Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 215, pp. 1–27.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH DERIVATIVE TERMS

CUIPING CHENG, ZHAOSHENG FENG, YOUHUI SU

Abstract. In this article, we are concerned with the existence of positive solutions for nonlinear fractional differential equation whose nonlinearity contains the first-order derivative,

$$D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0,\quad t\in(0,1),\ n-1<\alpha\le n,$$
$$u^{(i)}(0)=0,\quad i=0,1,2,\dots,n-2,$$
$$[D_{0+}^{\beta}u(t)]_{t=1}=0,\quad 2\le\beta\le n-2,$$

where $n>4$ ($n\in\mathbb{N}$), D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α and $f(t,u,u'):[0,1]\times[0,\infty)\times(-\infty,+\infty)\to[0,\infty)$ satisfies the Carathéodory type condition. Sufficient conditions are obtained for the existence of at least one or two positive solutions by using the nonlinear alternative of the Leray-Schauder type and Krasnosel'skii's fixed point theorem. In addition, several other sufficient conditions are established for the existence of at least triple, n or $2n-1$ positive solutions. Two examples are given to illustrate our theoretical results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. Although the notion of fractional derivative dates back to the time when Leibnitz and Newton invented differential calculus and the tools of fractional calculus have been available and applicable to various fields of study, the qualitative and quantitative investigation of the theory of fractional differential equations has been started recently [1, 2, 3]. In the past two decades, we have seen that differential equations involving Riemann-Liouville differential operators of fractional order arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, etc [4, 5, 6, 7, 8]. In consequence, the subject of fractional differential equations is gaining diverse and continuous attention. For more details of some recent theoretical results on fractional differential equations and their applications, we refer the reader to [9, 10, 11, 12, 13, 14, 15, 16] and the references therein.

2000 *Mathematics Subject Classification.* 34A08, 34B18, 34K37.
Key words and phrases. Positive solution; equicontinuity; fractional differential equation; fixed point theorem; Carathéodory type condition.

Submitted August 15, 2012. Published November 29, 2012.

In order to better understand the background of our fractional differential system, let us briefly review some related studies on the topic. Xu et al [17] considered the equation

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad t\in(0,1), \\ u(0)=u'(0)=u(1)=u'(1)=0, \end{gathered} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order α ($2<\alpha\leq 3$) and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous. The existence of multiple positive solutions to system (1.1) is established by applying the fixed point theorems.

Goodrich [18] considered (1.1) subject to the boundary conditions

$$\begin{gathered} u^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2, \\ [D_{0+}^{\beta}u(t)]_{t=1}=0, \quad 2\leq\beta\leq n-2, \end{gathered}$$

where $n-1<\alpha\leq n$ and $n>4$ ($n\in\mathbb{N}$). The existence of one positive solution was explored.

It is notable that the nonlinear term $f(t,u(t))$ in (1.1) does not involve the derivative. Apparently, the nonlinear term $f(t,u(t),u'(t))$ containing the derivative is a more general case, and the study of such fractional differential equations is of significance theoretically and practically [7]. So far, to the best of our knowledge, it appears that there is only a few articles concerning the existence of positive solutions to fractional differential equations with nonlinear terms involving the derivative [19]. In the present article, we restrict our attention to this problem in some respects. More precisely, we consider the problem

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0, \quad t\in(0,1),\ n-1<\alpha\leq n, \\ u^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2, \\ [D_{0+}^{\beta}u(t)]_{t=1}=0, \quad 2\leq\beta\leq n-2, \end{gathered} \tag{1.2}$$

where $u^{(i)}$ represents the ith derivative of u, $n>4$ ($n\in\mathbb{N}$), D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $n-1<\alpha\leq n$ and $f(t,u,u'):[0,1]\times[0,\infty)\times(-\infty,+\infty)\to[0,\infty)$ satisfies Carathéodory type conditions. Our goal is to establish the existence of at least one or two positive solutions by using the nonlinear alternative of Leray-Schauder type and the Krasnosel'skii's fixed point theorem, and find sufficient conditions of the existence of at least n or $2n-1$ distinct positive solutions by means of the Leggett-Williams fixed point theorem, the generalized Avery-Henderson fixed point theorem as well as the Avery-Peterson fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we presents some basic definitions and several fixed point theorems. In Section 3, we state properties of the associated Green's function. In Section 4, we discuss the completely continuous operator of fractional differential (1.2). In Section 5, by using the nonlinear alternative of Leray-Schauder type and the Krasnosel'skii's fixed point theorem, some new sufficient conditions of the existence of at least *one* or *two* positive solutions of fractional differential (1.2) are obtained. In Section 6, the existence criteria for at least three or arbitrary n or $2n-1$ positive solutions of fractional differential (1.2) are established. In Section 7, we present two examples.

In this study, we assume that $f(t,u_1,u_2):[0,1]\times[0,\infty)\times(-\infty,+\infty)\to[0,\infty)$ satisfies the following conditions of Carathéodory type:

(S1) $f(t, u_1, u_2)$ is Lebesgue measurable with respect to t on $[0, 1]$;
(S2) for a.e. $t \in [0, 1]$, $f(t, ., .)$ is continuous on $[0, 1] \times [0, \infty) \times (-\infty, +\infty)$.

2. Preliminaries

In this section, we present some basic definitions and the fixed point theorems which help us to better understand discussions presented in next a few sections.

Definition 2.1 ([1]). The Riemann-Liouville fractional integral of order $a > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([1]). The Riemann-Liouville fractional derivative of order $a > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{0^+} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$.

The following are two fixed point theorems. The former one is the so-called nonlinear alternative of Leray-Schauder type and the latter one is the Krasnosel'skii's fixed point theorem [20, 21].

