $v \in W^{1,1}(K_\rho(y))$ and let $k, l \in \mathbb{R}$, with $k < l$. There exists a constant $\gamma = \gamma(N,p)$ independent of k, l, v, y, ρ such that*

$$(l-k)|\{v > l\}| \le \gamma \frac{\rho^{N+1}}{|\{v < k\}|} \int_{\{k<v<l\}} |Dv|\, dx. \tag{3.1}$$

Let us state now a lemma on fast geometric convergence one can find in [2]; for a simple proof see again [4] and [6].

Lemma 3.4. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying*

$$Y_{n+1} \le C b^n Y_n^{1+\alpha},$$

being $C, b > 1$ and $\alpha > 0$. If

$$Y_0 \le C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then Y_n converges to 0, as n tends to $+\infty$.

Let us prove energy estimates we will need later. We start with estimates for super-solutions, then we will state the analogous ones for sub-solutions.

Proposition 3.5 (Energy estimates for super-solutions)**.** *Let u be a local, weak super-solution to* (2.1)-(2.2) *in E_T. There exists a positive constant γ, depending only upon the data, such that for every cylinder $(y,s) + Q_R(\lambda) \subset E_T$, every level $k \in \mathbb{R}$ and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(y)$,*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\lambda<t\le s} \int_{K_R(y)} \Big(\int_u^k (k-s)_+ s^{n-1} ds\Big) \zeta^2(x,t)\, dx \\
&+ \iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1} \big|D[(u-k)_-\zeta]\big|^2\, dx\, d\tau \\
&\le \gamma \Big\{ \int_{K_R(y)} \Big(\int_u^k (k-s)_+ s^{n-1} ds\Big) \zeta^2(x, s-\lambda)\, dx \\
&\quad + \iint_{(y,s)+Q_R(\lambda)} \Big(\int_u^k (k-s)_+ s^{n-1} ds\Big) \zeta |\zeta_\tau|\, dx\, d\tau \\
&\quad + \iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1} (u-k)_-^2 |D\zeta|^2\, dx\, d\tau \Big\}.
\end{aligned} \tag{3.2}$$

Proof. After a translation we may assume that (y,s) coincides with the origin and it suffices to prove (3.2) for the cylinder $Q_R(\lambda)$. In the weak formulation of (2.2), take the test function

$$\varphi = -(u-k)_-\zeta^2$$

over $Q_t = K_R \times (-\lambda\,,t]$, where $-\lambda < t \le 0$.

Taking into account that

$$\frac{\partial}{\partial\tau}\Big(\int_u^k (k-s)_+ s^{n-1} ds\Big) = -u^{n-1}(u-k)_- u_\tau,$$

and estimating the various terms separately, we have first

$$\begin{aligned}
-\iint_{Q_t} (u^n)_\tau (u-k)_-\zeta^2\,dx\,d\tau &= n\iint_{Q_t} \frac{\partial}{\partial\tau}\Big(\int_u^k (k-s)_+ s^{n-1} ds\Big)\zeta^2\,dx\,d\tau \\
&\ge n\int_{K_R}\Big(\int_u^k (k-s)_+ s^{n-1} ds\Big)\zeta^2(x,t)\,dx \\
&\quad - n\int_{K_R}\Big(\int_u^k (k-s)_+ s^{n-1} ds\Big)\zeta^2(x,-\lambda)\,dx \\
&\quad - 2n\iint_{Q_t}\Big(\int_u^k (k-s)_+ s^{n-1} ds\Big)\zeta|\zeta_\tau|\,dx\,d\tau.
\end{aligned}$$

From the structure conditions (2.1) and Young's inequality it follows that

$$\begin{aligned}
&-\iint_{Q_t} \widetilde{\mathcal{A}}(x,\tau,u,Du) D\big[(u-k)_-\zeta^2\big]\,dx\,d\tau \\
&= -\iint_{Q_t} \widetilde{\mathcal{A}}(x,\tau,u,Du) D(u-k)_-\zeta^2\,dx\,d\tau \\
&\quad - 2\iint_{Q_t} \widetilde{\mathcal{A}}(x,\tau,u,Du)(u-k)_-\zeta D\zeta\,dx\,d\tau \\
&\ge nmC_0\iint_{Q_t} |u|^{nm-1}|D(u-k)_-|^2\zeta^2\,dx\,d\tau \\
&\quad - 2nmC_1\iint_{Q_t} |u|^{nm-1}|D(u-k)_-|(u-k)_-\zeta|D\zeta|\,dx\,d\tau \\
&\ge nm\frac{C_0}{2}\iint_{Q_t} |u|^{nm-1}\big|D[(u-k)_-\zeta]\big|^2\,dx\,d\tau \\
&\quad - 2nm\frac{C_1^2}{C_0}\iint_{Q_t} |u|^{nm-1}(u-k)_-^2|D\zeta|^2\,dx\,d\tau.
\end{aligned}$$

Combining these estimates and taking the supremum over $t \in (-\lambda, 0]$, completes the proof. $\square$

Proposition 3.6 (Energy estimates for sub-solutions)**.** *Let u be a local, weak sub-solution to* (2.1)-(2.2) *in E_T. There exists a positive constant γ, depending only upon the data, such that for every cylinder $(y,s) + Q_R(\lambda) \subset E_T$, every level $k \in \mathbb{R}$*

and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(y)$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\lambda<t\le s}\int_{K_R(y)}\Big(\int_k^u (s-k)_+ s^{n-1}ds\Big)\zeta^2(x,t)\,dx \\
&+\iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1}\big|D[(u-k)_+\zeta]\big|^2\,dx\,d\tau \\
&\le \gamma\Big\{\int_{K_R(y)}\Big(\int_k^u (s-k)_+ s^{n-1}ds\Big)\zeta^2(x,s-\lambda)\,dx \\
&\quad+\iint_{(y,s)+Q_R(\lambda)}\Big(\int_k^u (s-k)_+ s^{n-1}ds\Big)|\zeta_\tau|\,dx\,d\tau \\
&\quad+\iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1}(u-k)_+^2|D\zeta|^2\,dx\,d\tau\Big\}.
\end{aligned}\tag{3.3}$$

Proof. The proof is analogous to the previous one; we just need to take the test function $\varphi=(u-k)_+\zeta^2$ and observe that

$$\frac{\partial}{\partial\tau}\Big(\int_k^u (s-k)_+ s^{n-1}ds\Big)=u^{n-1}(u-k)_+u_\tau. \qquad \square$$

