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SIMULTANEOUS AND NON-SIMULTANEOUS BLOW-UP AND UNIFORM BLOW-UP PROFILES FOR REACTION-DIFFUSION SYSTEM

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Abstract. This article concerns the blow-up solutions of a reaction-diffusion system with nonlocal sources, subject to the homogeneous Dirichlet boundary conditions. The criteria used to identify simultaneous and non-simultaneous blow-up of solutions by using the parameters p and q in the model are proposed. Also, the uniform blow-up profiles in the interior domain are established.

1. Introduction and description of results

In this article, we investigate the following reaction-diffusion system with nonlocal sources

$$u_t = \Delta u + \|uv\|_\alpha^p, \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$v_t = \Delta v \ + \|uv\|_\beta^q, \quad (x,t) \in \Omega \times (0,T) \tag{1.2}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.3}$$

$$u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.4}$$

where $\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^N$ $(N \geq 1)$, $\alpha, \beta \geq 1$, $p, q > 0$, and the continuous functions $u_0(x), v_0(x)$ are nonnegative, nontrivial, radially symmetric, decreasing with $|x|$, and vanish on ∂B_R, where $\| \cdot \|_\alpha^\alpha = \int_\Omega |\cdot|^\alpha \, \mathrm{d}x$.

Nonlinear parabolic systems (1.1)-(1.4) can be used to describe some reaction diffusion phenomena, Such as heat propagations in a two-component combustible mixture [3], chemical reactions [6], interaction of two biological groups without self-limiting [10], etc., where u and v represent the temperatures of two different materials during a propagation, the thicknesses of two kinds of chemical reactants, the densities of two biological groups during a migration, etc. Using the methods of [7, 12, 4] we know that (1.1)-(1.4) has a local nonnegative classical solution. Moreover, if $p, q \geq 1$, then the uniqueness holds.

In recent years, many results on blow-up solutions have been obtained for the nonlinear parabolic system. We will recall several results in the following. As for the other related works on the global existence and blow-up of solutions of the nonlinear parabolic system, they can be found in [15, 1, 5, 14] and references therein.

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Li, Huang and Xie in [8] and Deng, Li and Xie in [2] considered the following two systems, respectively,

$$u_t = \Delta u + \int_\Omega u^m(x,t)v^n(x,t)\,\mathrm{d}x, \quad v_t = \Delta v + \int_\Omega u^p(x,t)v^q(x,t)\,\mathrm{d}x,$$

with $x \in \Omega$, $t > 0$; and

$$u_t = \Delta u^m + a\|v\|_\alpha^p, \quad v_t = \Delta v^n + b\|u\|_\beta^q, \quad (x,t) \in \Omega \times (0,T).$$

The authors showed some results on the global solutions, the blow-up solutions and the blow-up profiles. In 2002, Zheng, Zhao and Chen in [18] studied the problem

$$u_t = \Delta u + f_1(u,v), \quad v_t = \Delta v + f_2(u,v), \quad (x,t) \in \Omega \times (0,T) \tag{1.5}$$

with homogeneous Dirichlet boundary conditions, where

$$f_1(u,v) = e^{mu(x,t)+pv(x,t)}, \quad f_2(u,v) = e^{qu(x,t)+v(x,t)}.$$

The simultaneous blow-up rates are obtained for radially symmetric blow-up solutions in the exponent region $\{0 \le m < q, 0 \le n < p\}$.

Later, Zhao and Zheng in [17], Li and Wang in [9] studied the localized problem (1.5) with the more general $\Omega \subset \mathbb{R}^N$ and

$$f_1(u,v) = e^{mu(x_0,t)+pv(x_0,t)}, \quad f_2(u,v) = e^{qu(x_0,t)+nv(x_0,t)}, \quad x_0 \in \Omega.$$

The critical blow-up exponents were discussed. Uniform blow-up profiles for simultaneous blow-up solutions were proved in the exponent region $\{0 \le m \le q, 0 \le n \le p\}$.