Lemma 2.3. *Let X be a Banach space with $C \subset X$ being closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $A : \overline{U} \to C$ is a completely continuous operator, then either*

(i) *A has a fixed point in $\overline{U}$, or*
(ii) *there exists $u \in \partial U$ and $\gamma_1^* \in (0, 1)$ with $u = \gamma_1^* A u$.*

Lemma 2.4. *Let P be a cone in a Banach space E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) *$\|Ax\| \le \|x\|$ for all $x \in P \cap \partial\Omega_1$, and $\|Ax\| \ge \|x\|$ for all $x \in P \cap \partial\Omega_2$, or*
(ii) *$\|Ax\| \ge \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Ax\| \le \|x\|$ for all $x \in P \cap \partial\Omega_2$.*

Then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Define that $P_c = \{u \in P : \|u\| < c\}$ and $P(q, b, d) = \{u \in P : b \le q(u),\ \|u\| \le d\}$, where the map q is a nonnegative continuous concave functional on P. The following are two fixed-point theorems due to Leggett and Williams [22].

Lemma 2.5. *Suppose that $A : \overline{P}_c \to \overline{P}_c$ is completely continuous and there exists a concave positive functional q on P such that $q(u) \le \|u\|$ for $u \in \overline{P}_c$. Suppose that there exist constants $0 < a < b < d \le c$ such that*

(i) *$\{u \in P(q, b, d) : q(u) > b\} \neq \emptyset$ and $q(Tu) > b$ if $u \in P(q, b, d)$;*
(ii) *$\|Tu\| < a$ if $u \in P_a$;*
(iii) *$q(Tu) > b$ for $u \in P(q, b, c)$ with $\|Tu\| > d$.*

Then A has at least three fixed points u_1, u_2 and u_3 such that

$$\|u_1\| < a, \quad b < q(u_2) \text{ and } u_3 > a \text{ with } q(u_3) < b.$$

For each $d > 0$, let $P(\gamma, d) = \{x \in P : \gamma(x) < d\}$, where γ is a nonnegative continuous functional on a cone P of a real Banach space E.

Lemma 2.6. *Let P be a cone in a real Banach space E. Let α, β and γ be increasing, nonnegative continuous functionals on P such that for some $c > 0$ and $H > 0$, $\gamma(x) \le \beta(x) \le \alpha(x)$ and $\|x\| \le H\gamma(x)$ for all $x \in \overline{P}(\gamma, c)$. Suppose that there exist positive numbers a and b with $a < b < c$, and $A : \overline{P}(\gamma, c) \to P$ is a completely continuous operator such that:*

(i) *$\gamma(Ax) < c$ for all $x \in \partial P(\gamma, c)$;*
(ii) *$\beta(Ax) > b$ for all $x \in \partial P(\beta, b)$;*
(iii) *$P(\alpha, a) \ne \emptyset$ and $\alpha(Ax) < a$ for $x \in \partial P(\alpha, a)$.*

Then A has at least three fixed points x_1, x_2 and x_3 belonging to $\overline{P}(\gamma, c)$ such that

$$0 \le \alpha(x_1) < a < \alpha(x_2) \text{ with } \beta(x_2) < b < \beta(x_3) \text{ and } \gamma(x_3) < c.$$

Let β and ϕ be nonnegative continuous convex functionals on P, λ be a nonnegative continuous concave functional on P and φ be a nonnegative continuous functional on P. We define the following convex sets:

$$P(\phi, \lambda, b, d) = \{x \in P : b \le \lambda(x), \phi(x) \le d\},$$
$$P(\phi, \beta, \lambda, b, c, d) = \{x \in P : b \le \lambda(x), \beta(x) \le c, \phi(x) \le d\},$$

and

$$R(\phi, \varphi, a, d) = \{x \in P : a \le \varphi(x), \ \phi(x) \le d\}.$$

We are ready to recall the Avery-Peterson fixed point theorem [23].

Lemma 2.7. *Let P be a cone in a real Banach space E, and β, ϕ, λ and φ be defined as the above. Moreover, φ satisfies $\varphi(\lambda' x) \le \lambda' \varphi(x)$ for $0 \le \lambda' \le 1$ such that for some positive numbers h and d,*

$$\lambda(x) \le \varphi(x), \quad \|x\| \le h\phi(x) \tag{2.1}$$

holds for all $x \in \overline{P(\phi, d)}$. Suppose that A: $\overline{P(\phi, d)} \to \overline{P(\phi, d)}$ is completely continuous and there exist positive real numbers a, b, c, with $a < b$ such that:

(i) *$\{x \in P(\phi, \beta, \lambda, b, c, d) : \lambda(x) > b\} \ne \emptyset$ and $\lambda(A(x)) > b$ for x in the set $P(\phi, \beta, \lambda, b, c, d)$;*
(ii) *$\lambda(A(x)) > b$ for $x \in P(\phi, \lambda, b, d)$ with $\beta(A(x)) > c$;*
(iii) *$0 \notin R(\phi, \varphi, a, d)$ and $\lambda(A(x)) < a$ for all $x \in R(\phi, \varphi, a, d)$ with $\varphi(x) = a$.*

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\phi, d)}$ such that

$$\phi(x_i) \le d \text{ for i=1, 2, 3}, \ \ b < \lambda(x_1), \quad a < \varphi(x_2) \text{ and } \lambda(x_2) < b \text{ with } \varphi(x_3) < a.$$

3. Properties of Green's function

In this section, we present some properties of the Green's function which will be used in our discussions. Note that using a similar discussion as to proofs of Theorems 3.1 and 3.2 in [18], we have the following lemmas.