Let us introduce the logarithmic function

$$\psi(H^n,(u^n-k^n)_+,\nu^n)=\log^+\Big(\frac{H^n}{H^n-(u^n-k^n)_+ +\nu^n}\Big),$$

where

$$H^n=\operatorname*{ess\,sup}_{(y,s)+Q_R(\lambda)}(u^n-k^n)_+,\quad 0<\nu^n<\min\{1,H^n\},$$

and for $s>0$

$$\log^+ s=\max\{\log s,0\}.$$

Proposition 3.7 (Logarithmic estimates)**.** *Let u be a local, weak solution to* (2.1)-(2.2) *in E_T. There exists a positive constant γ, depending only upon the data, such that for every cylinder $(y,s)+Q_R(\lambda)\subset E_T$, every level $k\in\mathbb{R}$ and every non-negative, piecewise smooth cutoff function $\zeta=\zeta(x)$*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\lambda<t\le s}\int_{K_R(y)}\psi^2\big(H^n,(u^n-k^n)_+,\nu^n\big)(x,t)\,\zeta^2(x)\,dx \\
&\le \int_{K_R(y)}\psi^2\left(H^n,(u^n-k^n)_+,\nu^n\right)(x,s-\lambda)\zeta^2(x)\,dx \\
&\quad+\gamma\iint_{(y,s)+Q_R(\lambda)}|u|^{n(m-1)}\psi\left(H^n,(u^n-k^n)_+,\nu^n\right)|D\zeta|^2\,dx\,d\tau.
\end{aligned}\tag{3.4}$$

Proof. Again we assume that (y,s) coincides with the origin. Put $v=u^n$ and, in the weak formulation of (2.2), take the test function

$$\varphi=\frac{\partial\psi^2}{\partial v}\zeta^2=2\psi\psi'\zeta^2,$$

over $Q_t=K_R\times(-\lambda,t]$, where $-\lambda<t\le 0$.
By direct calculation

$$\big(\psi^2\big)''=2(1+\psi)(\psi')^2\in L^\infty_{\rm loc}(E_T),\tag{3.5}$$

which implies that such a φ is an admissible testing function. Estimating the various terms separately, we have

$$\begin{aligned}\iint_{Q_t} v_\tau \frac{\partial \psi^2}{\partial v}\zeta^2\,dx\,d\tau &= \iint_{Q_t} \frac{\partial}{\partial \tau}\psi^2\zeta^2\,dx\,d\tau \\ &= \int_{K_R}\psi^2(x,t)\zeta^2(x)\,dx - \int_{K_R}\psi^2(x,-\lambda)\zeta^2(x)\,dx;\end{aligned}$$

using (3.5) and the structure conditions (2.1)

$$\begin{aligned}&\iint_{Q_t}\widetilde{\mathcal{A}}(x,\tau,u,Du)D\Big(\frac{\partial\psi^2}{\partial v}\zeta^2\Big)\,dx\,d\tau\\ &= \iint_{Q_t}\widetilde{\mathcal{A}}(x,\tau,u,Du)Dv(\psi^2)''\zeta^2\,dx\,d\tau + 2\iint_{Q_t}(\psi^2)'\zeta\widetilde{\mathcal{A}}(x,\tau,u,Du)D\zeta\,dx\,d\tau\\ &= 2n\iint_{Q_t}u^{n-1}\widetilde{\mathcal{A}}(x,\tau,u,Du)Du(1+\psi)(\psi')^2\zeta^2\,dx\,d\tau\\ &\quad + 4\iint_{Q_t}\psi\psi'\zeta\widetilde{\mathcal{A}}(x,\tau,u,Du)D\zeta\,dx\,d\tau\\ &\geq 2n^2mC_0\iint_{Q_t}u^{n-1}|u|^{nm-1}|D(u-k)_+|^2(1+\psi)(\psi')^2\zeta^2\,dx\,d\tau\\ &\quad - 4nmC_1\iint_{Q_t}|u|^{nm-1}|D(u-k)_+|\zeta|D\zeta|\psi\psi'\,dx\,d\tau.\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}&\iint_{Q_t}\widetilde{\mathcal{A}}(x,\tau,u,Du)D\Big(\frac{\partial\psi^2}{\partial v}\zeta^2\Big)\,dx\,d\tau\\ &\geq 2nm(nC_0 - C_1\varepsilon^2)\iint_{Q_t}|u|^{nm-1}|u|^{n-1}|D(u-k)_+|^2\psi(\psi')^2\zeta^2\,dx\,d\tau\\ &\quad - 2nm\frac{C_1}{\varepsilon^2}\iint_{Q_t}|u|^{n(m-1)}|D\zeta|^2\psi\,dx\,d\tau.\end{aligned}$$

Combining these estimates, discarding the term with the gradient on the left-hand side, and taking the supremum over $t \in (-\lambda\,,0]$, proves the proposition. □

4. Reduction of the oscillation

To obtain the Hölder regularity, we argue as usual with this kind of estimate by a reduction-of-oscillation procedure. Let us state the basic result.

Theorem 4.1. *Let $(y,s)\in E_T$, and ρ, $\omega>0$ such that*

$$(y,s)+Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\subset E_T\,,\qquad \operatorname*{ess\,osc}_{(y,s)+Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)} u\leq\omega\,,$$

where

$$\theta=\omega^{\frac{1-n}{2}}.$$

Then, there exist η_*, $c_0\in(0,1)$, depending only upon data, such that

$$\operatorname*{ess\,osc}_{\mathcal{Q}^*} u\leq\eta_*\omega\,,$$

where

$$\mathcal{Q}^* = (y,s) + Q_{\theta\rho}\left(\theta_*\rho^2\right), \quad \theta_* = \frac{c_0}{2}\,\omega^{1-nm}.$$

As we show at the end, the local Hölder continuity of locally bounded solutions is a straightforward consequence of Theorem 4.1. The proof of this theorem splits into two alternatives.

Let $\epsilon \in (0,1), R > 0$, and $(y,s) \in E_T$. Consider the cylinder

$$Q_\epsilon := K_{R^{1-\epsilon\frac{n-1}{2}}}(y) \times (s - R^{2-\epsilon(nm-1)}, s] \subset E_T,$$

and set

$$\mu_+ \geq \operatorname*{ess\,sup}_{(y,s)+Q_\epsilon} u\,, \quad \mu_- \leq \operatorname*{ess\,inf}_{(y,s)+Q_\epsilon} u\,, \quad \omega = \mu_+ - \mu_-\,.$$

Let us recall that, without loss of generality, we can assume $\mu_+ > 0$, $\mu_- < 0$ and

$$\mu_+ \geq |\mu_-|.$$

Indeed, otherwise just change the sign of u and work with the new function.

If we take $2\rho < R$, and assume without loss of generality

$$\omega > R^\epsilon, \tag{4.1}$$

then we guarantee that

$$(y,s) + Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big) \subset Q_\epsilon.$$

5. The first alternative

We distinguish two alternatives; the first of them consists in assuming

$$\Big|\Big\{u < \mu_- + \frac{\omega}{2}\Big\} \cap \Big\{(y,s) + Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\Big\}\Big| \leq c_0\Big|Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\Big|, \tag{5.1}$$

being $c_0 \in (0,1)$ a constant to be determined later.

Let us prove now the following De Giorgi type lemma.