Our present work is motivated by the above mentioned papers, the main purpose of this paper is to identify the simultaneous and non-simultaneous blow-up of the solutions and establish the uniform blow-up profiles for the system (1.1)–(1.4).

For convenience, we introduce a pair of parameters σ and θ, the solution of

$$\begin{pmatrix} p-1 & p \\ q & q-1 \end{pmatrix} \begin{pmatrix} \sigma \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.6}$$

namely,

$$\sigma = \frac{p-(q-1)}{p+q-1}, \quad \theta = \frac{q-(p-1)}{p+q-1}\,. \tag{1.7}$$

This paper is organized as follows. In the next Section, we investigate the simultaneous and non-simultaneous blow-up of the solutions for the system (1.1)–(1.4), and give the blow-up criteria. In Section 3, we deal with the blow-up rates of the solutions.

2. Simultaneous and non-simultaneous blow-up

In this section, we discuss the simultaneous and non-simultaneous blow-up phenomena for the system (1.1)–(1.4), and propose a complete and optimal classification to identify the simultaneous and non-simultaneous blow-up solutions.

For problem (1.1)-(1.4), because of the nonlinear sources, there exist solution (u,v) that blow up in finite time, T, if and only if the exponents p, q verify any of conditions, $p > 1, q > 1$ or $pq > (q-1)(p-1)$. In particular, the component u(or v) can blow up for the large initial data if $p > q-1$(or $q > p-1$), see [9, 12]. So there may be non-simultaneous blow-up, that is to say that one component blows up while the other remains bounded. On the other hand, the simultaneous blow-up means that

$$\limsup_{t\to T}\|u(\cdot,t)\|_\infty=\limsup_{t\to T}\|v(\cdot,t)\|_\infty=+\infty.$$

Assume the initial data $u_0(x), v_0(x)$ satisfy

$$\Delta u_0(x)+\|u_0v_0\|_\alpha^p-\varepsilon\varphi(x)u_0^p(0)v_0^p(0)\ge 0,\quad x\in B_R, \tag{2.1}$$

$$\Delta v_0(x)+\|u_0v_0\|_\beta^q-\varepsilon\varphi(x)u_0^q(0)v_0^q(0)\ge 0,\quad x\in B_R \tag{2.2}$$

for some a constant $\varepsilon\in(0,1)$, where $\varphi(x)$ is the first eigenfunction of

$$-\Delta\varphi=\lambda\varphi,\ x\in B_R;\quad \varphi=0,\ x\in\partial B_R,$$

normalized by $\|\varphi\|_\infty=1,\ \varphi>0$ in B_R. In addition, by using the methods in [16], it is easy to check that $u_t, v_t\ge 0$ for $(x,t)\in B_R\times(0,T)$ by the comparison principle.

Our results about the simultaneous and non-simultaneous blow-up criteria are as follows.

Theorem 2.1. *If $p+q>1$, then there exists initial data such that the non-simultaneous blow-up occurs in* (1.1)–(1.4) *if and only if $\sigma<0$ (or $\theta<0$) (for v(or u) blowing up alone, respectively).*

Theorem 2.2. *If $p+q>1$, then any blow-up in* (1.1)–(1.4) *is non-simultaneous if and only if $\sigma\ge 0$ with $\theta<0$ (for u blowing up alone), or $\theta\ge 0$ with $\sigma<0$ (for v blowing up alone).*

Corollary 2.3. *If $p+q>1$, then any blow-up in* (1.1)–(1.4) *is simultaneous if and only if $\sigma\ge 0$ and $\theta\ge 0$.*

Similar to the study in[8], it is seen that

Corollary 2.4. *All solutions are global in* (1.1)–(1.4) *if and only if $\sigma<0$ and $\theta<0$(i.e., $p+q<1$).*

In summary, the complete and optimal classification for simultaneous and non-simultaneous blow-up solutions of the problem (1.1)-(1.4) can be described by Figure 1