Lemma 3.1. *Assume that $y(t) \in L[0, 1]$, then the fractional differential equation*

$$\begin{gathered} D^{\alpha}_{0^+} u(t) + y(t) = 0, \quad t \in (0, 1), \ n - 1 < \alpha \le n, \\ u^{(i)}(0) = 0, \quad i = 0, 1, 2, \dots, n - 2, \\ [D^{\beta}_{0^+} u(t)]_{t=1} = 0, \quad 2 \le \beta \le n - 2, \end{gathered} \tag{3.1}$$

has the unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1,\end{cases} \tag{3.2}$$

is the Green's function of problem (3.1) *with* $n>4$.

Lemma 3.2. *Let* $G(t,s)$ *be given as* (3.2)*, then:*

- (i) $G(t,s)$ *is a continuous function on the unit square* $[0,1]\times[0,1]$*;*
- (ii) $G(t,s)\ge 0$ *for* $(t,s)\in[0,1]\times[0,1]$*;*
- (iii) $\max_{t\in[0,1]} G(t,s)=G(1,s)$ *for each* $s\in[0,1]$*;*
- (iv) *there exists a constant* $\gamma\in(0,1)$ *such that*

$$\min_{t\in[1/2,1]} G(t,s)\ge\gamma\max_{t\in[0,1]} G(t,s)=\gamma G(1,s),$$

where

$$\gamma=\min\Big\{\frac{(1/2)^{\alpha-\beta-1}}{2^{\beta}-1},\,(1/2)^{\alpha-1}\Big\}. \tag{3.3}$$

Lemma 3.3. *Let* $G(t,s)$ *be given as* (3.2)*, then:*

- (i) $\frac{\partial G(t,s)}{\partial t}$ *is a continuous function on the unit square* $[0,1]\times[0,1]$*;*
- (ii) $\frac{\partial G(t,s)}{\partial t}\ge 0$ *for* $(t,s)\in[0,1]\times[0,1]$*;*
- (iii) $\max_{t\in[0,1]}\frac{\partial G(t,s)}{\partial t}=\frac{\partial G(1,s)}{\partial t}$ *for each* $s\in[0,1]$*;*
- (iv) $\max_{t\in[0,1]}\frac{\partial G(1,s)}{\partial t}\le(\alpha-1)\max_{t\in[0,1]} G(t,s)=(\alpha-1)G(1,s)$ *for each* $s\in[0,1]$.

Proof of Lemma 3.3. Note that

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}(\alpha-1)\frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ (\alpha-1)\frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \tag{3.4}$$

Let

$$G_1(t,s)=(\alpha-1)\frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-2}}{\Gamma(\alpha)},\quad 0\le s\le t\le 1;$$

$$G_2(t,s)=(\alpha-1)\frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)},\quad 0\le t\le s\le 1.$$

It is easy to see that property (i) holds since G_1 and G_2 are continuous on their domains and $G_1(s,s)=G_2(s,s)$.

When $0\le s\le t\le 1$, since $\alpha-2\ge\alpha-\beta-1$, we have

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t}&=(\alpha-1)\frac{t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-2}}{\Gamma(\alpha)}\\ &\ge(\alpha-1)t^{\alpha-2}\frac{(1-s)^{\alpha-\beta-1}-(1-\frac{s}{t})^{\alpha-2}}{\Gamma(\alpha)}\\ &\ge(\alpha-1)t^{\alpha-2}\frac{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-2}}{\Gamma(\alpha)}\ge 0.\end{aligned}$$

Hence, property (ii) is true in view of $G_2(t,s)\ge 0$ for $0\le t\le s\le 1$.

When $0 \le s \le t \le 1$, since $\alpha - 2 \ge \alpha - \beta - 1$, we have

$$\begin{aligned}\frac{\partial G_1(t,s)}{\partial t} &= (\alpha-1)(\alpha-3)\frac{t^{\alpha-3}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-3}}{\Gamma(\alpha)}\\ &\ge (\alpha-1)(\alpha-2)t^{\alpha-3}\frac{(1-s)^{\alpha-\beta-1} - (1-\frac{s}{t})^{\alpha-3}}{\Gamma(\alpha)}\\ &\ge (\alpha-1)(\alpha-2)t^{\alpha-3}\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-3}}{\Gamma(\alpha)} \ge 0.\end{aligned}$$

Similarly, when $0 \le t \le s \le 1$, we have

$$\frac{\partial G_2(t,s)}{\partial t} = (\alpha-1)(\alpha-2)\frac{t^{\alpha-3}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \ge 0.$$

This implies that $\frac{\partial G(t,s)}{\partial t}$ is increasing on its domain. Moreover,

$$\max_{t\in[0,1]} \frac{\partial G(t,s)}{\partial t} = \frac{\partial G(1,s)}{\partial t} \quad \text{for each} \quad s \in [0,1]. \tag{3.5}$$

So property (iii) holds.

When $0 \le s \le 1$, we have

$$\begin{aligned}\max_{t\in[0,1]} \frac{\partial G(1,s)}{\partial t} &= \frac{\partial G(1,s)}{\partial t}\\ &= (\alpha-1)\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-2}}{\Gamma(\alpha)}\\ &\le (\alpha-1)\frac{(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-1}}{\Gamma(\alpha)}\\ &= (\alpha-1)\max_{t\in[0,1]} G(t,s)\\ &= (\alpha-1)G(1,s).\end{aligned}$$

This implies that property (iv) holds. □

Lemma 3.4. *Let $G(t,s)$ be given as (3.2), then there exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t\in[1/2,1]} \frac{\partial G(t,s)}{\partial t} \ge \gamma^* \max_{t\in[0,1]} \frac{\partial G(t,s)}{\partial t} = \gamma^* \frac{\partial G(1,s)}{\partial t}.$$