Lemma 5.1. *There exists a number $c_0 \in (0,1)$, depending only upon data, such that if* (5.1) *holds, then*

$$u \geq \mu_- + \frac{\omega}{4} \quad \text{a.e. in } (y,s) + Q_{\theta\rho}\Big(\frac{\rho^2}{\omega^{nm-1}}\Big). \tag{5.2}$$

Proof. Without loss of generality we may assume $(y,s) = (0,0)$ and for $k = 0,1,\dots$, set

$$\rho_k = \rho + \frac{\rho}{2^k}, \quad \widetilde{K}_k = K_{\theta\rho_k}, \quad \widetilde{Q}_k = \widetilde{K}_k \times \big(-\frac{\rho_k^2}{\omega^{nm-1}}, 0\big].$$

Let ζ_k be a piecewise smooth cutoff function in $\widetilde{Q}_k$ vanishing on the parabolic boundary of $\widetilde{Q}_k$ such that $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ in $\widetilde{Q}_{k+1}$ and

$$|D\zeta_k| \leq \frac{2^{k+2}}{\rho}\omega^{\frac{n-1}{2}}, \quad 0 \leq \zeta_{k,t} \leq \frac{2^k}{\rho^2}\,\omega^{nm-1}.$$

We consider the following levels

$$\begin{aligned} h_k &= \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{k+2}} \quad \text{if } \mu_- \geq -\frac{\omega}{8}, \\ h_k &= \mu_- + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} \quad \text{if } \mu_- < -\frac{\omega}{8}. \end{aligned} \tag{5.3}$$

We first treat the least favorable case in which u might be close to zero; i.e., we assume first that

$$\mu_- \geq -\frac{\omega}{8}. \tag{5.4}$$

Write down the energy estimates (3.2) for $(u-h_k)_-$ over the cylinder $\widetilde{Q}_k$, to obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} \int_{\widetilde{K}_k} \Big(\int_u^{h_k} (h_k-s)_+ s^{n-1} ds\Big)\zeta_k^2(x,t)\,dx \\
&+ \iint_{\widetilde{Q}_k} |u|^{nm-1}\big|D[(u-h_k)_-\zeta_k]\big|^2\,dx\,d\tau \\
&\leq \gamma\Big\{ \iint_{\widetilde{Q}_k} \Big(\int_u^{h_k} (h_k-s)_+ s^{n-1} ds\Big)|\zeta_{k,\tau}|\,dx\,d\tau \\
&\quad + \iint_{\widetilde{Q}_k} |u|^{nm-1}(u-h_k)_-^2|D\zeta_k|^2\,dx\,d\tau\Big\}.
\end{aligned}$$

Let us introduce the truncation

$$v = \max\Big(u, \frac{\omega}{2^4}\Big),$$

in order to estimate the terms with the integral over $[u, h_k]$; we have

$$\begin{aligned}
\int_u^{h_k} (h_k-s)_+ s^{n-1} ds &\geq \int_v^{h_k} (h_k-s)_+ s^{n-1} ds \\
&\geq v^{n-1}\frac{(v-h_k)_-^2}{2} \geq \Big(\frac{\omega}{2^4}\Big)^{n-1}\frac{(v-h_k)_-^2}{2}\,.
\end{aligned} \tag{5.5}$$

On the other hand, as $(u-h_k)_- \leq \omega$ and $-\frac{\omega}{8} \leq \mu_- < 0$, we have

$$\int_u^{h_k} (h_k-s)_+ s^{n-1} ds \leq h_k^{n-1}\,\frac{(u-h_k)_-^2}{2} \leq \frac{\omega^{n+1}}{2}\,. \tag{5.6}$$

By the definition of v, we obtain

$$\begin{aligned}
&\iint_{\widetilde{Q}_k} v^{nm-1}\big|D[(v-h_k)_-\zeta_k]\big|^2\,dx\,d\tau \\
&= \iint_{\widetilde{Q}_k\cap\{u>\frac{\omega}{2^4}\}} |u|^{nm-1}\big|D[(u-h_k)_-\zeta_k]\big|^2\,dx\,d\tau \\
&\quad + \iint_{\widetilde{Q}_k\cap\{u\leq\frac{\omega}{2^4}\}} \Big(\frac{\omega}{2^4}\Big)^{nm-1}\Big(\frac{\omega}{2^4}-h_k\Big)_-^2|D\zeta_k|^2\,dx\,d\tau \\
&\leq \iint_{\widetilde{Q}_k} |u|^{nm-1}\big|D[(u-h_k)_-\zeta_k]\big|^2\,dx\,d\tau + \frac{2^{2(k+1)}}{\rho^2}\,\omega^{n(m+1)}|A_k|\,,
\end{aligned} \tag{5.7}$$

where

$$A_k = \{u < h_k\} \cap \widetilde{Q}_k.$$

Let us observe that

$$A_k = \widetilde{A}_k := \{v < h_k\} \cap \widetilde{Q}_k. \tag{5.8}$$

Indeed, the inclusion $A_k \supseteq \widetilde{A}_k$ follows by the definition of v; let us now prove the other one: if $v = u$ there is nothing to prove; if $v = \frac{\omega}{2^4}$, by (5.4) we have

$$h_k = \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{k+2}} \geq \frac{\omega}{8} + \frac{\omega}{2^{k+2}} \geq \frac{\omega}{2^4}.$$

Taking into account that $|u| \le \omega$, (5.5)-(5.8) yield

$$\Big(\frac{\omega}{2^4}\Big)^{n-1} \operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0} \int_{\widetilde{K}_k} (v-h_k)_-^2 \zeta_k^2(x,t)\,dx + \iint_{\widetilde{Q}_k} v^{nm-1}\big|D[(v-h_k)_-\zeta_k]\big|^2\,dx\,d\tau$$
$$\le \gamma \frac{2^{2k}}{\rho^2}\,\omega^{n(m+1)}|\widetilde{A}_k|,$$

and again, thanks to the definition of v, it follows that

$$\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0} \int_{\widetilde{K}_k} (v-h_k)_-^2 \zeta_k^2(x,t)\,dx + \Big(\frac{\omega}{2^4}\Big)^{n(m-1)} \iint_{\widetilde{Q}_k} \big|D[(v-h_k)_-\zeta_k]\big|^2\,dx\,d\tau$$
$$\le \gamma \frac{2^{2k}}{\rho^2}\,\omega^{nm+1}|\widetilde{A}_k|. \tag{5.9}$$

The change of variables

$$\bar{x} = x\,\theta^{-1}, \quad \bar{t} = \omega^{nm-1}\tau$$

maps the cube $\widetilde{K}_k$ into K_{ρ_k}, and the cylinder $\widetilde{Q}_k$ into $Q_k = K_{\rho_k}\times(-\rho_k^2,0]$. With $(\bar{x},\bar{t}) \to u(\bar{x},\bar{t})$ denoting again the transformed function, the assumption (5.1) of the lemma implies

$$\Big|\Big\{v<\mu_- + \frac{\omega}{2}\Big\}\cap Q_0\Big| \le c_0|Q_0|. \tag{5.10}$$