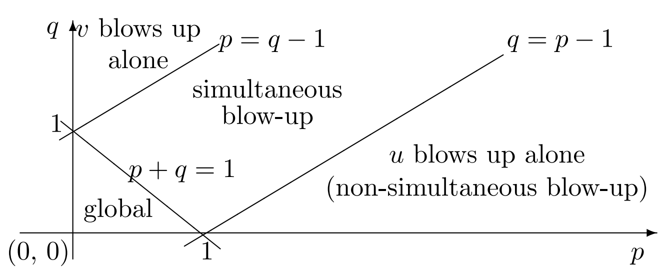

Figure 1. Regions of simultaneous and non-simultaneous blow-up

The key clues for the classification of simultaneous and non-simultaneous blow-up solutions are the signs of $p-(q-1)$, $q-(p-1)$ and $p+q-1$. The conditions $p>q-1$ and $p+q>1$ imply that u may blow up by itself but cannot provide sufficient help to the blow-up of v (with small v_0), while $q < p-1$ ensures that v can provide effective help to the blow-up of u, but v remains bounded.

Before we give the proof of Theorem 2.1, we first introduce the following lemma. Let $\phi(x,t)$ satisfy

$$\phi_t = \Delta\phi,\ (x,t) \in B_R \times (0,T); \quad \phi = 0,\ (x,t) \in \partial B_R \times (0,T)$$

with

$$\phi(x,0) = \varphi(x), \quad x \in B_R.$$

Lemma 2.5. *Under conditions* (2.1) *and* (2.2)*, the solution* (u,v) *of* (1.1)–(1.4) *satisfies*

$$u_t(x,t) \ge \varepsilon\phi(x,t)u^p(0,t)v^p(0,t), \quad (x,t) \in B_R \times [0,T), \tag{2.3}$$

$$v_t(x,t) \ge \varepsilon\phi(x,t)u^q(0,t)v^q(0,t), \quad (x,t) \in B_R \times [0,T). \tag{2.4}$$

Proof. Since that the proofs of the inequalities (2.3) and (2.4) are similar, we prove only (2.3). Let

$$J(x,t) = u_t(x,t) - \varepsilon\phi(x,t)u^p(0,t)v^p(0,t).$$

It is easy to check that for ε small enough since $u_t, v_t \ge 0$, we obtain

$$\begin{gathered} J_t - \Delta J = \big(\|uv\|_\alpha^p\big)_t - \varepsilon\phi\big(u^p(0,t)v^p(0,t)\big)_t \ge 0, \quad (x,t) \in B_R \times (0,T), \\ J(x,t) = 0, \quad (x,t) \in \partial B_R \times (0,T), \\ J(x,0) = \Delta u_0(x) + \|u_0v_0\|_\alpha^p - \varepsilon\varphi(x)u_0^p(0)v_0^p(0) \ge 0, \quad x \in B_R. \end{gathered}$$

Consequently, (2.3) is true by the comparison principle. □

Proof of Theorem 2.1. Without loss of generality, we only prove that there exist suitable initial data such that u blows up while v remains bounded if and only if $\theta < 0$.

Assume $\theta < 0$, namely, $p-1 > q$ and $p > 1$ by Figure 1 and (1.7). From (2.3), we obtain that

$$u_t(0,t) \ge \varepsilon\phi(0,T)u^p(0,t)v_0^p(0), \quad t \in [0,T). \tag{2.5}$$

Integrating the above inequality (2.5) from t to T, we have the estimate for u as follows

$$u(0,t) \le \Big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\Big)^{-1/(p-1)}(T-t)^{-1/(p-1)}, \quad t \in [0,T). \tag{2.6}$$

At the same time, since the initial data (u_0,v_0) is radially symmetric and non-increasing, therefore the (u,v) is also radial symmetrical and non-increasing; i.e., $u_r(r,t), v_r(r,t) \le 0$ for $r \in [0,R)$. Thus, $u(x,t)$ and $v(x,t)$ always reach their maxima at $x = 0$, which means that

$$\Delta u(0,t) \le 0, \quad \Delta v(0,t) \le 0.$$

Hence, from (1.1) and (1.2), we know that there exist constants $C_1, C_2 > 0$ such that

$$\begin{gathered} u_t(0,t) \le \|uv\|_\alpha^p \le C_1u^p(0,t)v^p(0,t), \quad t \in [0,T) \\ v_t(0,t) \le \|uv\|_\beta^q \le C_2u^q(0,t)v^q(0,t), \quad t \in [0,T). \end{gathered} \tag{2.7}$$