Proof of Lemma 3.4. It follows from the proof of Lemma 3.3 that

$$\begin{aligned}\min_{t\in[1/2,1]} \frac{\partial G(t,s)}{\partial t} &= \begin{cases} G_1(\frac{1}{2}, s), & s \in (0, \frac{1}{2}]\\ G_2(\frac{1}{2}, s), & s \in [\frac{1}{2}, 1)\end{cases}\\ &= \begin{cases} (\alpha-1)\frac{(1/2)^{\alpha-2}(1-s)^{\alpha-\beta-1} - (\frac{1}{2}-s)^{\alpha-2}}{\Gamma(\alpha)}, & s \in (0, \frac{1}{2}],\\ (\alpha-1)\frac{(1/2)^{\alpha-2}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & s \in [\frac{1}{2}, 1).\end{cases}\end{aligned}$$

L'Hopital's rule applies:

$$
\begin{aligned}
&\lim_{s\to 0+}\frac{(\frac{1}{2})^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\frac{1}{2}-s)^{\alpha-2}}{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-2}}\\
&=\lim_{s\to 0+}\frac{(\frac{1}{2})^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\frac{1}{2}-s)^{\alpha-2}}{(1-s)^{\alpha-\beta-1}[1-(1-s)^{\beta-1}]}\\
&=\lim_{s\to 0+}\frac{-(\alpha-\beta-1)(\frac{1}{2})^{\alpha-2}(1-s)^{\alpha-\beta-2}+(\alpha-2)(\frac{1}{2}-s)^{\alpha-3}}{-(\alpha-\beta-1)(1-s)^{\alpha-\beta-2}+(\alpha-2)(1-s)^{\alpha-3}}\\
&=\frac{(\frac{1}{2})^{\alpha-2}(\alpha+\beta-3)}{\beta-1}>0.
\end{aligned}
\tag{3.6}
$$

When $0<s\le 1/2$, a straightforward calculation gives

$$
\begin{aligned}
\frac{(1/2)^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\frac{1}{2}-s)^{\alpha-2}}{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-2}}
&=\frac{(\frac{1}{2})^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\frac{1}{2}-s)^{\alpha-2}}{(1-s)^{\alpha-\beta-1}[1-(1-s)^{\beta-1}]}\\
&\ge\frac{(1/2)^{\alpha-\beta-2}}{2^{\beta-1}-1}.
\end{aligned}
$$

Similarly, when $\frac{1}{2}\le s\le 1$, we have

$$
\frac{(\frac{1}{2})^{\alpha-2}}{1-(1-s)^{\beta}}\ge(\frac{1}{2})^{\alpha-2}.
$$

Define

$$
\overline{\gamma}(s):=\begin{cases}
\frac{(1/2)^{\alpha-2}(1-s)^{\alpha-\beta-1}-(\frac{1}{2}-s)^{\alpha-2}}{(1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-2}}, & s\in(0,\frac{1}{2}],\\
\frac{(1/2)^{\alpha-2}}{1-(1-s)^{\beta}}, & s\in[1/2,1],
\end{cases}
$$

where $\overline{\gamma}(0)=\lim_{s\to 0+}\overline{\gamma}(s)>0$ due to (3.6). Let

$$
\gamma^*=\Big\{\frac{(\frac{1}{2})^{\alpha-\beta-2}}{2^{\beta-1}-1},\,(\frac{1}{2})^{\alpha-2}\Big\}.
$$

It is obvious that $0<\gamma^*<1$. Consequently, we have

$$
\min_{t\in[1/2,1]}\frac{\partial G(t,s)}{\partial t}=\overline{\gamma}(s)\max_{t\in[0,1]}\frac{\partial G(t,s)}{\partial t}\ge\gamma^*\max_{t\in[0,1]}\frac{\partial G(t,s)}{\partial t}=\gamma^*\frac{\partial G(1,s)}{\partial t}.
$$

□

4. Completely Continuous Operator

In this section, we construct the completely continuous operator for our system, and show that finding the solution of fractional differential (1.2) is equivalent to finding the fixed points of the associated completely continuous operator.

Let us denote $E_1=C^1([0,1],\mathbb{R})$. Then E_1 is a Banach space endowed with norm

$$
\|u\|=\max\{\|u\|_1,\ \|u\|_2\},
$$

where

$$
\|u\|_1=\sup_{t\in[0,1]}|u(t)|,\quad \|u\|_2=\sup_{t\in[0,1]}|u'(t)|.
$$

The cone $P_1\subset E_1$ is defined by

$$
P_1=\big\{u\in E_1: u(0)=0 \text{ and } u(t)\ge 0 \text{ for } t\in[0,1]\big\}.
$$

Assume that $y(t) = f(t, u(t), u'(t))$, then it follows from Lemma 3.1 that the solutions of fractional differential (1.2) are the corresponding fixed points of the operator $A : E_1 \to E_1$, which is defined by

$$Au = \int_0^1 G(t,s) f(s, u(s), u'(s))\, ds. \tag{4.1}$$

Lemma 4.1. *Suppose that conditions* (S1) *and* (S2) *hold. For* $t \in [0,1]$ *and all* $(u_1, u_2) \in [0,+\infty) \times (-\infty,+\infty)$*, we assume that there exist two nonnegative real-value functions* a_1*,* $a_2 \in L[0,1]$ *such that*

$$f(t, u_1, u_2) \le a_1(t) + a_2(t) \max_{t\in[0,1]} u_1(t), \tag{4.2}$$

or

$$f(t, u_1, u_2) \le a_1(t) + a_2(t) \max_{t\in[0,1]} |u_2(t)|. \tag{4.3}$$

Then the operator $A : P_1 \to P_1$ *is completely continuous.*

Proof of Lemma 4.1. Firstly, we show that A: $P_1 \to P_1$ is continuous.