Performing such a change of variables in (5.9), we have

$$\operatorname*{ess\,sup}_{-\rho_k^2<t\le 0} \int_{K_{\rho_k}} (v-h_k)_-^2 \zeta_k^2(\bar{x},t)\,d\bar{x} + \iint_{Q_k} \big|D[(v-h_k)_-\zeta_k]\big|^2 d\bar{x}d\bar{t}$$
$$\le \gamma \frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|,$$

where

$$\bar{A}_k = \{v<h_k\}\cap Q_k.$$

This implies

$$\big\|(v-h_k)_-\zeta_k\big\|^2_{V^2(Q_k)} \le \gamma \frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|. \tag{5.11}$$

Then from Proposition 3.2 with $r=2$ and (5.11), one obtains

$$\begin{aligned}
\iint_{Q_{k+1}} (v-h_k)_-^2\,d\bar{x}d\bar{t} &\le \iint_{Q_k} (v-h_k)_-^2 \zeta_k^2\,d\bar{x}d\bar{t}\\
&\le \gamma|\{v<h_k\}\cap Q_k|^{\frac{2}{N+2}}\,\big\|(v-h_k)_-\zeta_k\big\|^2_{V^2(Q_k)}\\
&\le \gamma\frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|^{1+\frac{2}{N+2}};
\end{aligned}$$

the left-hand side is estimated by

$$\begin{aligned}
\iint_{Q_{k+1}} (v-h_k)_-^2\,d\bar{x}d\bar{t} &= \iint_{Q_{k+1}\cap\{v<h_k\}} (h_k-v)^2 d\bar{x}d\bar{t}\\
&\ge \iint_{Q_{k+1}\cap\{v<h_{k+1}\}} (h_k-v)^2 d\bar{x}d\bar{t}\\
&\ge (h_k-h_{k+1})^2|\bar{A}_{k+1}|
\end{aligned}$$

$$= \Big(\frac{\omega}{2^{k+3}}\Big)^2 |\bar{A}_{k+1}|.$$

Combining the previous estimates yields

$$|\bar{A}_{k+1}| \le \gamma \frac{2^{4k}}{\rho^2} |\bar{A}_k|^{1+\frac{2}{N+2}},$$

and setting

$$Y_k = \frac{|\bar{A}_k|}{|Q_k|},$$

it follows that

$$Y_{k+1} \le \gamma\, 2^{4k}\, Y_k^{1+\frac{2}{N+2}}.$$

Thanks to Lemma 3.4, we deduce that Y_k tends to zero as $k \to \infty$, provided

$$Y_0 = \frac{|\{v < h_0\} \cap Q_0|}{|Q_0|} = \frac{|\{v < \mu_- + \frac{\omega}{2}\} \cap Q_0|}{|Q_0|} \le \gamma^{-\frac{N+2}{2}}\, 2^{-(N+2)^2},$$

that is (5.10), with $c_0 := \gamma^{-\frac{N+2}{2}}\, 2^{-(N+2)^2}$.

Therefore,

$$v \ge \mu_- + \frac{\omega}{4} \quad \text{a.e. in } K_\rho \times (-\rho^2, 0].$$

Returning to the variables x, t, we have

$$v \ge \mu_- + \frac{\omega}{4} \quad \text{a.e. in } Q_{\theta\rho}\Big(\frac{\rho^2}{\omega^{nm-1}}\Big); \tag{5.12}$$

this implies that $u = v$ in $Q_{\theta\rho}\big(\frac{\rho^2}{\omega^{nm-1}}\big)$ and, consequently, (5.2). In fact, by contradiction, if there were a point $(x,t) \in Q_{\theta\rho}\big(\frac{\rho^2}{\omega^{nm-1}}\big)$ such that $v(x,t) = \frac{\omega}{2^4}$, by (5.12) and (5.4), we would obtain

$$\frac{\omega}{2^4} \ge \mu_- + \frac{\omega}{4} \ge \frac{\omega}{8}.$$

Assume now that (5.4) is violated; that is, $\mu_- < -\frac{\omega}{8}$. Choosing the levels h_k according to (5.3), we have

$$h_k = \mu_- + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} < -\frac{\omega}{8} + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} \le -\frac{\omega}{2^5}.$$

Thus on the set $\{u \le h_k\}$, one has

$$|u|^{nm-1} \ge \Big(\frac{\omega}{2^5}\Big)^{nm-1}.$$

It follows that $|u|^{nm-1}$ can be estimate above and below by ω^{nm-1} up to a constant, depending only upon the data; the proof can be repeated as before, but in this case there is no need to introduce the truncated function v. □

Therefore under assumption (5.1),

$$-\operatorname*{ess\,inf}_{Q_{\theta\rho}(\frac{\rho^2}{\omega^{nm-1}})} u \le -\mu_- - \frac{\omega}{4};$$

adding $\operatorname*{ess\,sup}_{Q_{\theta\rho}(\frac{\rho^2}{\omega^{nm-1}})} u$, gives

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}(\frac{\rho^2}{\omega^{nm-1}})} u \le \operatorname*{ess\,sup}_{Q_{\theta\rho}(\frac{\rho^2}{\omega^{nm-1}})} u - \mu_- - \frac{\omega}{4} \le \frac{3}{4}\,\omega.$$

6. The second alternative

Let us recall the two fundamental hypotheses we assume, namely

$$\mu_+ > 0\,, \quad \mu_- < 0\,, \quad \mu_+ \geq |\mu_-|.$$

Throughout this new section, let us assume that (5.1) does not hold; i.e.,

$$\Big|\Big\{u \geq \mu_- + \frac{\omega}{2}\Big\} \cap \Big\{(y,s) + Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\Big\}\Big| < (1-c_0)\Big|Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\Big|.$$

For simplicity in the following we assume $(y,s) = (0,0)$.

Lemma 6.1. *There exists a time level t^* in the interval $\big(-\frac{(2\rho)^2}{\omega^{nm-1}}, -\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}\big)$ such that*

$$\Big|\Big\{u(\cdot,t^*) < \mu_- + \frac{\omega}{2}\Big\} \cap K_{2\theta\rho}\Big| > \frac{c_0}{2}|K_{2\theta\rho}|. \tag{6.1}$$

This in turn implies

$$\Big|\Big\{u(\cdot,t^*) \geq \mu_+ - \frac{\omega}{4}\Big\} \cap K_{2\theta\rho}\Big| \leq \Big(1 - \frac{c_0}{2}\Big)|K_{2\theta\rho}|. \tag{6.2}$$

