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EXISTENCE OF SOLUTIONS TO SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS WITH IMPULSES

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Abstract. By constructing a weighted Banach space and a completely continuous operator, we establish the existence of solutions for singular fractional differential systems with impulses. Our results are proved using the Leray-Schauder nonlinear alternative, and are illustrated with examples.

1. Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [18]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [21]. There have been many excellent books and monographs available on this field [13], [20] and [22], the authors gave the most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [17].

Recently, the authors in papers [2, 3, 4, 5, 12, 27, 28] and the survey paper [1] studied the existence of solutions of the different initial value problems for the impulsive fractional differential equations.

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In [5], the author studied the existence of solutions of the following impulsive anti-periodic boundary value problem

$$\begin{gathered} {}^cD^q_{0^+}x(t) = f(t, x(t)), \quad 1 < q \le 2,\ t \in [0, T] \setminus \{t_1, \dots, t_p\}, \\ x(0) = -x(T), \\ x'(0) = -x'(T), \\ \Delta x(t_k) = I_k(x(t_k^-)), k = 1, \dots, p, \\ \Delta x'(t_k) = J_k(x(t_k^-)), k = 1, \dots, p, \end{gathered} \tag{1.1}$$

where ${}^cD^\alpha_{0^+}$ is the standard Caputo fractional derivative of order q, $0 < T < +\infty$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$, $\Delta x(t_k) = \lim_{t\to t_k^+} x(t) - \lim_{t\to t_k^-} x(t)$ and $\Delta x'(t_k) = \lim_{t\to t_k^+} x'(t) - \lim_{t\to t_k^-} x'(t)$, f defined on $[0, T] \times \mathbb{R}$ is continuous, $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are also continuous.

Boundary-value problems for second-order differential equations with integral boundary conditions constitute a very interesting and important class of problems. They include as special cases two, three, multi-point and nonlocal boundary-value problems as special cases. For such problems and comments on their importance, we refer the readers to the papers [11, 15, 16] and the references therein. Various problems arising in heat conduction [6, 7], chemical engineering [8], underground water flow [10], thermo-elasticity [26], and plasma physics [24] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [25, 29, 9] for parabolic equations and in [23] for hyperbolic equations.

Motivated by [5], in this paper, we discuss the anti-periodic type boundary value problem of the nonlinear fractional differential system

$$\begin{gathered} D^\alpha_{t_k^+}u(t) = m(t)f(t, u(t), v(t)), \quad t \in (t_k, t_{k+1}], k = 0, 1, \dots, p, \\ D^\beta_{t_k^+}v(t) = n(t)g(t, u(t), v(t)), \quad t \in (t_k, t_{k+1}], k = 0, 1, \dots, p, \\ \lim_{t\to 1} t^{1-\alpha}u(t) + \lim_{t\to 0} t^{1-\alpha}u(t) = \int_0^1 \phi(s)F(s, u(s), v(s))ds, \\ \lim_{t\to 1} t^{1-\beta}v(t) + \lim_{t\to 0} t^{1-\beta}v(t) = \int_0^1 \psi(s)G(s, u(s), v(s))ds, \\ \lim_{t\to t_k^+} (t - t_k)^{1-\alpha}u(t) - u(t_k) = I_k(t_k, u(t_k), v(t_k)), \quad k = 1, 2, \dots, p, \\ \lim_{t\to t_k^+} (t - t_k)^{1-\beta}v(t) - v(t_k) = J_k(t_k, u(t_k), v(t_k)), \quad k = 1, 2, \dots, p, \end{gathered} \tag{1.2}$$

where:

- $0 < \alpha, \beta \le 1$, D^α (or D^β) is the Riemann-Liouville fractional derivative of order α (or β),
- p is a positive integer, $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = 1$ are fixed impulsive points,
- $m, n : (0, 1) \to \mathbb{R}$ satisfy $m|_{(t_k,t_{k+1}]}, n|_{(t_k,t_{k+1}]} \in L^1(t_k, t_{k+1}]$ $(k = 0, 1, \dots, p)$, both m and n may be singular at $t = 0$ or $t = 1$, there exist constants $l_1 \ge 0$, $l_2 \ge 0$, $k_1 \ge -\alpha$, $k_2 \ge -\beta$ such that
$$|m(t)| \le l_1t^{k_1}, \quad |n(t)| \le l_2t^{k_2}, \quad t \in (0, 1),$$
- $\phi, \psi : (0, 1) \to \mathbb{R}$ satisfy $\phi|_{(t_k,t_{k+1}]}, \psi|_{(t_k,t_{k+1}]} \in L^1(t_k, t_{k+1}]$ $(k = 0, 1, \dots, p)$,

• f, g, F, G, I_k, J_k $(k = 1, 2, \dots, p)$ defined on $(0, 1] \times R \times \mathbb{R}$ are impulsive Carathéodory functions that may be singular at $t = 0$.

A pair of functions (x, y) with $x : (0, 1] \to \mathbb{R}$ and $y : (0, 1] \to \mathbb{R}$ is said to be a solution of (1.2), if $x|_{(t_k,t_{k+1}]}, y|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}]$ $(k = 0, 1, \dots, p)$ and $D^{\beta}_{0+}y, D^{\alpha}_{0+}x \in L^1(0, 1)$ and (x, y) satisfies all equations in (1.2). We will obtain at least one solution of (1.2).

Remark 1.1. When $\alpha = \beta = 1$, $F(t, x, y) = G(t, x, y) \equiv 0$ and all of the impulse effects disappears, i.e., ($I_k(t, x, y) = J_k(t, x, y) \equiv 0$ and $\lim_{t\to t_k^+}(t - t_k)^{1-\alpha}u(t) - u(t_k) = \Delta u(t_k) = 0$, $\lim_{t\to t_k^+}(t-t_k)^{1-\beta}v(t)-v(t_k) = \Delta v(t_k) = 0$ at this case), (1.2) becomes the anti-periodic boundary value problem for ordinary differential system

$$\begin{gathered} u'(t) = m(t)f(t, u(t), v(t)), \quad t \in (0, 1), \\ v'(t) = n(t)g(t, u(t), v(t)), \quad t \in (0, 1), \\ u(0) = -u(1), \quad v(0) = -v(1). \end{gathered}$$

So we call (1.2) the anti-periodic type boundary-value problem of the nonlinear singular fractional differential system with impulse effects.

The remainder of this paper is as follows: in Section 2, we present preliminary results. In Section 3, we state and prove the main theorems. In Section 4, we give an example to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the monograph [20] and [18]. Let the Gamma and beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p, q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \quad \mathbf{B}(p, q) = \int_0^1 x^{p-1}(1 - x)^{q-1}dx.$$

Definition 2.1 ([20])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}g(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2 ([20])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0+}g(t) = \frac{1}{\Gamma(n - \alpha)}\frac{d^n}{dt^n}\int_0^t \frac{g(s)}{(t - s)^{\alpha-n+1}}ds,$$

where $n - 1 \le \alpha < n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. Let X and Y be Banach spaces. $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero if $\operatorname{Im} L$ is closed in X and $\dim\ker L = \operatorname{codim}\operatorname{Im} L < +\infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist the projectors $P : X \to X$, and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

If $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero, the inverse of

$$L|_{D(L)\cap \ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

is denoted by K_p.

Definition 2.4. *Suppose that $L : D(L) \subset X \to Y$ is called a Fredholm operator of index zero. The continuous map $N : X \to Y$ is called L-compact if both $QN(\overline{\Omega})$ and $K_p(I - Q)N : \overline{\Omega} \to X$ are compact for each nonempty open subset Ω of X satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$.*

To obtain the main results, we need the following abstract existence theorem, the Leray-Schauder Nonlinear Alternative.

Lemma 2.5 ([19])**.** *Let X, Y be Banach spaces and $L : D(L) \cap X \to Y$ a Fredholm operator of index zero with $\ker L = \{0 \in X\}$, $N : X \to Y$ L-compact. Suppose Ω is a nonempty open subset of X satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$. Then either there exists $x \in \partial\Omega$ and $\theta \in (0, 1)$ such that $Lx = \theta Nx$ or there exists $x \in \overline{\Omega}$ such that $Lx = Nx$.*

Definition 2.6 ([14])**.** *An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0, +\infty)$ onto itself which supports Φ in the sense that for all $v_1, v_2 \geq 0$ it holds*

$$\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2). \tag{2.1}$$

The function ω is called the supporting function of Φ.

Remark 2.7. Note that any sup-multiplicative function is sup-multiplicative-like function. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u, \quad u \in \mathbb{R}$$

is sup-multiplicative-like, provided that $c_j \geq 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geq 0$.

Remark 2.8. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and moreover their inverses Φ^{-1} and ν respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{2.2}$$

for all $w_1, w_2 \geq 0$ and ν is called the supporting function of Φ^{-1}.

In this article we assume that Φ is a sup-multiplicative-like function with its supporting function ω, the inverse function Φ^{-1} has its supporting function ν.