Let

$$\Gamma(x,y,t,s) = \frac{1}{[4\pi(t-s)]^{N/2}}\exp\Big\{-\frac{|x-y|^2}{4(t-s)}\Big\}$$

be the fundamental solution of the heat equation. Suppose that $(\tilde{u}_0, \tilde{v}_0)$ is a pair of initial data such that the solution of (1.1)–(1.4) blows up. Fix radially symmetrical

$v_0(\geq \tilde v_0)$ in B_R and take constant $M_1 > v_0(x)$. By the proof of [11, Theorem 1.1], we know that if u_0 is large with v_0 fixed then T becomes small. Therefore, let $u_0(\geq \tilde u_0)$ be large such that T becomes small and satisfies

$$M_1 \geq v_0(0) + \frac{p-1}{p-1-q}\big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\big)^{-\frac{q}{p-1}} T^{\frac{p-1-q}{p-1}} \|M_1\|_\beta^q,$$

where $\|M_1\|_\beta^q = (\int_\Omega M_1^\beta \,\mathrm{d}x)^{q/\beta}$. Consider the following auxiliary problem

$$\begin{gathered}\bar v_t = \Delta \bar v + \big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\big)^{-\frac{q}{p-1}}(T-t)^{-\frac{q}{p-1}}\|M_1\|_\beta^q, \quad (x,t)\in B_R\times(0,T),\\ \bar v(x,t) = 0, \quad (x,t)\in \partial B_R\times(0,T),\\ \bar v(x,0) = v_0(x), \quad x\in B_R.\end{gathered}$$

Since $p-1>q$, we obtain by Green's identity that

$$\bar v \leq v_0(0) + \frac{p-1}{p-1-q}\Big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\Big)^{-\frac{q}{p-1}} T^{\frac{p-1-q}{p-1}} \|M_1\|_\beta^q \leq M_1,$$

and hence $\bar v$ satisfies

$$\bar v_t \geq \Delta \bar v + \big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\big)^{-\frac{q}{p-1}}(T-t)^{-\frac{q}{p-1}}\|\bar v(x,t)\|_\beta^q.$$

On the other hand, v satisfies

$$v_t \leq \Delta v + \big(\varepsilon(p-1)\phi(0,T)v_0^p(0)\big)^{-\frac{q}{p-1}}(T-t)^{-\frac{q}{p-1}}\|v(x,t)\|_\beta^q.$$

Therefore, by the comparison principle, we conclude $v \leq \bar v \leq M_1$.

Now assume that u blows up while v remains bounded. By (2.7) we have

$$u_t(0,t) \leq Cu^p(0,t), \quad \text{for } t\in[0,T).$$

This implies $p>1$ and the estimate for u that

$$u(0,t) \geq \big(C(p-1)\big)^{-1/(p-1)}(T-t)^{-1/(p-1)}.$$

Therefore, by using (2.4), we have

$$v_t(0,t) \geq \varepsilon\phi(0,T)\big(C(p-1)\big)^{-\frac{q}{p-1}} v_0^q(0)(T-t)^{-\frac{q}{p-1}}.$$

By integrating, we obtain that

$$v(0,t) \geq v_0(0) + \varepsilon\phi(0,T)\big(C(p-1)\big)^{-\frac{q}{p-1}} v_0^q(0)\int_0^t (T-s)^{-\frac{q}{p-1}}\,\mathrm{d}s. \tag{2.8}$$

The boundedness of v requires $p-1>q$ from (2.8), that is $\theta<0$. Thus, the proof is complete. □

Proof of Theorem 2.2. We only treat the case of u blowing up and v remains bounded.