Proof. By contradiction, suppose that (6.1) does not hold for any t^* in the indicated range; then

$$\begin{aligned}
\Big|\Big\{u < \mu_- + \frac{\omega}{2}\Big\} \cap Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big)\Big| &= \int_{-\frac{(2\rho)^2}{\omega^{nm-1}}}^{-\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}} \Big|\Big\{u(\cdot,t^*) < \mu_- + \frac{\omega}{2}\Big\} \cap K_{2\theta\rho}\Big| dt^* \\
&\quad + \int_{-\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}}^{0} \Big|\Big\{u(\cdot,t^*) < \mu_- + \frac{\omega}{2}\Big\} \cap K_{2\theta\rho}\Big| dt^* \\
&\leq \frac{c_0}{2}|K_{2\theta\rho}|\Big(1 - \frac{c_0}{2}\Big)\frac{(2\rho)^2}{\omega^{nm-1}} + |K_{2\theta\rho}|\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}} \\
&< c_0\Big|Q_{2\theta\rho}\big(\frac{(2\rho)^2}{\omega^{nm-1}}\big)\Big|.
\end{aligned}$$

This proves (6.1); (6.2) follows by the fact that (6.1) is equivalent to

$$\Big|\Big\{u(\cdot,t^*) \geq \mu_- + \frac{\omega}{2}\Big\} \cap K_{2\theta\rho}\Big| < \Big(1 - \frac{c_0}{2}\Big)|K_{2\theta\rho}|,$$

and $\mu_- + \frac{\omega}{2} \leq \mu_+ - \frac{\omega}{4}$. □

The next lemma asserts that a property similar to (6.2) continues to hold for all time levels from t^* up to zero.

Lemma 6.2. *There exists a positive integer j^*, depending upon the data and c_0, such that*

$$\Big|\Big\{u(\cdot,t) > \mu_+ - \frac{\omega}{2^{j^*}}\Big\} \cap K_{2\theta\rho}\Big| < \big(1 - \frac{c_0^2}{4}\big)|K_{2\theta\rho}|,$$

for all times $t^* < t < 0$.

Proof. Consider the logarithmic estimates (3.4) written over the cylinder $K_{2\theta\rho} \times (t^*, 0)$ for the function $(u^n - k^n)_+$ and for the level $k = \big(\mu_+^n - (\frac{\omega}{4})^n\big)^{1/n}$. Notice that, thanks to our assumptions, $\mu_+ > \frac{\omega}{4}$, so $k > 0$. The number ν in the definition of the logarithmic function is taken as $\nu = \frac{\omega}{2^{j+2}}$, where j is a positive number to be chosen. Thus we have

$$\psi\,(H^n, (u^n - k^n)_+, \nu^n) = \log^+\Big(\frac{H^n}{H^n - (u^n - k^n)_+ + \frac{\omega^n}{2^{(j+2)n}}}\Big),$$

where

$$H^n = \operatorname*{ess\,sup}_{K_{2\theta\rho}\times(t^*,0)} \Big[u^n - \big(\mu_+^n - (\frac{\omega}{4})^n\big)\Big]_+ .$$

The cutoff function $x \to \zeta(x)$ is taken such that

$$\zeta = 1 \quad \text{on } K_{(1-\sigma)2\theta\rho} \text{ for } \sigma \in (0,1), \quad |D\zeta| \le \frac{1}{\sigma\theta\rho}\,.$$

With these choices, inequality (3.4) yields

$$\begin{aligned} &\int_{K_{(1-\sigma)2\theta\rho}} \psi^2(x,t)\,dx \\ &\le \int_{K_{2\theta\rho}} \psi^2(x,t^*)\,dx + \gamma \int_{t^*}^0 \int_{K_{2\theta\rho}} |u|^{n(m-1)}\psi|D\zeta|^2\,dx\,d\tau, \end{aligned} \tag{6.3}$$

for all $t^* \le t \le 0$. Let us observe that

$$\psi \le \log\Big(\frac{\frac{\omega^n}{2^{2n}}}{\frac{\omega^n}{2^{(j+2)n}}}\Big) = jn\log 2.$$

To estimate the first integral on the right-hand side of (6.3), observe that ψ vanishes on the set $\{u^n < k^n\}$ and that $\mu_+^n - \big(\frac{\omega}{4}\big)^n \ge \big(\mu_+ - \frac{\omega}{4}\big)^n$; therefore by (6.2)

$$\int_{K_{2\theta\rho}} \psi^2(x,t^*)\,dx \le j^2n^2\log^2 2\Big(1 - \frac{c_0}{2}\Big)|K_{2\theta\rho}|.$$

The remaining integral is estimated as follows

$$\begin{aligned} &\gamma \int_{t^*}^0 \int_{K_{2\theta\rho}} |u|^{n(m-1)}\psi|D\zeta|^2\,dx\,d\tau \\ &\le \frac{\gamma}{(\sigma\theta\rho)^2}\,jn\log 2\,\frac{(2\rho)^2}{\omega^{nm-1}}\,\omega^{n(m-1)}|K_{2\theta\rho}| = \frac{\gamma}{\sigma^2}\,jn|K_{2\theta\rho}|. \end{aligned}$$

Combining the previous estimates,

$$\int_{K_{(1-\sigma)2\theta\rho}} \psi^2(x,t)\,dx \le \Big\{j^2n^2\log^2 2\Big(1 - \frac{c_0}{2}\Big) + \frac{\gamma}{\sigma^2}\,jn\Big\}\,|K_{2\theta\rho}| \tag{6.4}$$

for all $t^* \le t \le 0$. The left-hand side of (6.4) is estimated below by integrating over the smaller set

$$\Big\{u^n > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\};$$

on such a set, since ψ is a decreasing function of H^n, we have

$$\psi^2 \ge \log^2\Big(\frac{\frac{\omega^n}{2^{2n}}}{\frac{\omega^n}{2^{(j+1)n}}}\Big) = (j-1)^2n^2\log^2 2;$$

hence, for all $t^* \le t \le 0$, we obtain

$$\Big|\Big\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\} \cap K_{(1-\sigma)2\theta\rho}\Big| \le \Big\{\big(\frac{j}{j-1}\big)^2\Big(1 - \frac{c_0}{2}\Big) + \frac{\gamma}{\sigma^2 j}\Big\}|K_{2\theta\rho}|.$$

On the other hand,

$$\begin{aligned} &\Big|\Big\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\} \cap K_{2\theta\rho}\Big| \\ &\le \Big|\Big\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\} \cap K_{(1-\sigma)2\theta\rho}\Big| + |K_{2\theta\rho} \setminus K_{(1-\sigma)2\theta\rho}| \end{aligned}$$

$$\leq \Big|\Big\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\} \cap K_{(1-\sigma)2\theta\rho}\Big| + N\sigma|K_{2\theta\rho}|.$$

Then

$$\Big|\Big\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big\} \cap K_{2\theta\rho}\Big| \leq \Big\{\Big(\frac{j}{j-1}\Big)^2\Big(1-\frac{c_0}{2}\Big) + \frac{\gamma}{\sigma^2 j} + N\sigma\Big\}|K_{2\theta\rho}|.$$

for all $t^* \leq t \leq 0$. Now choose σ so small and then j so large as to obtain

$$\Big|\{u(\cdot,t) > \Big(\mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big)^{1/n}\} \cap K_{2\theta\rho}\Big| \leq \big(1-\frac{c_0^2}{4}\big)|K_{2\theta\rho}| \qquad \forall t^* \leq t \leq 0.$$