Let $u \in P_1$. It is obvious that $Au(0) = 0$ because of $G(0,s) = 0$. Suppose that $\{u_n\}_{n=1}^{\infty} \subset \overline{P}_1$ and $u_n(t)$ converges to $u(t)$ uniformly on $[0,1]$ as $n \to \infty$; that is,

$$\lim_{n\to\infty} \|u_n - u\| = 0.$$

So we have

$$\lim_{n\to\infty} \|u_n - u\|_1 = 0 \text{ and } \lim_{n\to\infty} \|u_n - u\|_2 = 0,$$

which implies that

$$\lim_{n\to\infty} u_n(t) = u(t) \text{ and } \lim_{n\to\infty} u_n'(t) = u'(t), \quad t \in [0,1].$$

It follows from (S1) that

$$\lim_{n\to\infty} f(t, u_n(t), u_n'(t)) = f(t, u(t), u'(t)), \quad t \in [0,1],$$

which gives

$$|Au_n(t) - Au(t)| \le \Big| \int_0^1 G(1,s)(f(s, u_n(s), u_n'(s)) - f(s, u(s), u'(s)))\, ds\Big| \to 0 \tag{4.4}$$

as $n \to \infty$, and

$$\begin{aligned} &|A'u_n(t) - A'u(t)| \\ &= \Big|\int_0^1 \frac{\partial G(t,s)}{\partial t}(f(s, u_n(s), u_n'(s)) - f(s, u(s), u'(s)))\, ds\Big| \\ &\le \int_0^1 \frac{\partial G(1,s)}{\partial t}\big|f(s, u_n(s), u_n'(s)) - f(s, u(s), u'(s))\big| ds \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{4.5}$$

By (4.4) and (4.5), we have

$$\|(Au_n)(t) - (Au)(t)\| \to 0 \quad \text{as } n \to \infty,$$

which means that A is continuous.

Secondly, we show that A maps bounded sets into bounded sets in P_1. It suffices to show that for any $\eta > 0$, there is a positive constant $l > 0$ such that for each $u \in B_\eta = \{u \in P_1 : \|u\| \le \eta\}$, we have $\|Au\| \le l$. Let

$$l = (\alpha - 1)\Big(\int_0^1 a_1(t) G(1,s) ds + \eta \int_0^1 a_2(t) G(1,s) ds\Big) > 0.$$

Using (4.2) yields

$$
\begin{aligned}
|Au(t)| &= \Big|\int_0^1 G(t,s)f(s,u(s),u'(s))ds\Big| \\
&\le \int_0^1 (a_1(t)+a_2(t)\max_{t\in[0,1]} u(t))G(1,s)ds \\
&\le \int_0^1 a_1(t)G(1,s)ds + \int_0^1 a_2(t)G(1,s)ds\|u(t)\| \\
&\le \int_0^1 a_1(t)G(1,s)ds + \eta\int_0^1 a_2(t)G(1,s)ds < l.
\end{aligned}
$$

Using (4.3) yields

$$
\begin{aligned}
|Au(t)| &= \Big|\int_0^1 G(t,s)f(s,u(s),u'(s))ds\Big| \\
&\le \int_0^1 (a_1(t)+a_2(t)\max_{t\in[0,1]} |u'(t)|)G(1,s)ds \\
&\le \int_0^1 a_1(t)G(1,s)ds + \int_0^1 a_2(t)G(1,s)ds\|u(t)\| \\
&\le \int_0^1 a_1(t)G(1,s)ds + \eta\int_0^1 a_2(t)G(1,s)ds < l.
\end{aligned}
$$

In view of Lemma 3.3, we have

$$
\begin{aligned}
|A'u(t)| &= \Big|\int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))ds\Big| \\
&\le (\alpha-1)\int_0^1 G(1,s)f(s,u(s),u'(s))ds \\
&\le (\alpha-1)\Big(\int_0^1 a_1(t)G(1,s)ds + \eta\int_0^1 a_2(t)G(1,s)ds\Big) < l.
\end{aligned}
$$

Hence, we have $\|Au\| \le l$.

Thirdly, we consider that A maps bounded sets into equicontinuous sets of P_1. It follows from Lemma 3.4 that $\frac{\partial G(t,s)}{\partial t}$ is continuous in $[0,1]\times[0,1]$. In addition, $G(t,s)$ is continuous in $[0,1]\times[0,1]$, then $\frac{\partial G(t,s)}{\partial t}$ and $G(t,s)$ are uniformly continuous in $[0,1]\times[0,1]$. Take t_1, $t_2 \in [0,1]$. For any $\varepsilon > 0$, there exists $\delta > 0$, whenever $|t_1 - t_2| < \delta$, we have

$$
|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{1 + a_1(s) + \eta a_2(s)},
$$

and

$$
\Big|\frac{\partial G(t_2,s)}{\partial t} - \frac{\partial G(t_1,s)}{\partial t}\Big| < \frac{\varepsilon}{1 + a_1(s) + \eta a_2(s)}.
$$

For convenience, we assume $t_1 < t_2$. For any $u \in B_\eta$, according to (4.2) and (4.3), we obtain

$$
\begin{aligned}
|Au(t_2) - Au(t_1)| &= \Big|\int_0^1 (G(t_2,s) - G(t_1,s))f(s,u(s),u'(s))\,ds\Big| \\
&\le \int_0^1 |G(t_2,s) - G(t_1,s)|[a_1(s) + \eta a_2(s)]\,ds < \varepsilon,
\end{aligned}
$$

and

$$\begin{aligned}|A'u(t_2)-A'u(t_1)| &= \Big|\int_0^1 [\frac{\partial G(t_2,s)}{\partial t}-\frac{\partial G(t_1,s)}{\partial t}]f(s,u(s),u'(s))\,ds\Big| \\ &\le \int_0^1 \Big|\frac{\partial G(t_2,s)}{\partial t}-\frac{\partial G(t_1,s)}{\partial t}\Big|[a_1(s)+\eta a_2(s)]\,ds<\varepsilon.\end{aligned}$$

Consequently,

$$\|Au(t_2)-Au(t_1)\|<\varepsilon,$$

which implies that the family of functions $\{Au : u\in B_\eta\}$ is equicontinuous. By the Arzela-Ascoli theorem, we conclude that the operator $A : P_1 \to P_1$ is completely continuous. □

Remark 4.2. If $f(t,u_1,u_2) : [0,1]\times[0,\infty)\times(-\infty,+\infty)\to[0,\infty)$ is continuous, we can see that $A : P_1\to P_1$ is completely continuous by using a similar argument as the above.