Assume $\sigma \geq 0$ with $\theta < 0$; that is $p \geq q-1, q < p-1$ and $p>1$ by Figure 1 and (1.7). From (2.3) and (2.7), we have

$$v^{p-q}(0,t)v_t(0,t) \leq \frac{C_2}{\varepsilon\phi(0,T)}u^{q-p}(0,t)u_t(0,t), \quad t\in[0,T). \tag{2.9}$$

By Theorem 2.1, it is impossible for v blowing up alone under $\sigma \geq 0$ with $\theta<0$. Then we show that v is bounded. In fact, by integrating the inequality (2.9) from 0 to t, we have

$$v^{p-q+1}(0,t) \leq C - Cu^{-(p-q-1)}(0,t)$$

for some a $C>0$. Therefore, we can get the boundedness of $v(0,t)$.

Now, assume that any blow-up must be the case for u blowing up alone. This requires $\theta < 0$ by Theorem 2.1. Again by Theorem 2.1, if in addition $\sigma < 0$, there exists the initial data such that v blows up alone. Therefore, it has to be satisfied that $\sigma \geq 0$. Then, the proof is complete. □

3. Uniform Blow-up Profiles

In this section, we study the uniform blow-up profiles for system (1.1)–(1.4). At first, the following result of Souplet for a single diffusion equation with nonlocal nonlinear sources [13, Theorem 4.1] will play a basic role in our discussion.

Lemma 3.1. *Let $u \in C^{2,1}(\bar{\Omega} \times (0, T^*))$ be a solution of the problem*

$$\begin{gathered} u_t = \Delta u + g(t), \quad (x,t) \in \Omega \times (0, T^*), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0, T^*), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered}$$

where $g(t)$ is nonnegative and may depend on the solution u. Then

$$\lim_{t \to T^*} \|u(\cdot,t)\|_\infty = +\infty \tag{3.1}$$

if and only if $\int_0^t g(s)\,\mathrm{d}s = +\infty$. Furthermore, if (3.1) is fulfilled, then

$$\lim_{t \to T^*} \frac{u(x,t)}{G(t)} = \lim_{t \to T^*} \frac{\|u(\cdot,t)\|_\infty}{G(t)} = 1$$

uniformly on compact subsets of Ω, where $G(t) = \int_0^t g(s)\,\mathrm{d}s$.

For convenience, we denote

$$f(t) = \|uv\|_\alpha^p, \quad g(t) = \|uv\|_\beta^q, \quad F(t) = \int_0^t f(s)\,\mathrm{d}s, \quad G(t) = \int_0^t g(s)\,\mathrm{d}s.$$

According to the Lemma 3.1, we have the following result.

Lemma 3.2. *Assume $u, v \in C^{2,1}(\bar{\Omega} \times [0, T))$ are the solutions of (1.1)–(1.4). If u and v blow up simultaneously in the finite time T^*, then we have*

$$\lim_{t \to T^*} \frac{u(x,t)}{F(t)} = 1, \quad \lim_{t \to T^*} \frac{v(x,t)}{G(t)} = 1$$

uniformly on compact subsets of Ω, and

$$\lim_{t \to T^*} F(t) = \lim_{t \to T^*} G(t) = \infty.$$

We remark that if we assume that only u (or v) blows up in finite time T^*, then the above conclusions about u (or v) and F (or G) are also valid.

Throughout this section the notation $f(t) \sim g(t)$ is used to describe such functions $f(t)$ and $g(t)$ satisfying $f(t)/g(t) \to 1$ as $t \to T^*$. When u and v blow up simultaneously, we have the following results about the uniform blow-up profiles for u and v.

Theorem 3.3. *Let (u, v) be a solution of (1.1)–(1.4) with simultaneous blow-up time T^*. Then the following limits hold uniformly on any compact subset of Ω:*

(1) *If $\sigma > 0$ and $\theta > 0$, then*

$$\lim_{t \to T^*} u(x,t)(T^* - t)^\sigma = \Big(\frac{|\Omega|^{p/\alpha}}{\sigma}(|\Omega|^{\frac{q}{\beta} - \frac{p}{\alpha}}\frac{\sigma}{\theta})^{p/(p+1-q)}\Big)^{-\sigma}, \tag{3.2}$$

$$\lim_{t\to T^*} v(x,t)(T^*-t)^\theta = \Big(\frac{|\Omega|^{q/\beta}}{\theta}(|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{\theta}{\sigma})^{q/(q+1-p)}\Big)^{-\theta}. \tag{3.3}$$