Notice that our hypotheses imply $\mu_+ \geq \frac{\omega}{2}$, $\mu_+ < \omega$; therefore,

$$\begin{aligned}\Big(\mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\Big)^{1/n} &< \Big(\mu_+^n - \frac{\mu_+^n}{2^{(j+2)n}}\Big)^{1/n} = \mu_+\Big(1-\frac{1}{2^{(j+2)n}}\Big)^{1/n}\\ &\leq \mu_+\Big(1-\frac{1}{2^{(j+2)n}\,n}\Big) \leq \mu_+ - \frac{\omega}{2^{(j+2)n+1}\,n}.\end{aligned}$$

The proof is completed once we choose j^* as the smallest integer such that

$$\mu_+ - \frac{\omega}{2^{(j+2)n+1}\,n} \leq \mu_+ - \frac{\omega}{2^{j^*}}. \qquad \square$$

Corollary 6.3. *For all $j \geq j^*$ and for all times $-\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}} < t < 0$,*

$$\Big|\Big\{u(\cdot,t) > \mu_+ - \frac{\omega}{2^j}\Big\} \cap K_{2\theta\rho}\Big| < \Big(1-\frac{c_0^2}{4}\Big)|K_{2\theta\rho}|. \tag{6.5}$$

Motivated by Corollary 6.3, introduce the cylinder

$$Q_* = K_{2\theta\rho} \times \big(-\theta_*(2\rho)^2, 0\big], \quad \text{with } \theta_* = \frac{c_0}{2}\omega^{1-nm}.$$

Lemma 6.4. *For every $\nu_* \in (0,1)$, there exists a positive integer $q_* = q_*(data, \nu_*)$ such that*

$$\Big|\Big\{u \geq \mu_+ - \frac{\omega}{2^{j_*+q_*}}\Big\} \cap Q_*\Big| \leq \nu_*|Q_*|.$$

Proof. Write down the energy estimates (3.3) for the truncated functions $(u-k_j)_+$, with $k_j = \mu_+ - \frac{\omega}{2^j}$, for $j = j_*, \dots, j_* + q_*$ over the cylinder

$$\widetilde{Q} = K_{4\theta\rho} \times \Big(-c_0\frac{(2\rho)^2}{\omega^{nm-1}}, 0\Big] \supset Q_*;$$

the cutoff function ζ is taken to be one on Q_*, vanishing on the parabolic boundary of $\widetilde{Q}$ and such that

$$|D\zeta| \leq \frac{1}{\theta\rho}, \quad 0 \leq \zeta_t \leq \frac{\omega^{nm-1}}{c_0\rho^2}.$$

Thanks to these choices, the energy estimates (3.3) take the form

$$\begin{aligned}&\iint_{\tilde{Q}} |u|^{nm-1}|D(u-k_j)_+|^2\zeta^2\,dx\,d\tau\\ &\leq \gamma\Big\{\frac{\omega^{nm-1}}{c_0\rho^2}\iint_{\tilde{Q}}\Big(\int_{k_j}^u (s-k_j)_+ s^{n-1}ds\Big)\,dx\,d\tau\\ &\quad + \frac{\omega^{n-1}}{\rho^2}\iint_{\tilde{Q}} |u|^{nm-1}(u-k_j)_+^2\,dx\,d\tau\Big\}.\end{aligned}$$

Estimating

$$\int_{k_j}^{u} (s-k_j)_+ s^{n-1} ds \le u^{n-1}\frac{(u-k_j)_+^2}{2} \le \omega^{n-1}\frac{(u-k_j)_+^2}{2}\,,$$

and taking into account that $(u-k_j)_+ \le \frac{\omega}{2^j}$, yields

$$\iint_{\tilde{Q}} |u|^{nm-1}|D(u-k_j)_+|^2\zeta^2\,dx\,d\tau \le \gamma\Big(\frac{\omega}{2^j}\Big)^2\omega^{n-1}\frac{\omega^{nm-1}}{c_0\,\rho^2}|Q_*|.$$

Note that $u > k_j \ge \frac{\omega}{4}$: indeed the second inequality is equivalent to

$$\mu_+ \ge |\mu_-|\Big(\frac{1}{4}+\frac{1}{2^j}\Big)\Big(\frac{3}{4}-\frac{1}{2^j}\Big)^{-1},$$

and this is implied by our assumptions. Thus we can estimate

$$\begin{aligned}\iint_{\tilde{Q}} |u|^{nm-1}|D(u-k_j)_+|^2\zeta^2\,dx\,d\tau &\ge \iint_{Q_*} |u|^{nm-1}|D(u-k_j)_+|^2\,dx\,d\tau\\ &\ge \Big(\frac{\omega}{4}\Big)^{nm-1}\iint_{Q_*} |D(u-k_j)_+|^2\,dx\,d\tau;\end{aligned}$$

it follows that

$$\iint_{Q_*} |D(u-k_j)_+|^2\,dx\,d\tau \le \gamma\Big(\frac{\omega}{2^j}\Big)^2\omega^{n-1}\frac{1}{c_0\rho^2}|Q_*|. \tag{6.6}$$

Next, apply the isoperimetric inequality (3.1) to the function $u(\cdot,t)$, for t in the range $(-\theta_*(2\rho)^2, 0]$, over the cube $K_{2\theta\rho}$, and for the levels

$$k = k_j < l = k_{j+1};$$

in this way $(l-k) = \frac{\omega}{2^{j+1}}$.

Taking into account (6.5), this gives

$$\begin{aligned}&\frac{\omega}{2^{j+1}}|\{u(\cdot,t) > k_{j+1}\}\cap K_{2\theta\rho}|\\ &\le \frac{(2\theta\rho)^{N+1}}{|\{u(\cdot,t) < k_j\}\cap K_{2\theta\rho}|}\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_{2\theta\rho}} |Du|\,dx\\ &\le \frac{8\theta\rho}{c_0^2}\int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_{2\theta\rho}} |Du|\,dx;\end{aligned}$$

integrating in dt over the indicated interval and applying the Hölder inequality, one gets

$$\frac{\omega}{2^{j+1}}|A_{j+1}| \le \frac{8\theta\rho}{c_0^2}\Big(\iint_{Q_*} |D(u-k_j)_+|^2\,dx\,dt\Big)^{1/2}(|A_j|-|A_{j+1}|)^{1/2},$$

where

$$A_j = \{u > k_j\}\cap Q_*.$$

Square both sides of this inequality and estimate above the term containing $|D(u-k_j)_+|$ by inequality (6.6), to obtain

$$|A_{j+1}|^2 \le \frac{\gamma}{c_0^5}|Q_*|\,(|A_j|-|A_{j+1}|)\,.$$

Add these recursive inequalities for $j = j_* + 1, \dots, j_* + q_* - 1$, where q_* is to be chosen. Majorizing the right-hand side with the corresponding telescopic series, gives

$$(q_* - 2)|A_{j_*+q_*}|^2 \le \sum_{j=j_*+1}^{j_*+q_*-1} |A_{j+1}|^2 \le \frac{\gamma}{c_0^5}|Q_*|^2.$$

From this

$$|A_{j_*+q_*}| \le \frac{1}{\sqrt{q_* - 2}}\sqrt{\frac{\gamma}{c_0^5}}\,|Q_*|.$$

The number ν_* being fixed, choose q_* from

$$\frac{1}{\sqrt{q_* - 2}}\sqrt{\frac{\gamma}{c_0^5}} = \nu_*. \qquad \square$$

Now let $\xi \in (0, \frac{1}{2})$, $a \in (0,1)$ be fixed numbers.