Since $u(t)=u(0)+\int_0^t u'(s)ds$, it leads to

$$\max_{t\in[0,1]} u(t)=u(0)+\max_{t\in[0,1]}\int_0^t u'(s)ds \le \max_{t\in[0,1]}|u'(s)|.$$

That is,

Lemma 4.3. *If $u\in P_1$, then $\max_{t\in[0,1]} u(t)\le \max_{t\in[0,1]}|u'(t)|$.*

5. Existence of one or two solutions

In this section, we discuss the existence of single or twin positive solutions to problem (1.2) by the nonlinear alternative of Leray-Schauder type and the Krasnosel'skii's fixed point theorem, respectively.

Theorem 5.1. *Assume that all assumptions of Lemma 4.1 hold, and*

$$(\alpha-1)\int_0^1 G(1,s)a_2(s)\,ds<1,$$

Then (1.2) *has at least one positive solution.*

Proof. Let $U=\{u\in P_1 : \|u\|<r\}$, where

$$r=\frac{(\alpha-1)\int_0^1 G(1,s)a_1(s)ds}{1-(\alpha-1)\int_0^1 G(1,s)a_2(s)ds}>0.$$

It is easily seen that the operator $A : \overline{U}\to P_1$ defined by (4.1) is completely continuous.

Assume that there exist $u\in P_1$ and $\gamma_1^*\in(0,1)$ such that $u=\gamma_1^* Au$. The we find that

$$\begin{aligned}u(t) &= \gamma_1^* Au \\ &= \gamma_1^*\int_0^1 G(t,s)f(s,u(s),u'(s))\,ds \\ &\le \gamma_1^*\int_0^1 G(1,s)[a_1(s)+a_2(s)\|u\|]\,ds\end{aligned}$$

$$\le (\alpha-1)\gamma_1^*[\int_0^1 G(1,s)a_1(s)ds + \|u\| \int_0^1 G(1,s)a_2(s)\,ds],$$

and

$$\begin{aligned}
|u'(t)| &= |\gamma_1^* A'u(t)| \\
&= |\gamma_1^* \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))\,ds| \\
&\le (\alpha-1)\gamma_1^* \int_0^1 G(1,s)[a_1(s)+a_2(s)\|u\|]\,ds \\
&\le (\alpha-1)\gamma_1^*[\int_0^1 G(1,s)a_1(s)ds + \|u\| \int_0^1 G(1,s)a_2(s)\,ds].
\end{aligned}$$

Thus, we have

$$\|u\| < (\alpha-1)\gamma_1^*[\int_0^1 G(1,s)a_1(s)ds + r\int_0^1 G(1,s)a_2(s)\,ds] = \gamma_1^* r,$$

which means that $u \notin \partial U$ and $\|u\| \ne r$.

By Lemma 2.3, we conclude that (1.2) has at least one positive solution. □

Remark 5.2. In Theorem 5.1, "All assumptions of Lemma 4.1 hold" can be replaced by "$f(t,u_1,u_2) : [0,1]\times[0,\infty)\times(-\infty,+\infty)\to[0,\infty)$ is continuous".

Theorem 5.3. *Assume that all assumptions of Lemma 4.1 and the following conditions hold:*

(i) *there exists a constant $p > 0$ such that $f(t,u_1,u_2) \le p\Lambda_1$ for $(t,u_1,u_2) \in [0,1]\times[0,p]\times[-p,p]$, where $\Lambda_1 = ((\alpha-1)\int_0^1 G(1,s)\,ds)^{-1}$;*

(ii) *there exists a constant $q > 0$ such that $f(t,u_1,u_2) \ge q\Lambda_2$ for $(t,u_1,u_2) \in [1/2,1]\times[0,q]\times[-q,q]$, where $\Lambda_2 = (\gamma\int_{1/2}^1 G(1,s)\,ds)^{-1}$, and $p \ne q$.*

Then problem (1.2) *has at least one positive solution u such that $\|u\|$ lies in between p and q.*

Proof. Without loss of generality, we assume that $p < q$. Let

$$\Omega_p = \{u \in E_1 : \|u\| < p\}.$$

For any $u \in P_1 \cap \partial\Omega_p$, we see that

$$\max_{t\in[0,1]} |u'(t)| \le \|u\| < p \quad \text{and} \quad \max_{t\in[0,1]} u(t) \le \|u\| < p.$$

It follows from Lemma 3.3 and condition (i) that

$$\begin{aligned}
&\|Au\| \\
&= \max\{\|Au\|_1, \|Au\|_2\} \\
&= \max\Big\{\max_{t\in[0,1]}\int_0^1 G(t,s)f(s,u(s),u'(s))\,ds, \max_{t\in[0,1]}\int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,u(s),u'(s))\,ds\Big\} \\
&= (\alpha-1)\int_0^1 G(1,s)f(s,u(s),u'(s))ds \\
&< (\alpha-1)p\Lambda_1 \int_0^1 G(1,s)\,ds,
\end{aligned} \tag{5.1}$$

which implies that

$$\|Au\| \le \|u\| \quad \text{for } u \in P_1 \cap \partial\Omega_p. \tag{5.2}$$