Definition 2.9. We call $K : (0, 1] \times R^2 \to \mathbb{R}$ an *impulsive Caratheodory function* if it satisfies the following:

(i) $t \to K\big(t, (t - t_k)^{\alpha-1}x, (t - t_k)^{\beta-1}y\big)$ is continuous on $(t_k, t_{k+1}]$ for $k = 0, 1, \dots, p$, and there exist the following limits:

$$\lim_{t\to t_k^+} K\big(t, (t - t_k)^{\alpha-1}x, (t - t_k)^{\beta-1}y\big) \quad (k = 0, 1, \dots, p) \text{ for any } (x, y) \in \mathbb{R}^2,$$

(ii) $(x,y) \to K\big(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y\big)$ is continuous on R^2 for all $t \in (t_k,t_{k+1}]$ $(k=0,1,\dots,p)$.

We use the Banach spaces

$$X = \big\{x:(0,1]\to\mathbb{R} : x|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}],\ k=0,1,\dots,p,\\ \text{there exist the limits } \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t), k=0,1,\dots,p\big\}$$

with the norm

$$\|x\| = \|x\|_\infty = \max\big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|x(t)|,\ k=0,1,\dots,p\big\}.$$

$$Y = \big\{y:(0,1]\to\mathbb{R} : y|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}], k=0,1,\dots,p,\\ \text{there exist the limits } \lim_{t\to t_k^+}(t-t_k)^{1-\beta}y(t), k=0,1,\dots,p\big\}$$

with the norm

$$\|y\| = \|y\|_\infty = \max\big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|y(t)|, k=0,1,\dots,p\big\}.$$

$L^1[0,1]$ with the norm

$$\|u\|_1 = \int_0^1 |u(s)|ds.$$

Choose $E = X\times Y$ with the norm $\|(x,y)\| = \max\{\|x\|_\infty,\|y\|_\infty\}$, and choose $Z = L^1(0,1)\times L^1(0,1)\times R^{2p+2}$ with the norm

$$\left\|\begin{pmatrix} u\\ v\\ a\\ b\\ c_k\ (k=1,2,\dots,p)\\ d_k\ (k=1,2,\dots,p)\end{pmatrix}^T\right\| = \|(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\|\\ = \max\{\|u\|_1,\|v\|_1,|a|,|b|,|c_1|,\dots,|d_p|,|d_1|,\dots,|c_p|\}.$$

Define L to be the linear operator from $D(L)\cap E$ to Z with

$$D(L) = \{(x,y)\in E : D_{t_k^+}^{\alpha}x, D_{t_k^+}^{\beta}y \in L^1(0,1)\}$$

and

$$L(x,y)(t) = \begin{pmatrix} D_{t_k^+}^{\alpha}x(t)\\ D_{t_k^+}^{\beta}y(t)\\ \lim_{t\to1}t^{1-\alpha}x(t)+\lim_{t\to0}t^{1-\alpha}x(t)\\ \lim_{t\to1}t^{1-\beta}y(t)+\lim_{t\to0}t^{1-\beta}y(t)\\ \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t)-x(t_k), k=1,\dots,p\\ \lim_{t\to t_k^+}(t-t_k)^{1-\beta}y(t)-y(t_k), k=1,\dots,p\end{pmatrix}^T$$

for $(x,y)\in E\cap D(L)$. Define $N:E\to Z$ by

$$N(x,y)(t)=\begin{pmatrix} m(t)f(t,x(t),y(t))\\ n(t)g(t,x(t),y(t))\\ \int_0^1\phi(t)F\left(t,x(t),y(t)\right)dt\\ \int_0^1\psi(t)G\left(t,x(t),y(t)\right)dt\\ I_k\left(t_k,x(t_k),y(t_k)\right),k=1,\dots,p\\ J_k\left(t_k,x(t_k),y(t_k)\right),k=1,\dots,p\end{pmatrix}^T$$

for $(x,y)\in E$. Then (1.2) can be written as

$$L(x,y)=N(x,y),\quad (x,y)\in E.$$

Lemma 2.10. *Suppose that f,g,F,G,I_k,J_k $(k=1,2,\dots,p)$ are impulsive Caratheodory functions. Then L is a Fredholm operator of index zero and $N:X\to Y$ is L-compact.*

Proof. To prove that L is a Fredholm operator of index zero, we should do the following six steps.

Step (i) Prove that $\ker L=\{(0,0)\in E\}$. We know that $(x,y)\in\ker L$ if and only if

$$\begin{gathered} D^{\alpha}_{t_k^+}x(t)=0,\quad D^{\beta}_{t_k^+}y(t)=0,\\ \lim_{t\to1}t^{1-\alpha}x(t)+\lim_{t\to0}t^{1-\alpha}x(t)=0,\\ \lim_{t\to1}t^{1-\beta}y(t)+\lim_{t\to0}t^{1-\beta}y(t)=0,\\ \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t)-x(t_k)=0,k=1,\dots,p,\\ \lim_{t\to t_k^+}(t-t_k)^{1-\beta}y(t)-y(t_k))=0,k=1,\dots,p. \end{gathered}$$

Hence $(x,y)\in\ker L$ if and only if $x(t)=0$ and $y(t)=0$. Thus $\ker L=\{(0,0)\in E\}$.

Step (ii) Prove that $\operatorname{Im}L=Z$. First, we have $\operatorname{Im}L\subseteq Z$. Second, we know that $(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\in\operatorname{Im}L$ if and only if there exist $(x,y)\in D(L)\cap E$ such that

$$\begin{gathered} D^{\alpha}_{t_k^+}x(t)=u(t),\quad D^{\beta}_{t_k^+}y(t)=v(t),\\ \lim_{t\to1}t^{1-\alpha}x(t)+\lim_{t\to0}t^{1-\alpha}x(t)=a,\\ \lim_{t\to1}t^{1-\beta}y(t)+\lim_{t\to0}t^{1-\beta}y(t)=b,\\ \lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t)-x(t_k)=c_k,k=1,\dots,p,\\ \lim_{t\to t_k^+}(t-t_k)^{1-\beta}y(t)-y(t_k))=d_k,k=1,\dots,p. \end{gathered}\tag{2.3}$$

If (x,y) satisfies (2.3), then there exist two numbers $\overline{M}_k$ $(k=0,1,\dots,p)$ such that

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}u(s)ds+\overline{M}_k(t-t_k)^{\alpha-1},\tag{2.4}$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \dots, p$. By the boundary condition $\lim_{t\to 1} t^{1-\alpha}x(t) + \lim_{t\to 0} t^{1-\alpha}x(t) = a$, we obtain

$$\int_{t_p}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + \overline{M}_p(1-t_p)^{\alpha-1} + \overline{M}_0 = a. \tag{2.5}$$

By the impulse conditions $\lim_{t\to t_k^+}(t-t_k)^{1-\alpha}x(t) - x(t_k) = c_k$, we obtain

$$\overline{M}_k - \Big(\int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + \overline{M}_{k-1}(t_k-t_{k-1})^{\alpha-1}\Big) = c_k, \tag{2.6}$$

for $k = 1, \dots, p$. It follows from (2.6) that

$$\overline{M}_p - \prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\overline{M}_0 = \sum_{k=1}^{p}\Big(c_k + \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\gamma(\alpha)} u(s)ds\Big) \prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}.$$

By this equality and (2.5), we obtain

$$\begin{aligned}
&\overline{M}_0\\
&= \frac{a - \int_{t_p}^{1} \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\alpha-1}}\\
&\quad + \frac{\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}\Big(c_k + \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\alpha-1}},
\end{aligned}$$

$$\begin{aligned}
\overline{M}_p &= \frac{\sum_{k=1}^{p}\Big(c_k + \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\alpha-1}}\\
&\quad + \frac{\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\Big(a - \int_{t_p}^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds\Big)}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\alpha-1}}.
\end{aligned} \tag{2.7}$$

Then (2.6) implies that

$$\overline{M}_k = c_k + \int_{t_{k-1}}^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} u(s)ds + \overline{M}_{k-1}(t_k-t_{k-1})^{\alpha-1}, \quad k = 1, \dots, p-1. \tag{2.8}$$

Hence (2.4) is proved and $\overline{M}_k$ $(k = 0, 1, 2, \dots, p)$ are given by (2.7) and (2.8).

Similarly we obtain

$$y(t) = \frac{1}{\Gamma(\beta)}\int_{t_k}^{t}(t-s)^{\beta-1}v(s)ds + \overline{N}_k(t-t_k)^{\beta-1}, \tag{2.9}$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, \dots, p$, where $\overline{N}_k$ $(k = 0, 1, \dots, p)$ are given by

$$\begin{aligned}
&\overline{N}_0\\
&= \frac{b - \int_{t_p}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} v(s)ds}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\beta-1}}\\
&\quad + \frac{\prod_{k=1}^{p}(t_k-t_{k-1})^{\beta-1}\sum_{k=1}^{p}\Big(d_k + \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\beta-1}}{1 + \prod_{k=1}^{p+1}(t_k-t_{k-1})^{\beta-1}},
\end{aligned}$$

$$\begin{aligned}\overline{N}_p &= \frac{\sum_{k=1}^{p}\Big(d_k+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\beta-1}}{1+\prod_{k=1}^{p+1}(t_k-t_{k-1})^{\beta-1}}\\ &\quad+\frac{\prod_{k=1}^{p}(t_k-t_{k-1})^{\beta-1}\Big(b-\int_{t_p}^{1}\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds\Big)}{1+\prod_{k=1}^{p+1}(t_k-t_{k-1})^{\beta-1}},\end{aligned} \tag{2.10}$$

and

$$\overline{N}_k = d_k+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds+\overline{N}_{k-1}(t_k-t_{k-1})^{\beta-1},\quad k=1,\dots,p-1. \tag{2.11}$$

It is easy to show that $(x,y)\in D(L)\cap E$. Hence $(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\in \operatorname{Im}L$. Then $\operatorname{Im}L=Z$.