Lemma 6.5. *There exists a number $c_* \in (0,1)$, depending upon the data, ξ, and a, such that if*

$$|\{u \ge \mu_+ - \xi\omega\} \cap Q_*| \le c_*|Q_*|, \tag{6.7}$$

then

$$u \le \mu_+ - a\xi\omega \quad \textit{a.e. in } Q_{\theta\rho}(\theta_*\rho^2).$$

Proof. For $k = 0, 1, \dots$, set

$$\rho_k = \rho + \frac{\rho}{2^k}, \quad K_k = K_{\theta\rho_k}, \quad Q_k = K_k \times (-\theta_*\rho_k^2, 0].$$

Let $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ be a piecewise smooth cutoff function in Q_k such that

$$\zeta_1 = \begin{cases} 1 & \text{in } K_{k+1} \\ 0 & \text{in } \mathbb{R}^N \setminus K_k, \end{cases} \quad |D\zeta_1| \le \frac{2^{k+2}}{\theta\rho},$$

$$\zeta_2 = \begin{cases} 1 & \text{if } t \ge -\frac{\rho_{k+1}^2}{\omega^{nm-1}} \\ 0 & \text{if } t < -\frac{\rho_k^2}{\omega^{nm-1}}, \end{cases} \quad 0 \le \zeta_{2,t} \le \frac{2^k}{\theta_*\rho^2}.$$

Choose the sequence of truncating levels

$$h_k = \mu_+ - \xi_k\omega, \quad \text{where } \xi_k = a\xi + \frac{1-a}{2^k}\,\xi,$$

and write down the energy estimates (3.3) for $(u - h_k)_+$ over the cylinder Q_k,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}} < t \le 0} \int_{K_k} \Big(\int_{h_k}^u (s - h_k)_+ s^{n-1} ds\Big)\zeta^2(x,t)\,dx \\
&+ \iint_{Q_k} |u|^{nm-1}\big|D[(u - h_k)_+\zeta]\big|^2\,dx\,d\tau \\
&\le \gamma\Big\{\iint_{Q_k}\Big(\int_{h_k}^u (s - h_k)_+ s^{n-1} ds\Big)|\zeta_t|\,dx\,d\tau \\
&\quad + \iint_{Q_k} |u|^{nm-1}(u - h_k)_+^2|D\zeta|^2\,dx\,d\tau\Big\}.
\end{aligned}$$

Let us estimate

$$\int_{h_k}^u (s - h_k)_+ s^{n-1} ds \ge h_k^{n-1}\frac{(u - h_k)_+^2}{2},$$

$$\int_{h_k}^{u}(s-h_k)_+ s^{n-1}ds \le u^{n-1}\frac{(u-h_k)_+^2}{2} \le \omega^{n-1}\frac{(u-h_k)_+^2}{2}\,.$$

Taking into account that $(u-h_k)_+ \le \xi\omega$ and the definition of θ and θ_*, one has

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0} h_k^{n-1}\int_{K_k}\frac{(u-h_k)_+^2}{2}\zeta^2(x,t)\,dx \ + \iint_{Q_k}|u|^{nm-1}\big|D[(u-h_k)_+\zeta]\big|^2\,dx\,d\tau\\
&\le \gamma(\xi\omega)^2\Big\{\omega^{n-1}\frac{2^k}{\theta_*\rho^2}+\omega^{nm-1}\frac{2^{2k}}{(\theta\rho)^2}\Big\}|A_k|\\
&= \gamma\frac{2^{2k}}{\rho^2}(\xi\omega)^2\omega^{n-1}\,\omega^{nm-1}|A_k|,
\end{aligned}$$

where

$$A_k=\{u<h_k\}\cap Q_k.$$

Note that $u>h_k\ge\Big(\frac{1}{2}-\xi\Big)\omega$: indeed the last inequality is equivalent to

$$\mu_+\ge|\mu_-|\Big(\frac{1}{2}-\xi+\xi_k\Big)\Big(\frac{1}{2}+\xi-\xi_k\Big)^{-1},$$

and this follows by our hypotheses. Therefore, we obtain

$$\begin{gathered}
\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0}\int_{K_k}(u-h_k)_+^2\zeta^2(x,t)\,dx\le\gamma\frac{2^{2k}}{\rho^2}\Big(\frac{1}{2}-\xi\Big)^{1-n}(\xi\omega)^2\,\omega^{nm-1}|A_k|,\\
\iint_{Q_k}|D[(u-h_k)_+\zeta]|^2\,dx\,d\tau\le\gamma\,\frac{2^{2k}}{\rho^2}\Big(\frac{1}{2}-\xi\Big)^{1-nm}(\xi\omega)^2\omega^{n-1}|A_k|.
\end{gathered}\tag{6.8}$$

By $(u-h_k)_+\ge\frac{1-a}{2^{k+1}}\,\xi\omega$, applying the Hölder inequality, and then Proposition 3.1, (6.8) yields

$$\begin{aligned}
\frac{(1-a)^2}{2^{2(k+1)}}(\xi\omega)^2|A_{k+1}| &\le \iint_{Q_{k+1}}(u-h_k)_+^2\,dx\,d\tau\le\iint_{Q_k}(u-h_k)_+^2\zeta^2\,dx\,d\tau\\
&\le\Big(\iint_{Q_k}[(u-h_k)_+\zeta]^{\frac{2(N+2)}{N}}\,dx\,d\tau\Big)^{N/(N+2)}|A_k|^{2/(N+2)}\\
&\le\gamma\Big(\iint_{Q_k}\big|D[(u-h_k)_+\zeta]\big|^2\,dx\,d\tau\Big)^{N/(N+2)}\\
&\quad\times\Big(\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0}\int_{K_k}[(u-h_k)_+\zeta]^2dx\Big)^{2/(N+2)}|A_k|^{2/(N+2)}\\
&\le\gamma\frac{2^{2k}}{\rho^2}(\xi\omega)^2\omega^{\frac{2(nm-1)+N(n-1)}{N+2}}\Big(\frac{1}{2}-\xi\Big)^{\frac{N(1-nm)+2(1-n)}{N+2}}|A_k|^{1+\frac{2}{N+2}}.
\end{aligned}$$