(2) *If $\sigma = 0$, then*

$$\lim_{t\to T^*} u^2(x,t)|\ln(T^*-t)|^{-1} = \frac{2}{p}|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}, \tag{3.4}$$

$$\lim_{t\to T^*} v^p(x,t)\big(\ln v(x,t)\big)^{\frac{q}{2}}(T^*-t) = \frac{1}{p}|\Omega|^{-q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{-q/2}. \tag{3.5}$$

(3) *If $\theta = 0$, then we have*

$$\lim_{t\to T^*} u^q(x,t)\big(\ln u(x,t)\big)^{\frac{p}{2}}(T^*-t) = \frac{1}{q}|\Omega|^{-p/\alpha}\big(2|\Omega|^{\frac{q}{\beta}-\frac{p}{\alpha}}\big)^{-p/2}, \tag{3.6}$$

$$\lim_{t\to T^*} v^2(x,t)|\ln(T^*-t)|^{-1} = \frac{2}{q}|\Omega|^{\frac{q}{\beta}-\frac{p}{\alpha}}. \tag{3.7}$$

Proof. From Lemma 3.2, we know that $u(x,t)\sim F(t)$ and $v(x,t)\sim G(t)$, then

$$\lim_{t\to T^*}\frac{u^\alpha(x,t)}{F^\alpha(t)} = \lim_{t\to T^*}\frac{v^\alpha(x,t)}{G^\alpha(t)} = 1,$$
$$\lim_{t\to T^*}\frac{u^\beta(x,t)}{F^\beta(t)} = \lim_{t\to T^*}\frac{v^\beta(x,t)}{G^\beta(t)} = 1.$$

By the Lebesgue dominated convergence theorem, we find that

$$F'(t) = f(t) = \|uv\|_\alpha^p \sim |\Omega|^{p/\alpha}F^p(t)G^p(t), \tag{3.8}$$
$$G'(t) = g(t) = \|uv\|_\beta^q \sim |\Omega|^{q/\beta}F^q(t)G^q(t). \tag{3.9}$$

Hence,

$$F^{q-p}\,\mathrm{d}F \sim |\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}G^{p-q}\,\mathrm{d}G. \tag{3.10}$$

(1) Note that the conditions $\sigma > 0$ and $\theta > 0$ imply that $p+1 > q, q+1 > p$ since $p+q > 1$. Integrating (3.10) from 0 to t, we obtain

$$F^{q+1-p}(t) \sim |\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{q+1-p}{p+1-q}G^{p+1-q}(t) = |\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{\theta}{\sigma}G^{p+1-q}(t). \tag{3.11}$$

Combining (3.9) and (3.11), we can obtain

$$G'(t) \sim |\Omega|^{q/\beta}\big(|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{\theta}{\sigma}\big)^{\frac{q}{q+1-p}}G^{\frac{2q}{q+1-p}}(t). \tag{3.12}$$

Since

$$1-\frac{2q}{q+1-p} = -\frac{p+q-1}{q+1-p} = -\frac{1}{\theta} < 0$$

and $\lim_{t\to T^*}G(t)=\infty$, by integrating (3.12), we obtain

$$G(t) \sim \Big(\frac{|\Omega|^{q/\beta}}{\theta}\big(|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{\theta}{\sigma}\big)^{\frac{q}{q+1-p}}\Big)^{-\theta}(T^*-t)^{-\theta}. \tag{3.13}$$

From (3.13) and Lemma 3.2, we have

$$\lim_{t\to t^*} v(x,t)(T^*-t)^\theta = \Big(\frac{|\Omega|^{q/\beta}}{\theta}\big(|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\frac{\theta}{\sigma}\big)^{\frac{q}{q+1-p}}\Big)^{-\theta},$$

which holds uniformly on the compact subsets of Ω.