On the other hand, we can prove that (x,y) is a solution of (2.3) if $x\in E$ satisfies (2.4) and $y\in Y$ satisfies (2.9).

Step (iii) Prove that $\operatorname{Im}L$ is closed in X and $\dim\ker L=\operatorname{codim}\operatorname{Im}L<+\infty$. From Step (ii) $\operatorname{Im}L=Z$ is closed in Z. It follows from $\ker L=\{(0,0)\in E\}$ that $\dim\ker L=0$. Define the projector $P:E\to E$ by

$$P(x,y)(t)=(0,0)\quad\text{for } (x,y)\in E. \tag{2.12}$$

It is easy to prove that

$$\operatorname{Im}P=\ker L,\quad X=\ker L\oplus\ker P. \tag{2.13}$$

Define the projector $Q:Z\to Z$ by

$$Q(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)(t)=(0,0,0,0,0,\dots,0,0,\dots,0) \tag{2.14}$$

for $(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\in Z$. It is easy to show that

$$\operatorname{Im}L=\ker Q,\quad Y=\operatorname{Im}Q\oplus\operatorname{Im}L. \tag{2.15}$$

From above discussion, we see that $\dim\ker L=\operatorname{codim}\operatorname{Im}L=0<+\infty$. So L is a Fredholm operator of index zero.

Now, we prove that N is L-compact. This is divided into three steps (Steps (iv)-(vi)).

Step (iv) We prove that N is continuous. Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We will show that $N(x_n,y_n)\to N(x_0,y_0)$ as $n\to\infty$. In fact, we have

$$\begin{aligned}&\|(x_n,y_n)\|\\ &=\sup_{n=0,1,2,\dots}\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|x_n(t)|,\\ &\quad\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\beta}|y_n(t)|:k=0,1,\dots,p\Big\}=r<+\infty\end{aligned}$$

and

$$\begin{aligned}&\max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|x_n(t)-x_0(t)|,\ k=0,1,\dots,p\Big\}\to0,\quad n\to\infty,\\ &\max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\beta}|y_n(t)-y_0(t)|,\ k=0,1,\dots,p\Big\}\to0,\quad n\to\infty.\end{aligned} \tag{2.16}$$

By

$$N(x_n,y_n)(t)=\begin{pmatrix} m(t)f(t,x_n(t),y_n(t)) \\ n(t)g(t,x_n(t),y_n(t)) \\ \int_0^1 \phi(t)F\left(t,x_n(t),y_n(t)\right)dt \\ \int_0^1 \psi(t)G\left(t,x_n(t),y_n(t)\right)dt \\ I_k\left(t_k,x_n(t_k),y_n(t_k)\right)\ (k=1,2,\dots,p) \\ J_k\left(t_k,x_n(t_k),y_n(t_k)\right)\ (k=1,2,\dots,p) \end{pmatrix}^T$$

for $(x,y)\in E$, for any $\epsilon>0$, since f,F,I_k $(k=1,\dots,p)$ are impulsive Caratheodory functions, we know that $f\left(t,(t-t_k)^{\alpha-1}u,(t-t_k)^{\beta-1}v\right)$ is continuous on $[t_k,t_{k+1}]\times[-r,r]^2$ $(k=0,1\dots,p)$ respectively, so $f\left(t,(t-t_k)^{\alpha-1}u,(t-t_k)^{\beta-1}v\right)$ is uniformly continuous on $[t_k,t_{k+1}]\times[-r,r]^2$ respectively. Similarly, F,I_k $(k=1,\dots,p)$ are uniformly continuous on $[t_k,t_{k+1}]\times[-r,r]^2$ respectively. Then there exists $\delta>0$ such that

$$\begin{gathered}\left|f\left(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1\right)-f\left(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2\right)\right|<\epsilon,\\ t\in(t_k,t_{k+1}],\\ \left|F\left(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1\right)-F\left(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2\right)\right|<\epsilon,\\ t\in(t_k,t_{k+1}],\\ \big|I_k\left(t_k,(t_k-t_{k-1})^{\alpha-1}u_1,(t_k-t_{k-1})^{\beta-1}v_1\right)\\ -I_k\left(t_k,(t_k-t_{k-1})^{\alpha-1}u_2,(t_k-t_{k-1})^{\beta-1}v_2\right)\big|<\epsilon\end{gathered}$$

for all $k=0,1,\dots,p$, $|u_1-u_2|<\delta$ and $|v_1-v_2|<\delta$ with $u_1,u_2,v_1,v_2\in[-,r,r]$.

From (2.16), there exists N such that

$$\begin{aligned}&(t-t_k)^{1-\alpha}|x_n(t)-x_0(t)|<\delta,\quad t\in(t_k,t_{k+1}],\ k=0,1,\dots,p,\ n>N,\\ &(t-t_k)^{1-\beta}|y_n(t)-y_0(t)|<\delta,\quad t\in(t_k,t_{k+1}],\ k=0,1,\dots,p,\ n>N.\end{aligned}\tag{2.17}$$

Hence using (2.17), we obtain

$$\begin{aligned}&\int_0^1|m(t)f\left(t,x_n(t),y_n(t)\right)-m(t)f\left(t,x_0(t),y_0(t)\right)|\,dt\\ &=\sum_{k=0}^{p}\int_{t_k}^{t_{k+1}}\big|m(t)f\left(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x_n(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y_n(t)\right)\\ &-m(t)f\left(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x_0(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y_0(t)\right)\big|dt\\ &<\sum_{k=0}^{p}\int_{t_k}^{t_{k+1}}\epsilon m(t)dt=\epsilon\int_0^1 m(t)dt,n>N.\end{aligned}$$

It follows that

$$\left|\int_0^1 m(t)f\left(t,x_n(t),y_n(t)\right)dt-\int_0^1 f\left(t,x_0(t),y_0(t)\right)dt\right|<\epsilon\int_0^1 m(t)dt,\tag{2.18}$$

for $n>N$. Similarly,

$$\left|\int_0^1 \phi(t)F\left(t,x_n(t),y_n(t)\right)dt-\int_0^1 F\left(t,x_0(t),y_0(t)\right)dt\right|<\epsilon\int_0^1 \phi(t)dt,\tag{2.19}$$

for $n>N$, and

$$|I_k\left(t_k,x_n(t_k),y_n(t_k)\right)-I_k\left(t_k,x_0(t_k),y_0(t_k)\right)|<\epsilon,\quad n>N,\ k=1,\dots,p\tag{2.20}$$

We can also show that

$$\Big|\int_0^1 n(t)g\left(t,x_n(t),y_n(t)\right)dt-\int_0^1 n(t)g\left(t,x_0(t),y_0(t)\right)dt\Big|<\epsilon\int_0^1 n(t)dt, \tag{2.21}$$

for $n>N$. Similarly,

$$\left|\int_0^1 \psi(t)G\left(t,x_n(t),y_n(t)\right)dt-\int_0^1 G\left(t,x_0(t),y_0(t)\right)dt\right|<\epsilon\int_0^1 \psi(t)dt, \tag{2.22}$$

for $n>N$, and

$$|J_k\left(t_k,x_n(t_k),y_n(t_k)\right)-J_k\left(t_k,x_0(t_k),y_0(t_k)\right)|<\epsilon,\quad n>N,k=1,\dots,p \tag{2.23}$$

Then (2.18)–(2.23) imply that

$$\|N(x_n,y_n)-N(x_0,y_0)\|\to 0,\quad n\to\infty.$$

It follows that N is continuous.

Let $P:X\to X$ and $Q:Y\to Y$ be defined by (2.12) and (2.14). For $(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\in \operatorname{Im}L=Z$, let

$$K_P(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)(t)=\left(x_1(t),y_1(t)\right), \tag{2.24}$$

where

$$\begin{aligned}x_1(t)&=\int_{t_k}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds+\overline{M}_kt^{\alpha-1},t\in(t_k,t_{k+1}],\quad k=0,1,\dots,p;\\ y_1(t)&=\int_{t_k}^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}v(s)ds+\overline{N}_kt^{\alpha-1},t\in(t_k,t_{k+1}],\quad k=0,1,\dots,p.\end{aligned}$$

Here $\overline{M}_k,\overline{N}_k$ $(k=0,1,\dots,p)$ are given by (2.7), (2.8), (2.9) and (2.11).