It follows that

$$|A_{k+1}|\le\gamma\,\frac{2^{4k}}{(1-a)^2\rho^2}\omega^{\frac{2(nm-1)+N(n-1)}{N+2}}\Big(\frac{1}{2}-\xi\Big)^{\frac{N(1-nm)+2(1-n)}{N+2}}|A_k|^{1+\frac{2}{N+2}}.$$

Setting

$$Y_k=\frac{|A_k|}{|Q_k|},$$

we obtain

$$\begin{aligned} Y_{k+1} &\le \gamma \frac{2^{4k}}{(1-a)^2\rho^2}\omega^{\frac{2(nm-1)+N(n-1)}{N+2}}\Big(\frac{1}{2}-\xi\Big)^{\frac{N(1-nm)+2(1-n)}{N+2}}\rho^2(\theta^N\theta_*)^{\frac{2}{N+2}}Y_k^{1+\frac{2}{N+2}} \\ &= \gamma\frac{2^{4k}}{(1-a)^2}\Big(\frac{1}{2}-\xi\Big)^{\frac{N(1-nm)+2(1-n)}{N+2}}Y_k^{1+\frac{2}{N+2}}. \end{aligned}$$

Applying Lemma 3.4, Y_k tends to zero as $k\to\infty$, provided

$$\begin{aligned} Y_0 &= \frac{|\{u>h_0\}\cap Q_0|}{|Q_0|} = \frac{|\{u>\mu_+-\xi\omega\}\cap Q_0|}{|Q_0|} \\ &\le \frac{\gamma^{-\frac{N+2}{2}}}{(1-a)^{-(N+2)}}\Big(\frac{1}{2}-\xi\Big)^{\frac{N(nm-1)+2(n-1)}{2}}2^{-(N+2)^2}, \end{aligned}$$

which is (6.7) with $c_* := \frac{\gamma^{-\frac{N+2}{2}}}{(1-a)^{-(N+2)}}\big(\frac{1}{2}-\xi\big)^{\frac{N(nm-1)+2(n-1)}{2}}2^{-(N+2)^2}$. This completes the proof. □

Thanks to Lemma 6.4, we can apply Lemma 6.5 with $\xi = \frac{1}{2^{j_*+q_*}}$ and $a=\frac{1}{2}$, getting

$$u \le \mu_+ - \frac{\omega}{2^{j_*+q_*+1}} \quad \text{a.e. in } Q_{\theta\rho}\big(\theta_*\rho^2\big),$$

which implies

$$\operatorname*{ess\,sup}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \mu_+ - \frac{\omega}{2^{j_*+q_*+1}}\,.$$

Hence

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \mu_+ - \operatorname*{ess\,inf}_{Q_{\theta\rho}(\theta_*\rho^2)} u - \frac{\omega}{2^{j_*+q_*+1}} \le \omega\Big(1-\frac{1}{2^{j_*+q_*+1}}\Big).$$

7. Conclusion

The two alternatives just discussed can be combined to prove Theorem 4.1.

Proof of Theorem 4.1. The concluding statement of the first alternative says that

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}(\frac{\rho^2}{\omega^{nm-1}})} u \le \frac{3}{4}\,\omega;$$

analogously, the conclusion of the second alternative is that

$$\operatorname*{ess\,osc}_{\mathcal{Q}^*} u = \operatorname*{ess\,osc}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \omega\Big(1-\frac{1}{2^{j_*+q_*+1}}\Big).$$

Recalling the definition of θ_*, we observe that

$$\mathcal{Q}^* = Q_{\theta\rho}(\theta_*\rho^2) \subset Q_{\theta\rho}\Big(\frac{\rho^2}{\omega^{nm-1}}\Big).$$

The thesis follows by defining

$$\eta_* := 1-\frac{1}{2^{j_*+q_*+1}}\,.$$

□

We are now ready to prove the local Hölder regularity.

Proof of Theorem 1.1. Let us remind that we fixed $\epsilon \in (0,1)$, $R > 0$, $(y,s) \in E_T$, and we considered the cylinder

$$Q_\epsilon = K_{R^{1-\epsilon\frac{n-1}{2}}}(y) \times (s - R^{2-\epsilon(nm-1)}, s] \subset E_T.$$

Let now $\beta, \delta \in (0,1)$ to be chosen, and let us introduce the sequences

$$R_k := \beta^k R, \quad \omega_k := \delta^k \omega, \quad \theta_k := \omega_k^{\frac{1-n}{2}}, \quad \mathcal{Q}_k := (y,s) + Q_{\theta_k R_k}\Big(\frac{R_k^2}{\omega_k^{nm-1}}\Big),$$

for $k \in \mathbb{N}$. The thesis follows by standard arguments once we prove that

$$\begin{gathered} \mathcal{Q}_{k+1} \subset \mathcal{Q}_k \subset Q_\epsilon \subset E_T \quad \forall k \in \mathbb{N}, \\ \operatorname*{ess\,osc}_{\mathcal{Q}_k} u \le \omega_k. \end{gathered} \tag{7.1}$$

The inclusion $\mathcal{Q}_0 \subset Q_\epsilon$ immediately follows by assumption (4.1), while $\mathcal{Q}_{k+1} \subset \mathcal{Q}_k$ is equivalent to

$$\beta \le \min\{\delta^{\frac{n-1}{2}}, \delta^{\frac{nm-1}{2}}\} = \delta^{\frac{n-1}{2}}.$$

To prove (7.1), we will argue by induction. The validity for $k = 0$ is true by construction since

$$\operatorname*{ess\,osc}_{\mathcal{Q}_0} u \le \operatorname*{ess\,osc}_{Q_\epsilon} u \le \omega.$$

Assume that (7.1) holds for k and apply Theorem 4.1 taking $\rho = \frac{R_k}{2}$ and $\omega = \omega_k$; thanks to these choices

$$\theta = \theta_k\,, \quad (y,s) + Q_{2\theta\rho}\Big(\frac{(2\rho)^2}{\omega^{nm-1}}\Big) = \mathcal{Q}_k\,.$$

The assumptions of Theorem 4.1 are satisfied because (7.1) holds for k; hence, we have $\operatorname{ess\,osc}_{\mathcal{Q}^*} u \le \eta_* \omega_k$, where in this setting

$$\mathcal{Q}^* = (y,s) + Q_{\theta_k \frac{R_k}{2}}\Big(\frac{c_0}{8}\,\omega_k^{1-nm} R_k^2\Big)\,.$$

This leads us to choose $\delta = \eta_* \in (0,1)$, so that $\eta_* \omega_k = \omega_{k+1}$. It remains only to check $\mathcal{Q}_{k+1} \subset \mathcal{Q}^*$, which by a simple calculation is equivalent to

$$\beta \le \min\Big\{\frac{1}{2}\,\delta^{\frac{n-1}{2}}\,, \sqrt{\frac{c_0}{8}}\,\delta^{\frac{nm-1}{2}}\Big\}.$$

We conclude by choosing β small enough. □

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Simona Puglisi
Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
E-mail address: spuglisi@dmi.unict.it