Combining (3.8) and (3.11), and applying the similar proofs of F and u, we obtain that

$$\lim_{t\to T^*} u(x,t)(T^*-t)^{\sigma} = \Big(\frac{|\Omega|^{p/\alpha}}{\sigma}\big(|\Omega|^{\frac{q}{\beta}-\frac{p}{\alpha}}\frac{\sigma}{\theta}\big)^{\frac{p}{p+1-q}}\Big)^{-\sigma}$$

holds uniformly on the compact subsets of Ω.

(2) When $\sigma = 0$, or $p+1 = q$, noticing (3.9) and (3.10), we see that

$$G'(t) \sim |\Omega|^{q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{q/2} G^q(t)\big(\ln G(t)\big)^{q/2}. \tag{3.14}$$

Note that $\lim_{t\to T^*} G(t) = \infty$, integrating (3.14) from $t(>0)$ to T^* asserts

$$\int_{G(t)}^{\infty} \frac{1}{s^q(\ln s)^{q/2}}\,\mathrm{d}s \sim |\Omega|^{q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{q/2}(T^*-t). \tag{3.15}$$

Furthermore,

$$\lim_{t\to T^*} \frac{\int_{G(t)}^{\infty} s^{-q}(\ln s)^{-q/2}\,\mathrm{d}s}{G^{1-q}(t)\big(\ln G(t)\big)^{-q/2}} = \lim_{G\to\infty} \frac{\int_{G}^{\infty} s^{-q}(\ln s)^{-q/2}\,\mathrm{d}s}{G^{1-q}\big(\ln G\big)^{-q/2}} = \frac{1}{q-1} = \frac{1}{p}.$$

That is to say that

$$p\int_{G(t)}^{\infty} s^{-q}(\ln s)^{-q/2}\,\mathrm{d}s \sim G^{1-q}(t)(\ln G(t))^{-q/2} = G^{-p}(t)(\ln G(t))^{-q/2}. \tag{3.16}$$

By (3.15) and (3.16), it indicates

$$G^{-p}(t)(\ln G(t))^{-q/2} \sim p|\Omega|^{q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{q/2}(T^*-t). \tag{3.17}$$

Since $\lim_{t\to T^*} v(x,t) = \infty$ uniformly on the compact subset of Ω and $\lim_{t\to T^*} G(t) = \infty$, we may claim that the following equivalent is valid uniformly on the compact subset of Ω,

$$v(x,t) \sim G(t) \;\Rightarrow\; \ln v(x,t) \sim \ln G(t).$$

And thus by (3.17), we reach the conclusion

$$v^{-p}(x,t)(\ln v(x,t))^{-q/2} \sim p|\Omega|^{q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{q/2}(T^*-t).$$

Then uniformly on the compact subsets of Ω, it yields

$$\lim_{t\to T^*} v^p(x,t)(\ln v(x,t))^{q/2}(T^*-t) = \frac{1}{p}|\Omega|^{-q/\beta}\big(2|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}\big)^{-q/2}.$$

Since

$$\ln G(t) \sim \frac{1}{2}|\Omega|^{\frac{q}{\beta}-\frac{p}{\alpha}}F^2(t),$$

it follows from (3.8) and (3.17) that

$$F'(t)F^{-p}(t) \sim |\Omega|^{p/\alpha}G^p(t) \sim \frac{F^{-q}(t)}{p(T^*-t)}|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}. \tag{3.18}$$

In view of (3.18), we have

$$\frac{1}{2}F^2(t) \sim \frac{1}{p}|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}|\ln(T^*-t)|.$$

Therefore, by Lemma 3.2, we obtain

$$u^2(x,t) \sim \frac{2}{p}|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}|\ln(T^*-t)|;$$

that is to say

$$\lim_{t\to T^*} u^2(x,t)|\ln(T^*-t)|^{-1} = \frac{2}{p}|\Omega|^{\frac{p}{\alpha}-\frac{q}{\beta}}$$

holds uniformly on the compact subsets of Ω.

(3) When $\theta = 0$, the proof is similar to that of the case (2). Then, the proof is completed. □

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