One sees that $K_P(u,v,a,b,c_1,\dots,c_p,d_1,\dots,d_p)\in D(L)\cap E$ and $K_P:\operatorname{Im}L\to D(L)\cap\ker P$ is the inverse of $L:D(L)\cap\ker P\to\operatorname{Im}L$. The isomorphism $\wedge:\ker L\to Y/\operatorname{Im}L$ is given by

$$\wedge(0,0)=(0,0,0,0,0,\dots,0,0\dots,0).$$

Furthermore, one has

$$QN(x,y)(t)=(0,0,0,0,0,\dots,0,0\dots,0), \tag{2.25}$$

and

$$K_p(I-Q)N(x,y)(t)=K_pN(x,y)(t)=\left(x_2(t),y_2(t)\right),$$

where

$$x_2(t)=\int_{t_k}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds+M_kt^{\alpha-1},t\in(t_k,t_{k+1}], \tag{2.26}$$

for $k=0,1,\dots,p$, and

$$y_2(t)=\int_{t_k}^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds+N_kt^{\alpha-1},t\in(t_k,t_{k+1}], \tag{2.27}$$

for $k=0,1,\dots,p$. Here M_k, N_k $(k=0,1,\dots,p)$ are given by

$$
\begin{aligned}
M_0 = \frac{1}{\lambda}\Big(&\int_0^1 \phi(s)F(s,x(s),y(s))ds - \int_{t_p}^{t_{p+1}} \frac{(t_{p+1}-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds \\
&+ \prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1} \\
&\times \big(I_k(t_k,x(t_k),y(t_k)) + \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds\big)\Big), \\
&M_1 = I_1(t_1,x(t_1),y(t_1)) \\
&\quad + \int_{t_0}^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds + (t_1-t_0)^{\alpha-1}M_0, \\
&\dots \\
M_{p-1} = I_{p-1}&(t_{p-1},x(t_{p-1}),y(t_{p-1})) + \int_{t_{p-2}}^{t_{p-1}}\frac{(t_{p-1}-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds \\
&+ (t_{p-1}-t_{p-2})^{\alpha-1}M_{p-2}, \\
M_p = \frac{1}{\lambda}\Big[&\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\Big(\int_0^1\phi(s)F(s,x(s),y(s))ds \\
&- \int_{t_p}^{t_{p+1}}\frac{(t_{p+1}-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds\Big) + \sum_{k=1}^{p}\Big(I_k(t_k,x(t_k),y(t_k)) \\
&+ \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
N_0 = \frac{1}{\lambda}\Big(&\int_0^1 \psi(s)G(s,x(s),y(s))ds - \int_{t_p}^{t_{p+1}} \frac{(t_{p+1}-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds \\
&+ \prod_{k=1}^{p}(t_k-t_{k-1})^{\beta-1}\sum_{k=1}^{p}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\beta-1}\times \\
&\Big(J_k(t_k,x(t_k),y(t_k)) + \int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds\Big)\Big), \\
N_1 = J_1(t_1&,x(t_1),y(t_1)) + \int_{t_0}^{t_1}\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds + (t_1-t_0)^{\alpha-1}N_0, \\
&\dots \\
N_{p-1} = J_{p-1}&(t_{p-1},x(t_{p-1}),y(t_{p-1})) + \int_{t_{p-2}}^{t_{p-1}}\frac{(t_{p-1}-s)^{\beta-1}}{\Gamma(\beta}n(s)g(s,x(s),y(s))ds \\
&+ (t_{p-1}-t_{p-2})^{\beta-1}N_{p-2},
\end{aligned}
$$

$$\begin{aligned}N_p &= \frac{1}{\lambda}\Big(\prod_{k=1}^{p}(t_k-t_{k-1})^{\beta-1}\Big(\int_0^1 \psi(s)G(s,x(s),y(s))ds\\&\quad-\int_{t_p}^{t_{p+1}}\frac{(t_{p+1}-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds\Big)+\sum_{k=1}^{p}(J_k(t_k,x(t_k),y(t_k))\\&\quad+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\beta-1}\Big).\end{aligned}$$

Let Ω be a bounded open subset of E satisfying $D(L)\cap\Omega\neq 30\emptyset$. We have

$$\begin{aligned}\|(x,y)\| &= \max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|x(t)|,\\&\quad \sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\beta}|y(t)|: k=0,1,\dots,p\Big\}\\&= r<+\infty,\quad (x,y)\in\Omega.\end{aligned}\tag{2.28}$$

Since f,g,F,G,I_k,J_k are impulsive Caratheodory functions, together with (2.28), there exists $M>0$ such that

$$\begin{aligned}|f(t,x(t),y(t))| &= \big|f\big(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y(t)\big)\big|\\&\le M\end{aligned}$$

holds for $t\in(t_k,t_{k+1}]$ $(k=0,1,\dots,p)$. Hence

$$|f(t,x(t),y(t))|\le M,\quad t\in(0,1].$$

Similarly,

$$\begin{gathered}|g(t,x(t),y(t))|\le M,\\|F(t,x(t),y(t))|\le M\quad\text{for all } t\in(0,1],\\|G(t,x(t),y(t))|\le M\quad\text{for all } t\in(0,1],\\|I_k(t_k,x(t_k),y(t_k))|\le M,\quad k=1,2,\dots,p\\|J_k(t_k,x(t_k),y(t_k))|\le M,\quad k=1,2,\dots,p.\end{gathered}$$

Step (v) Prove that $QN(\overline{\Omega})$ is bounded. It is easy to see from (2.25) that $QN(\overline{\Omega})$ is bounded.

Step (vi) Prove that $K_P(I-Q)N:\overline{\Omega}\to E$ is compact; i.e., prove that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. We must prove that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded and equi-continuous on each subinterval $[e,f]\subseteq(t_k,t_{k+1}]$ $(k=0,1,\dots,p)$, respectively and equi-convergent at $t=t_k$ $(k=0,1,\dots,p)$, respectively.

Substep (vi1) Prove that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded. We have

$$(t-t_k)^{1-\alpha}x_2(t)=(t-t_k)^{1-\alpha}\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds+M_k,\tag{2.29}$$

for $t\in(t_k,t_{k+1}]$. By the definition of M_k, we have

$$\begin{aligned}&|M_0|\\&\le\frac{1}{\lambda}\Big(\int_0^1|\phi(s)F(s,x(s),y(s))|ds+\int_{t_p}^{t_{p+1}}\frac{(t_{p+1}-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds\end{aligned}$$

$$
\begin{aligned}
&+\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&\times\Big(|I_k(t_k,x(t_k),y(t_k))|+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds\Big)\Big)\\
\le{}&\frac{M}{\lambda}\Big(\|\phi\|_1+l_1t_{p+1}^{\alpha+k_1}\int_{\frac{t_p}{t_{p+1}}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw\\
&+\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&\times\Big(1+l_1t_k^{\alpha+k_1}\int_{\frac{t_{k-1}}{t_k}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw\Big)\Big)<+\infty.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|M_1|&\le M+Ml_1t_1^{\alpha+k_1}\int_{\frac{t_0}{t_1}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw+(t_1-t_0)^{\alpha-1}|M_0|<+\infty,\\
&\dots\\
|M_{p-1}|&\le M+Ml_1t_{p-1}^{\alpha+k_1}\int_{\frac{t_{p-2}}{t_{p-1}}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw+(t_{p-1}-t_{p-2})^{\alpha-1}|M_{p-2}|\\
&<+\infty,\\
|M_p|&\le\frac{M}{\lambda}\Big(\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\Big(\|\phi\|_1+l_1t_{p+1}^{\alpha+k_1}\int_{\frac{t_p}{t_{p+1}}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw\Big)\\
&\quad+\sum_{k=1}^{p}\Big(1+l_1t_k^{\alpha+k_1}\int_{\frac{t_{k-1}}{t_k}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw\Big)\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\Big)\\
&\quad<+\infty.
\end{aligned}
$$

First, use (2.29), for $t\in(t_0,t_1]$ we have

$$
\begin{aligned}
(t-t_0)^{1-\alpha}|x_2(t)|&\le(t-t_0)^{1-\alpha}\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds+|M_0|\\
&\le Ml_1(t_1-t_0)^{1-\alpha}t_1^{\alpha+k_1}\int_{\frac{t_0}{t}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw+|M_0|<+\infty.
\end{aligned}
$$

Second, for $t\in(t_k,t_{k+1}]$ $(k=1,\dots,p-1)$, we have

$$
\begin{aligned}
&(t-t_0)^{1-\alpha}|x_2(t)|\\
&\le(t-t_k)^{1-\alpha}\int_{t_k}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds+|M_k|\\
&\le Ml_1(t_{k+1}-t_k)^{1-\alpha}t_k^{\alpha+k_1}\int_{\frac{t_{k-1}}{t_k}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw+|M_k|<+\infty.
\end{aligned}
$$