We define

$$\Omega_q = \{u \in E_1 : \|u\| < q\}$$

for arbitrary $u \in P_1 \cap \partial\Omega_q$, and find

$$\max_{t\in[0,1]} |u'(t)| \le \|u\| < q \quad \text{and} \quad \max_{t\in[0,1]} u(t) \le \|u\| < q.$$

On the other hand, it follows from Lemma 3.2 and condition (ii) that

$$\begin{aligned}
\|Au\| &\ge \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s),u'(s))\,ds\\
&\ge \int_0^{\frac{1}{2}} G(t,s) f(s,u(s),u'(s))\,ds + \int_{1/2}^1 G(t,s) f(s,u(s),u'(s))\,ds\\
&\ge \int_{1/2}^1 G(t,s) f(s,u(s),u'(s))\,ds\\
&\ge \int_{1/2}^1 \min_{t\in[1/2,1]} G(t,s) f(s,u(s),u'(s))\,ds\\
&\ge \int_{1/2}^1 \gamma G(1,s) q\Lambda_2\,ds\\
&\ge \gamma q\Lambda_2 \int_{1/2}^1 G(1,s)\,ds,
\end{aligned}$$

which implies that

$$\|Au\| \ge \|u\| \quad \text{for } u \in P_1 \cap \partial\Omega_q. \tag{5.3}$$

In view of (5.2) and ((5.3), it follows from Lemma 2.4, that problem (1.2) has a positive solution u in $P_1 \cap (\overline{\Omega}_q \backslash \Omega_p)$. □

For $u_1, u_2 \in P_1$, we denote

$$\begin{gathered}
f^0 = \lim_{(u_1,u_2)\to(0,0)} \sup_{t\in[0,1]} \frac{f(t,u_1,u_2)}{|u_2|},\\
f_\infty = \lim_{u_1+|u_2|\to\infty} \inf_{t\in[0,1]} \frac{f(t,u_1,u_2)}{u_1+|u_2|},\\
f_0 = \lim_{(u_1,u_2)\to(0,0)} \inf_{t\in[0,1]} \frac{f(t,u_1,u_2)}{u_1+|u_2|},\\
f^\infty = \lim_{(u_1,u_2)\to(\infty,\infty)} \sup_{t\in[0,1]} \frac{f(t,u_1,u_2)}{|u_2|}.
\end{gathered}$$

Now, we have the following two theorems.

Theorem 5.4. *Assume that all assumptions of Lemma 4.1 are satisfied, and* $f^0 \in [0,\Lambda_1)$ *and* $f_\infty \in (\Lambda_2,\infty) \cup \{\infty\}$. *Then* (1.2) *has at least one positive solution.*

Proof. According to the assumption $f^0 < \Lambda_1$ and Lemma 4.3, there exists a sufficiently small $p > 0$ such that

$$f(t,u,u') \le \Lambda_1 |u'| \le \Lambda_1 p \quad \text{for } (t,u,u') \in [0,1]\times[0,p]\times[-p,p],$$

which implies that the condition (i) of Theorem 5.3 is true. That is, if we let

$$\Omega_p = \{u \in E_1 : \|u\| < p\},$$

then (5.2) is satisfied. It follows from $f_\infty > \Lambda_2$ that there exists an $H > 2p$ satisfying

$$f(t,u,u') \ge \Lambda_2(u + |u'|) \ge \Lambda_2\|u\|, \tag{5.4}$$

where $t \in [0,1]$ and $u + |u'| \ge H$. Set

$$\Omega_H = \{u \in E_1 : u + |u'| < H\},$$

then we see that $\overline{\Omega}_p \subset \Omega_H$.

For any $u \in P_1 \cap \partial\Omega_H$, we have $u + |u'| = H$. Using (5.4) gives

$$\begin{aligned}
\|Au\| &\ge \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s),u'(s))\,ds \\
&\ge \int_{1/2}^1 G(t,s) f(s,u(s),u'(s))\,ds \\
&\ge \int_{1/2}^1 \min_{t\in[1/2,1]} G(t,s) f(s,u(s),u'(s))\,ds \\
&\ge \int_{1/2}^1 \gamma G(1,s)\Lambda_2\|u\| ds \\
&\ge \gamma\Lambda_2 \int_{1/2}^1 G(1,s)\,ds\|u\| = \|u\|.
\end{aligned}$$

Consequently, by Lemma 2.4, we conclude that (1.2) has a positive solution u in $P_1 \cap (\overline{\Omega}_H \backslash \Omega_p)$. □

Theorem 5.5. *Assume that all assumptions of Lemma 4.1 hold, and that* $f_0 \in (\Lambda_2,\infty) \cup \{\infty\}$ *and* $f^\infty \in [0,\Lambda_1)$. *Then* (1.2) *has at least one positive solution.*

Proof. It follows from $f_0 > \Lambda_2$ that there exists a sufficiently small $q > 0$ such that

$$\begin{aligned}
f(t,u,u') &\ge \Lambda_2(u + |u'|) \\
&\ge \Lambda_2 \max\{\|u\|_1, \|u\|_2\} \quad \text{for } (t,u,u') \in [0,1] \times [0,q] \times [-q,q].
\end{aligned}$$

When $(t,u,u') \in [1/2,1] \times [0,q] \times [-q,q]$, we obtain

$$f(t,u,u') \ge \Lambda_2 \max\{\|u\|_1, \|u\|_2\} = \Lambda_2 q,$$

which implies that the condition (ii) in Theorem 5.3 is satisfied. Take

$$\Omega_q = \{u \in E_1 : \|u\| < q\}.$$

Then, inequality (5.3) holds.