Finally, for $t\in(t_p,t_{p+1}]$, we have

$$
(t-t_p)^{1-\alpha}|x_2(t)|
$$

$$
\begin{aligned}
&\le (t-t_p)^{1-\alpha}\int_{t_p}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds+|M_p|\\
&\le Ml_1(t_{p+1}-t_p)^{1-\alpha}t_{p+1}^{\alpha+k_1}\int_{\frac{t_p}{t_{p+1}}}^{1}\frac{(1-w)^{\alpha-1}}{\Gamma(\alpha)}w^{k_1}dw+|M_p|<+\infty.
\end{aligned}
$$

From above discussion, there exists $M_1>0$ such that

$$
\|x_2\|_\infty=\max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|x_2(t)|:k=0,1,\dots,p\Big\}\le M_1<+\infty.
$$

Similarly, we can show that there exist $M_2>0$ such that

$$
\|y_2\|_\infty=\max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\alpha}|y_2(t)|:k=0,1,\dots,p\Big\}\le M_2<+\infty.
$$

Hence $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

Substep (vi2) Prove that $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f]\subseteq(t_k,t_{k+1}]$ $(k=0,1,\dots,p)$, respectively. For each $[e,f]\subseteq(t_k,t_{k+1}]$, and $s_1,s_2\in[e,f]$ with $s_2\ge s_1$, use (2.26), we have

$$
\begin{aligned}
&|(s_1-t_k)^{1-\alpha}x_2(s_1)-(s_2-t_k)^{1-\alpha}x_2(s_2)|\\
&\le\frac{l_1M}{\Gamma(\alpha)}\left|(s_1-t_k)^{1-\alpha}-(s_2-t_k)^{1-\alpha}\right|s_1^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1)\\
&\quad+\frac{l_1M}{\Gamma(\alpha)}(t_{k+1}-t_k)^{1-\alpha}s_2^{\alpha+k_1}\int_{s_1/s_2}^{1}(1-w)^{\alpha-1}w^{k_1}dw\\
&\quad+\frac{l_1M}{\Gamma(\alpha)}(t_{k+1}-t_k)^{1-\alpha}\Big(s_2^{\alpha+k_1}\int_0^1(1-w)^{\alpha-1}w^{k_1}dw\\
&\quad-s_1^{\alpha+k_1}\int_0^{s_2/s_1}|(1-w)^{\alpha-1}w^{k_1}dw\Big)\to 0
\end{aligned}
$$

uniformly as $s_1\to s_2$. It follows that

$$
|(s_1-t_k)^{1-\alpha}x_2(s_1)-(s_2-t_k)^{1-\alpha}x_2(s_2)|\to 0 \tag{2.30}
$$

uniformly as $s_1\to s_2,\ s_1,s_2\in[e,f]\subseteq(t_k,t_{k+1}]$ $(k=0,1,\dots,p)$.

Similarly, we can prove that

$$
|(s_1-t_k)^{1-\beta}y_2(s_1)-(s_2-t_k)^{1-\beta}y_2(s_2)|\to 0 \tag{2.31}
$$

uniformly as $s_1\to s_2,\ s_1,s_2\in[e,f]\subseteq(t_k,t_{k+1}]$ $(k=0,1,\dots,p)$.

Substep (vi3) Prove that $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent at $t=t_k$ $(k=0,1,\dots,p)$, respectively. Since

$$
\begin{aligned}
&\left|(t-t_k)^{1-\alpha}x_2(t)-M_k\right|\\
&\le l_1M(t_{k+1}-t_k)^{1-\alpha}t_{k+1}^{\alpha+k_1}\int_{\frac{t_k}{t}}^{1}(1-w)^{\alpha-1}w^{k_1}dw\to 0
\end{aligned}
$$

uniformly as $t\to t_k$. Similarly we can show that

$$
\left|(t-t_k)^{1-\beta}y_2(t)-N_k\right|\to 0\quad\text{uniformly as } t\to t_k\ (k=0,1,\dots,p. \tag{2.32}
$$

From (2.31)–(2.32), we see that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Then N is L-compact. The proof is complete. □

3. Main Result

Now, we prove the main theorem in this article, using the following assumptions:

- (A) Φ is a sup-multiplicative-like function with its supporting function w, the inverse function of Φ is Φ^{-1} with supporting function ν.
- (B) f, g, F, G, I_k, J_k $(k = 1, 2, \dots, p)$ are impulsive Caratheodory functions and satisfy that there exist nonnegative constants c_i, b_i, a_i $(i = 1, 2)$, C_i, B_i, A_i and $C_{i,k}, B_{i,k}, A_{i,k}$ $(i = 1, 2, k = 1, 2, \dots, p)$ such that

$$
\begin{gathered}
|f(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le c_1 + b_1|x| + a_1\Phi^{-1}(|y|),\\
t\in(t_k,t_{k+1}], k=0,1,\dots,p,\\
|g(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le c_2 + b_2\Phi(|x|) + a_2|y|,\\
t\in(t_k,t_{k+1}], k=0,1,\dots,p,\\
|F(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le C_1 + B_1|x| + A_1\Phi^{-1}(|y|),\\
t\in(t_k,t_{k+1}], k=0,1,\dots,p,\\
|G(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le C_2 + B_2\Phi(|x|) + A_2|y|,\\
t\in(t_k,t_{k+1}], k=0,1,\dots,p,\\
|I_k(t,(t_{k+1}-t_k)^{\alpha-1}x,(t_{k+1}-t_k)^{\beta-1}y)| \le C_{1,k} + B_{1,k}|x| + A_{1,k}\Phi^{-1}(|y|),\\
k=1,2,\dots,p,\\
|J_k(t,(t_{k+1}-t_k)^{\alpha-1}x,(t_{k+1}-t_k)^{\beta-1}y)| \le C_{2,k} + B_{2,k}\Phi(|x|) + A_{2,k}|y|,\\
k=1,2,\dots,p.
\end{gathered}
$$

Also we introduce the following notation.

$$
\begin{aligned}
\lambda &= 1 + \prod_{k=1}^{p+1}(t_k - t_{k-1})^{\alpha-1},\\
M_{0,1} &= \frac{1}{\lambda}\Big[C_1\|\phi\|_1 + l_1c_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\\
&\quad + l_1c_1\mathbf{B}(\alpha,k_1+1)\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&\quad + \prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}C_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1} + l_1c_1\mathbf{B}(\alpha,k_1+1)\Big],\\
M_{0,2} &= \frac{1}{\lambda}\Big[B_1\|\phi\|_1 + l_1b_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\\
&\quad + l_1b_1\mathbf{B}(\alpha,k_1+1)\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&\quad + \prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}B_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1} + l_1b_1\mathbf{B}(\alpha,k_1+1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
M_{0,3} = \frac{1}{\lambda}\Big[&A_1\|\phi\|_1 + l_1a_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\\
&+ l_1a_1\mathbf{B}(\alpha,k_1+1)\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&+ \prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^{p}A_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1} + l_1a_1\mathbf{B}(\alpha,k_1+1)\Big],
\end{aligned}
$$

$$
\begin{gathered}
M_{1,1} = C_{1,1} + l_1c_1t_1^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_1-t_0)^{\alpha-1}M_{0,1},\\
M_{1,2} = B_{1,1} + l_1b_1t_1^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_1-t_0)^{\alpha-1}M_{0,2},\\
M_{1,3} = A_{1,1} + l_1a_1t_1^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_1-t_0)^{\alpha-1}M_{0,3},\\
\dots\\
M_{p-1,1} = C_{1,p-1} + l_1c_1t_{p-1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_{p-1}-t_{p-2})^{\alpha-1}M_{p-2,1},\\
M_{p-1,2} = B_{1,p-1} + l_1b_1t_{p-1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_{p-1}-t_{p-2})^{\alpha-1}M_{p-2,2},\\
M_{p-1,3} = A_{1,p-1} + l_1a_1t_{p-1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + (t_{p-1}-t_{p-2})^{\alpha-1}M_{p-2,3},
\end{gathered}
$$

$$
\begin{aligned}
M_{p,1} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\|\phi\|_1 + l_1c_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\\
&+ \sum_{k=1}^{p}C_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&+ l_1c_1\mathbf{B}(\alpha,k_1+1)\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\Big),\\
M_{p,2} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\|\phi\|_1 + l_1b_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\\
&+ \sum_{k=1}^{p}B_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&+ l_1b_1\mathbf{B}(\alpha,k_1+1)\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\Big),\\
M_{p,3} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\|\phi\|_1 + l_1a_1\mathbf{B}(\alpha,k_1+1)t_{p+1}^{\alpha+k_1}\prod_{k=1}^{p}(t_k-t_{k-1})^{\alpha-1}\\
&+ \sum_{k=1}^{p}A_{1,k}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\\
&+ l_1a_1\mathbf{B}(\alpha,k_1+1)\sum_{k=1}^{p}t_k^{\alpha+k_1}\prod_{s=k+1}^{p}(t_s-t_{s-1})^{\alpha-1}\Big),
\end{aligned}
$$

and

$$
\begin{gathered}
\sigma_{k,1} = l_1c_1(t_{k+1}-t_k)^{1-\alpha}t_{k+1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + M_{k,1}, \quad k=0,1,\dots,p,\\
\sigma_{k,2} = l_1b_1(t_{k+1}-t_k)^{1-\alpha}t_{k+1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + M_{k,2}, \quad k=0,1,\dots,p,\\
\sigma_{k,3} = l_1a_1(t_{k+1}-t_k)^{1-\alpha}t_{k+1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1) + M_{k,3}, \quad k=0,1,\dots,p,
\end{gathered}
$$

$$\sigma_1 = \max\{\sigma_{k,1} : k = 0, 1, \dots, p\},$$
$$\sigma_2 = \max\{\sigma_{k,2} : k = 0, 1, \dots, p\},$$
$$\sigma_3 = \max\{\sigma_{k,3} : k = 0, 1, \dots, p\}.$$

Denote

$$\begin{aligned}
N_{0,1} = \frac{1}{\lambda}\Big[&C_2\|\psi\|_1 + l_2c_2\mathbf{B}(\beta, k_2+1)t_{p+1}^{\beta+k_2} \\
&+ l_2c_2\mathbf{B}(\beta, k_2+1)\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} t_k^{\beta+k_2}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ \prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} C_{2,k}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} + l_2c_2\mathbf{B}(\beta, k_2+1)\Big], \\
N_{0,2} = \frac{1}{\lambda}\Big[&B_2\|\psi\|_1 + l_2b_2\mathbf{B}(\beta, k_2+1)t_{p+1}^{\beta+k_2} \\
&+ l_2b_2\mathbf{B}(\beta, k_2+1)\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} t_k^{\beta+k_2}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ \prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} B_{2,k}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} + l_2b_2\mathbf{B}(\beta, k_2+1)\Big], \\
N_{0,3} = \frac{1}{\lambda}\Big[&A_2\|\psi\|_1 + l_2a_2\mathbf{B}(\beta, k_2+1)t_{p+1}^{\beta+k_2} \\
&+ l_2a_2\mathbf{B}(\beta, k_2+1)\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} t_k^{\beta+k_2}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ \prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\sum_{k=1}^{p} A_{2,k}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} + l_2a_2\mathbf{B}(\beta, k_2+1)\Big],
\end{aligned}$$

$$\begin{aligned}
N_{1,1} &= C_{2,1} + l_2c_2t_1^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_1 - t_0)^{\beta-1}N_{0,1}, \\
N_{1,2} &= B_{2,1} + l_2b_2t_1^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_1 - t_0)^{\beta-1}N_{0,2}, \\
N_{1,3} &= A_{2,1} + l_2a_2t_1^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_1 - t_0)^{\beta-1}N_{0,3}, \\
&\dots \\
N_{p-1,1} &= C_{2,p-1} + l_2c_2t_{p-1}^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_{p-1} - t_{p-2})^{\beta-1}N_{p-2,1}, \\
N_{p-1,2} &= B_{2,p-1} + l_2b_2t_{p-1}^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_{p-1} - t_{p-2})^{\beta-1}N_{p-2,2}, \\
N_{p-1,3} &= A_{2,p-1} + l_2a_2t_{p-1}^{\beta+k_2}\mathbf{B}(\beta, k_2+1) + (t_{p-1} - t_{p-2})^{\beta-1}N_{p-2,3},
\end{aligned}$$

$$\begin{aligned}
N_{p,1} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\|\psi\|_1 + l_2c_2\mathbf{B}(\beta, k_2+1)t_{p+1}^{\beta+k_2}\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1} \\
&+ \sum_{k=1}^{p} C_{2,k}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ l_2c_2\mathbf{B}(\beta, k_2+1)\sum_{k=1}^{p} t_k^{\beta+k_2}\prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1}\Big),
\end{aligned}$$

$$
\begin{aligned}
N_{p,2} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\|\psi\|_1 + l_2 b_2 \mathbf{B}(\beta, k_2+1) t_{p+1}^{\beta+k_2} \prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1} \\
&+ \sum_{k=1}^{p} B_{2,k} \prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ l_2 b_2 \mathbf{B}(\beta, k_2+1) \sum_{k=1}^{p} t_k^{\beta+k_2} \prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1}\Big), \\
N_{p,3} = \frac{1}{\lambda}\Big(&\prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1}\|\psi\|_1 + l_2 a_2 \mathbf{B}(\beta, k_2+1) t_{p+1}^{\beta+k_2} \prod_{k=1}^{p}(t_k - t_{k-1})^{\beta-1} \\
&+ \sum_{k=1}^{p} A_{2,k} \prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1} \\
&+ l_2 a_2 \mathbf{B}(\beta, k_2+1) \sum_{k=1}^{p} t_k^{\beta+k_2} \prod_{s=k+1}^{p}(t_s - t_{s-1})^{\beta-1}\Big),
\end{aligned}
$$

and

$$
\begin{gathered}
\mu_{k,1} = l_2 c_2 (t_{k+1} - t_k)^{1-\beta} t_{k+1}^{\alpha+k_1} \mathbf{B}(\beta, k_2+1) + N_{k,1}, \quad k = 0, 1, \dots, p, \\
\mu_{k,2} = l_2 b_2 (t_{k+1} - t_k)^{1-\beta} t_{k+1}^{\alpha+k_1} \mathbf{B}(\beta, k_2+1) + N_{k,2}, \quad k = 0, 1, \dots, p, \\
\mu_{k,3} = l_2 a_2 (t_{k+1} - t_k)^{1-\beta} t_{k+1}^{\alpha+k_1} \mathbf{B}(\beta, k_2+1) + N_{k,3}, \quad k = 0, 1, \dots, p, \\
\mu_1 = \max\{\mu_{k,1} : k = 0, 1, \dots, p\}, \\
\mu_2 = \max\{\mu_{k,2} : k = 0, 1, \dots, p\}, \\
\mu_3 = \max\{\mu_{k,3} : k = 0, 1, \dots, p\}.
\end{gathered}
$$

Theorem 3.1. *Suppose that both* (A) *and* (B) *hold. Let* μ_2, μ_3 *and* σ_2, σ_3 *be defined above. Then* (1.2) *has at least one solution if*

$$
\sigma_2 < 1, \quad \mu_2 \frac{1}{w((1-\sigma_2)/(2\sigma_3))} + \mu_3 < 1. \tag{3.1}
$$

Proof. To apply Lemma 2.1, we should define an open bounded subset Ω of E centered at zero such that all assumptions in Lemma 2.1 hold. To obtain Ω.

Let $\Omega_1 = \{(x,y) \in E \cap D(L) \setminus \ker L,\ L(x,y) = \theta N(x,y) \text{ for some } \theta \in (0,1)\}$. We will prove that Ω_1 is bounded.

For $(x,y) \in \Omega_1$, we obtain $L(x,y) = \theta N(x,y)$ and $N(x,y) \in \operatorname{Im} L$. Then

$$
\begin{gathered}
D_{t_k^+}^{\alpha} x(t) = \theta m(t) f(t, x(t), y(t)), \\
D_{t_k^+}^{\beta} y(t) = \theta n(t) g(t, x(t), y(t)), \\
\lim_{t\to 1} t^{1-\alpha} x(t) + \lim_{t\to 0} t^{1-\alpha} x(t) = \theta \int_0^1 \phi(t) F(t, x(t), y(t))\, dt, \\
\lim_{t\to 1} t^{1-\beta} y(t) + \lim_{t\to 0} t^{1-\beta} y(t) = \theta \int_0^1 \psi(t) G(t, x(t), y(t))\, dt, \\
\lim_{t\to t_k^+} (t - t_k)^{1-\alpha} u(t) - u(t_k) = \theta I_k(t_k, u(t_k), v(t_k)), k = 1, 2, \dots, p, \\
\lim_{t\to t_k^+} (t - t_k)^{1-\beta} v(t) - v(t_k) = \theta J_k(t_k, u(t_k), v(t_k)), k = 1, 2, \dots, p.
\end{gathered} \tag{3.2}
$$

So

$$x(t)=\theta\int_{t_k}^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m(s)f(s,x(s),y(s))ds+\theta(t-t_k)^{\alpha-1}M_k,t\in(t_k,t_{k+1}],\tag{3.3}$$

$$y(t)=\theta\int_{t_k}^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}n(s)g(s,x(s),y(s))ds+\theta(t-t_k)^{\alpha-1}N_k,t\in(t_k,t_{k+1}],\tag{3.4}$$

for $k=0,1,\dots,p$. Here M_k,N_k $(k=0,1,\dots,p)$ are given in Step (iv) in the proof of Lemma 2.2.

By the definition of M_k, we have

$$\begin{aligned}
&|M_0|\\
&\le\frac{1}{\lambda}\Big(\int_0^1|\phi(s)F(s,x(s),y(s))|ds+\int_{t_p}^{t_{p+1}}\frac{(t_{p+1}-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds\\
&\quad+\prod_{k=1}^p(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^p\prod_{s=k+1}^p(t_s-t_{s-1})^{\alpha-1}\\
&\quad\times\Big(|I_k(t_k,x(t_k),y(t_k))|+\int_{t_{k-1}}^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}|m(s)f(s,x(s),y(s))|ds\Big)\\
&\le\frac{1}{\lambda}\Big[C_1\|\phi\|_1+l_1c_1t_{p+1}^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1)\\
&\quad+\prod_{k=1}^p(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^pC_{1,k}\prod_{s=k+1}^p(t_s-t_{s-1})^{\alpha-1}+l_1c_1t_k^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1)\Big]\\
&\quad+\frac{1}{\lambda}\Big[B_1\|\phi\|_1+l_1b_1\mathbf{B}(\alpha,k_1+1)\\
&\quad+\prod_{k=1}^p(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^pB_{1,k}\prod_{s=k+1}^p(t_s-t_{s-1})^{\alpha-1}+l_1b_1\mathbf{B}(\alpha,k_1+1)\Big]\|x\|\\
&\quad+\frac{1}{\lambda}\Big[B_1\|\phi\|_1+l_1b_1\mathbf{B}(\alpha,k_1+1)\\
&\quad+\prod_{k=1}^p(t_k-t_{k-1})^{\alpha-1}\sum_{k=1}^pB_{1,k}\prod_{s=k+1}^p(t_s-t_{s-1})^{\alpha-1}+l_1b_1\mathbf{B}(\alpha,k_1+1)\Big]\Phi^{-1}(\|y\|)\\
&=M_{0,1}+M_{0,2}\|x\|+M_{0,3}\Phi^{-1}(\|y\|).
\end{aligned}$$

Similarly,

$$\begin{gathered}
|M_1|\le M_{1,1}+M_{1,2}\|x\|+M_{1,3}\Phi^{-1}(\|y\|),\\
\dots\\
|M_{p-1}|\le M_{p-1,1}+M_{p-1,2}\|x\|+M_{p-1,3}\Phi^{-1}(\|y\|),\\
|M_p|\le M_{p,1}+M_{p,2}\|x\|+M_{p,3}\Phi^{-1}(\|y\|).
\end{gathered}$$

Similarly, we can prove that

$$\begin{gathered}
|N_0|\le N_{0,1}+N_{0,2}\Phi(\|x\|)+N_{0,3}\|y\|,\\
|N_1|\le N_{1,1}+N_{1,2}\Phi(\|x\|)+N_{1,3}\|y\|,\\
\dots
\end{gathered}$$

$$\begin{gathered}|N_{p-1}| \le N_{p-1,1} + N_{p-1,2}\Phi(\|x\|) + N_{p-1,3}\|y\|,\\ |N_p| \le N_{p,1} + N_{p,2}\Phi(\|x\|) + N_{p,3}\|y\|.\end{gathered}$$

First, using (3.3) for $t \in (t_0, t_1]$, we have

$$\begin{aligned}(t-t_0)^{1-\alpha}|x(t)| &\le \Big|(t-t_0)^{1-\alpha}\int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds + M_0\Big|\\ &\le l_1(t_1-t_0)^{1-\alpha} t_1^{\alpha+k_1}\mathbf{B}(\alpha,k_1+1)(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)),\end{aligned}$$

$$M_{0,1} + M_{0,2}\|x\| + M_{0,3}\Phi^{-1}(\|y\|) \le \sigma_{0,1} + \sigma_{0,2}\|x\| + \sigma_{0,3}\Phi^{-1}(\|y\|).$$

For $k = 1, 2, \dots, p-1$, we have

$$\begin{aligned}&(t-t_k)^{1-\alpha}|x(t)|\\ &\le \Big|(t-t_k)^{1-\alpha}\int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds + M_k\Big|\\ &\le l_1(t-t_k)^{1-\alpha}\int_{t_k}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}ds(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)) + |M_k|\\ &\le \sigma_{k,1} + \sigma_{k,2}\|x\| + \sigma_{k,3}\Phi^{-1}(\|y\|).\end{aligned}$$

For $t \in (t_p, t_{p+1}]$, we have

$$\begin{aligned}&(t-t_p)^{1-\alpha}|x(t)|\\ &\le \Big|(t-t_p)^{1-\alpha}\int_{t_p}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m(s)f(s,x(s),y(s))ds + M_p\Big|\\ &\le l_1(t-t_p)^{1-\alpha}\int_{t_p}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{k_1}ds(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)) + |M_p|\\ &\le \sigma_{p,1} + \sigma_{p,2}\|x\| + \sigma_{p,3}\Phi^{-1}(\|y\|).\end{aligned}$$

It follows that

$$\|x\| \le \sigma_1 + \sigma_2\|x\| + \sigma_3\Phi^{-1}(\|y\|). \tag{3.5}$$

Similarly, we can show that

$$\|y\| \le \mu_1 + \mu_2\Phi(\|x\|) + \mu_3\|y\|. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\|y\| \le \mu_1 + \mu_2\Phi\Big(\frac{\sigma_1}{1-\sigma_2} + \frac{\sigma_3\Phi^{-1}(\|y\|)}{1-\sigma_2}\Big) + \mu_3\|y\|.$$

Without loss of generality, assume that $\|y\| > \Phi(\frac{\sigma_1}{\sigma_3})$. Then (2.1) implies that

$$\begin{aligned}\|y\| &\le \mu_1 + \mu_2\Phi\Big(\frac{2\sigma_3\Phi^{-1}(\|y\|)}{1-\sigma_2}\Big) + \mu_3\|y\|\\ &\le \mu_1 + \mu_2\frac{\Phi\big(\Phi^{-1}(\|y\|)\big)}{w((1-\sigma_2)/(2\sigma_3))} + \mu_3\|y\|\\ &= \mu_1 + \Big(\mu_2\frac{1}{w((1-\sigma_2)/(2\sigma_3))} + \mu_3\Big)\|y\|.\end{aligned}$$

It follows that

$$\|y\| \le \frac{\mu_1}{1-\big(\mu_2\frac{1}{w((1-\sigma_2)/(2\sigma_3))} + \mu_3\big)}.$$

Then

$$\|x\| \le \sigma_1 + \sigma_2\|x\| + \sigma_3\Phi^{-1}\Big(\frac{\mu_1}{1-\big(\mu_2\frac{1}{w((1-\sigma_2)/(2\sigma_3))}+\mu_3\big)}\Big).$$

It follows that Ω_1 is bounded.

Now we show that all assumptions of Lemma 2.1 are satisfied. Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \overline{\Omega_1}$. By Lemma 2.2, L is a Fredholm operator of index zero, $\ker L = \{0 \in E\}$ and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have $Lx \neq \theta Nx$ for $x \in (D(L)\cap\partial\Omega$ and $\theta \in (0,1)$. Thus by Lemma 2.1, $L(x,y) = N(x,y)$ has at least one solution in $D(L)\cap\overline{\Omega}$. Then x is a solution of (1.2). The proof is complete. □

As an application of Theorem 3.1, we give the following theorem, under the assumption

(B') f, g, F, G, I_k, J_k $(k = 1,2,\dots,p)$ are impulsive Caratheodory functions and satisfy that there exist nonnegative constants $c_i, b_i, a_i (i = 1,2)$, C_i, B_i, A_i $(i = 1,2)$ and $C_{i,k}, B_{i,k}, A_{i,k}$ $(i = 1,2, k = 1,2,\dots,p)$ such that

$$\begin{gathered}
|f(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le c_1 + b_1|x| + a_1|y|,\\
t \in (t_k,t_{k+1}],\ k = 0,1,\dots,p,\\
|g(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le c_2 + b_2|x| + a_2|y|,\\
t \in (t_k,t_{k+1}],\ k = 0,1,\dots,p,\\
|F(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le C_1 + B_1|x| + A_1|y|,\\
t \in (t_k,t_{k+1}],\ k = 0,1,\dots,p,\\
|G(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \le C_2 + B_2|x| + A_2|y|,\\
t \in (t_k,t_{k+1}],\ k = 0,1,\dots,p,\\
|I_k(t,(t_{k+1}-t_k)^{\alpha-1}x,(t_{k+1}-t_k)^{\beta-1}y)| \le C_{1,k} + B_{1,k}|x| + A_{1,k}|y|,\\
k = 1,2,\dots,p,\\
|J_k(t,(t_{k+1}-t_k)^{\alpha-1}x,(t_{k+1}-t_k)^{\beta-1}y)| \le C_{2,k} + B_{2,k}|x| + A_{2,k}|y|,\\
k = 1,2,\dots,p.
\end{gathered}$$

Theorem 3.2. *Assume that* (B') *holds. Let μ_2, μ_3 and σ_2, σ_3 be defined at the beginning of this section. Then* (1.2) *has at least one solution if*

$$\sigma_2 < 1, \quad \mu_2\frac{2\sigma_3}{1-\sigma_2} + \mu_3 < 1.$$

For the proof of the above theorem, choose $\Phi(x) = x$ and then we obtain $\Phi^{-1}(x) = x$. The proof follows from Theorem 3.1 and is omitted.

4. An example

Now, we present an example that illustrates Theorem 3.1, and can not be covered by known results. Consider the boundary-value problem for the impulsive fractional differential equation

$$\begin{gathered}
D_{t_k^+}^{\frac{2}{3}}u(t) = t^{-1/4}f(t,u(t),v(t)), \quad t \in (t_k,t_{k+1}], k=0,1,\\
D_{t_k^+}^{1/2}v(t) = t^{-1/4}g(t,u(t),v(t)), \quad t \in (t_k,t_{k+1}], k=0,1,\\
\lim_{t\to 1}u(t) + \lim_{t\to 0}t^{1/3}u(t) = 0,\\
\lim_{t\to 1}v(t) + \lim_{t\to 0}t^{1/2}v(t) = 0,\\
\lim_{t\to \frac{1}{2}^+}(t-\frac{1}{2})^{1/3}u(t) - u(1/2) = 0,\\
\lim_{t\to \frac{1}{2}^+}(t-\frac{1}{2})^{1/2}v(t) - v(1/2) = 0,
\end{gathered} \tag{4.1}$$

where

$$\begin{gathered}
f(t,x,y) = \begin{cases} c_1 + b_1t^{-\frac{1}{3}}x + a_1t^{-3/2}y^3, & t \in (0,1/2],\\ c_1 + b_1(t-1/2)^{-\frac{1}{3}}x + a_1(t-1/2)^{-3/2}y^3, & t \in (1/2,1],\end{cases}\\
g(t,x,y) = \begin{cases} c_2 + b_2t^{-\frac{1}{3}}x^{1/3} + a_2t^{-3/2}y, & t \in (0,1/2],\\ c_2 + b_2(t-1/2)^{-\frac{1}{9}}x^{1/3} + a_2(t-1/2)^{-1/2}y, & t \in (1/2,1]\end{cases}
\end{gathered}$$

with $c_i,b_i,a_i \ge 0 (i=1,2)$ and $0=t_0<t_1=\frac{1}{2}<t_2=1$. Then (4.1) has at least one solution if

$$\begin{gathered}
2^{1/3}\mathbf{B}(2/3,3/4)b_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1 < 1,\\
\Big(2^{3/4}\mathbf{B}(1/4,3/4)b_2 + \frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)b_2\Big)\\
\times\Big(\frac{2^{7/3}\mathbf{B}(2/3,3/4)a_1 + \frac{2}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)a_1}{1-2^{1/3}\mathbf{B}(2/3,3/4)b_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1}\Big)^{1/3}\\
+ 2^{3/4}\mathbf{B}(1/4,3/4)a_2 + \frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)a_2 < 1.
\end{gathered} \tag{4.2}$$

Proof. Corresponding to (1.2), $\alpha=2/3$, $\beta=1/2$, $p=1$, $t_1=1/2$,

$$\begin{gathered}
m(t)=t^{-1/4}, \quad n(t)=t^{-1/4},\\
f\Big(t,(t-t_k)^{1/3}x,(t-t_k)^{1/2}y\Big) = c_1+b_1x+a_1y^3, \quad k=0,1,\\
g\Big(t,(t-t_k)^{1/3}x,(t-t_k)^{1/2}y\Big) = c_2+b_2x^{1/3}+a_2y, \quad k=0,1,\\
F\Big(t,(t-t_k)^{1/3}x,(t-t_k)^{1/2}y\Big) = \phi(t)=0, \quad k=0,1,\\
G(t,(t-t_k)^{1/3}x,(t-t_k)^{1/2}y) = \psi(t)=0, \quad k=0,1,\\
I_1(t_1,(t_2-t_1)^{1/3}x,(t_2-t_1)^{1/2}y)=0,\\
J_1(t_1,(t_2-t_1)^{1/3}x,(t_2-t_1)^{1/2}y)=0.
\end{gathered}$$

For $\Phi(x)=x^{1/3}$ with $\Phi^{-1}(x)=x^3$, the supporting function of Φ is $\omega(x)=x^{1/3}$ and the supporting function of Φ^{-1} is $\nu(x)=x^3$. It is easy to see that $m(t)\le l_1t^{k_1}$

with $l_1 = 1$ and $k_1 = -1/4$, $n(t) \le l_2 t^{k_2}$ with $l_2 = 1$ and $k_2 = -1/4$, $C_1 = B_1 = A_1 = C_2 = B_2 = A_2 = 0$, $C_{1,1} = B_{1,1} = A_{1,1} = C_{2,1} = B_{2,1} = A_{2,1} = 0$.

By direct computations, we show that

$$\lambda = 1 + \prod_{k=1}^{p+1} (t_k - t_{k-1})^{\alpha-1} = 1 + \sqrt[3]{4},$$
$$M_{0,1} = \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]c_1,$$
$$M_{0,2} = \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]b_1,$$
$$M_{0,3} = \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]a_1,$$
$$M_{1,1} = \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)c_1,$$
$$M_{1,2} = \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1,$$
$$M_{1,3} = \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)a_1$$

and

$$\sigma_{0,1} = 2^{-1/12}\mathbf{B}(2/3,3/4)c_1 + \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]c_1,$$
$$\sigma_{0,2} = 2^{-1/12}\mathbf{B}(2/3,3/4)b_1 + \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]b_1,$$
$$\sigma_{0,3} = 2^{-1/12}\mathbf{B}(2/3,3/4)a_1 + \frac{1}{1+\sqrt[3]{4}}[(1+2^{-1/12})\mathbf{B}(2/3,3/4)+1]a_1,$$
$$\sigma_{1,1} = 2^{1/3}\mathbf{B}(2/3,3/4)c_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)c_1,$$
$$\sigma_{1,2} = 2^{1/3}\mathbf{B}(2/3,3/4)b_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1,$$
$$\sigma_{1,3} = 2^{1/3}\mathbf{B}(2/3,3/4)a_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)a_1,$$
$$\begin{aligned}\sigma_1 &= \max\{\sigma_{k,1} : k = 0,1\}\\ &= 2^{1/3}\mathbf{B}(2/3,3/4)c_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)c_1,\\ \sigma_2 &= \max\{\sigma_{k,2} : k = 0,1\}\\ &= 2^{1/3}\mathbf{B}(2/3,3/4)b_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1,\\ \sigma_3 &= \max\{\sigma_{k,3} : k = 0,1\}\\ &= 2^{1/3}\mathbf{B}(2/3,3/4)a_1 + \frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)a_1.\end{aligned}$$

Denote

$$N_{0,1} = \frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)c_2,$$

$$
\begin{aligned}
N_{0,2} &= \frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)b_2,\\
N_{0,3} &= \frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)a_2,\\
N_{1,1} &= \frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)c_2,\\
N_{1,2} &= \frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)b_2,\\
N_{1,3} &= \frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)a_2
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_{0,1} &= 2^{3/4}\mathbf{B}(1/4,3/4)c_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)c_2,\\
\mu_{0,2} &= 2^{3/4}\mathbf{B}(1/4,3/4)b_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)b_2,\\
\mu_{0,3} &= 2^{3/4}\mathbf{B}(1/4,3/4)a_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)a_2,\\
\mu_{1,1} &= 2^{1/4}\mathbf{B}(1/4,3/4)c_2+\frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)c_2,\\
\mu_{1,2} &= 2^{1/4}\mathbf{B}(1/4,3/4)b_2+\frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)b_2,\\
\mu_{1,3} &= 2^{1/4}\mathbf{B}(1/4,3/4)a_2+\frac{1}{1+\sqrt[3]{4}}[2^{1/2}+2^{1/4}]\mathbf{B}(1/2,3/4)a_2,\\
\mu_1 = \max\{\mu_{k,1}:k=0,1\} &= 2^{3/4}\mathbf{B}(1/4,3/4)c_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)c_2,\\
\mu_2 = \max\{\mu_{k,2}:k=0,1\} &= 2^{3/4}\mathbf{B}(1/4,3/4)b_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)b_2,\\
\mu_3 = \max\{\mu_{k,3}:k=0,1\} &= 2^{3/4}\mathbf{B}(1/4,3/4)a_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)a_2.
\end{aligned}
$$

Then Theorem 3.1 implies that (4.1) has at least one solution if (4.2) holds. The proof is complete. □

Remark 4.1. Since

$$
\lim_{b_1\to 0}\big[2^{1/3}\mathbf{B}(2/3,3/4)b_1+\frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1\big]=0,
$$

and

$$
\begin{aligned}
&\lim_{a_1,b_1,a_2,b_2\to 0}\Big[\Big(2^{3/4}\mathbf{B}(1/4,3/4)b_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)b_2\Big)\\
&\times\Big(\frac{2^{\frac{7}{3}}\mathbf{B}(2/3,3/4)a_1+\frac{2}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)a_1}{1-2^{1/3}\mathbf{B}(2/3,3/4)b_1+\frac{1}{1+\sqrt[3]{4}}[2^{1/3}+2^{-5/12}]\mathbf{B}(2/3,3/4)b_1}\Big)^{1/3}\\
&+2^{3/4}\mathbf{B}(1/4,3/4)a_2+\frac{1}{1+\sqrt[3]{4}}[2+2^{3/4}]\mathbf{B}(1/2,3/4)a_2\Big]=0,
\end{aligned}
$$

we can see that (4.2) holds for sufficiently small b_1, a_1, b_2, a_2. Then (4.1) has at least one solution for sufficiently small b_1, a_1, b_2, a_2.

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