Let $\varepsilon_1 = \Lambda_1 - f^\infty$ (> 0). Since $f^\infty < \Lambda_1$, there exists a p_1 $(> q)$ such that

$$f(t,u,u') \le (\varepsilon_1 + f^\infty)|u'| = \Lambda_1|u'|, \tag{5.5}$$

where $(t,u,u') \in [0,1] \times [p_1,\infty) \times (-\infty,-p_1] \cup [p_1,+\infty)$. Note that

$$f \in C([0,1] \times [0,\infty) \times (-\infty,\infty), [0,\infty)).$$

So there exists a $C_4 > 0$ satisfying

$$f(t,u,u') \le C_4 \text{ for } (t,u,u') \in [0,1] \times [0,p_1] \times [-p_1,p_1]. \tag{5.6}$$

According to (5.5) and (5.6), we have

$$f(t,u,u') \le \max\{C_4, \Lambda_1 |u'|\} \quad \text{for } (t,u,u') \in [0,1]\times[0,\infty)\times(-\infty,\infty).$$

Let

$$p_2^* > \max\{C_4/\Lambda_1, 2q\},$$

and

$$\Omega_{p_2} = \{u \in E_1 : \|u\| < p_2^*\}.$$

If $u \in P_1 \cap \partial\Omega_{p_2}$, by (5.1), one has $\|u\| = p_2^*$ and

$$\begin{aligned}\|Au\| &\le (\alpha-1)\int_0^1 G(1,s) f(s,u(s),u'(s))\,ds\\ &\le (\alpha-1)\int_0^1 G(1,s)\max\{C_4, \Lambda_1|u'|\}\,ds\\ &\le (\alpha-1)\Lambda_1 p_2^* \int_0^1 G(1,s)\,ds = \|u\|.\end{aligned}$$

This implies our desired result. □

Next, we deal with the existence of at least two distinct positive solutions to problem (1.2).

Theorem 5.6. *Assume that all assumptions of Lemma 4.1 hold. Moreover, suppose that $f_0 = \infty$ and $f_\infty = \infty$, and the condition (i) in Theorem 5.3 is satisfied. Then problem (1.2) has at least two distinct positive solutions u_1, $u_2 \in P_1$.*

Proof. In view of $f_0 = \infty$, there exists an H_1 such that $0 < H_1 < p$ and

$$f(t,u,u') \ge m(u+|u'|) \ge m\|u\| \quad \text{for } (t,u,u') \in [0,1]\times(0,H_1]\times[-H_1,H_1], \tag{5.7}$$

where m is given by

$$\gamma m \int_{1/2}^1 G(1,s)ds \ge 1. \tag{5.8}$$

Take

$$\Omega_{H_1} = \{u \in E_1 : \|u\| < H_1\}.$$

If $u \in \Omega_{H_1}$ with $\|u\| = H_1$, it means that

$$\max_{t\in[0,1]} u(t) \le \max_{t\in[0,1]} |u'(t)| \le \|u\| = H_1 \text{ for } t \in [0,1].$$

It follows from (5.7) and (5.8) that

$$\begin{aligned}\|Au\| &\ge \int_{1/2}^1 \min_{t\in[1/2,1]} G(t,s) f(s,u(s),u'(s))\,ds\\ &\ge \int_{1/2}^1 \gamma G(1,s) m\|u\| ds\\ &\ge \gamma m \int_{1/2}^1 G(1,s)\,ds \|u\|,\end{aligned}$$

which implies that

$$\|Au\| \ge \|u\| \quad \text{for } u \in P_1 \cap \partial\Omega_{H_1}.$$

Let

$$\Omega_p = \{u \in E_1 : \|u\| < p\}.$$

Then, we obtain that (5.2) holds by using the condition

(i) of Theorem 5.3. According to Lemma 2.4, problem (1.2) has a positive solution u_1 in $P_1 \cap (\overline{\Omega}_p \backslash \Omega_{H_1})$.

It follows from $f_\infty = \infty$ that there exists an $H_2 > 4p$ such that

$$f(t,u,u') \geq k(u + |u'|) \geq k\|u\|, \tag{5.9}$$

where $t \in [0,1]$ and $u + |u'| \geq H_2$. Moreover, k satisfies

$$k\gamma \int_{1/2}^{1} G(1,s)dr \geq 1.$$

Let

$$\Omega_{H_2} = \{u \in \Omega_H : u + |u'| < H_2\},$$

then we see that $\overline{\Omega}_p \subset \Omega_{H_2}$.

For any $u \in P_1 \cap \partial\Omega_{H_2}$, we have $u + |u'| = H_2$. According to (5.9), we deduce that

$$\begin{aligned}
\|Au\| &\geq \int_{1/2}^{1} \min_{t\in[1/2,1]} G(t,s) f(s,u(s),u'(s))\,ds \\
&\geq \int_{1/2}^{1} \gamma G(1,s) k\|u\| ds \\
&\geq \gamma k \int_{1/2}^{1} G(1,s)\,ds \|u\| \geq \|u\|.
\end{aligned}$$

Thus, it follows from (i) of Lemma 2.4 that problem (1.2) has at least a single positive solution u_2 in $P_1 \cap (\overline{\Omega}_{H_2} \backslash \Omega_p)$ with

$$p \leq \|u_2\| \text{ and } u_2 + |u_2'| \leq H_2.$$

It is easily seen that u_1 and u_2 are distinct. □

By a closely similar way, we can obtain the following result.

Theorem 5.7. *Assume that all assumptions of Lemma 4.1 hold. Moreover, suppose that $f^0 = 0$ and $f^\infty = 0$, and the condition (ii) in Theorem 5.3 is satisfied, then problem (1.2) has at least two distinct positive solutions u_1, $u_2 \in P_1$.*

6. Existence of triple or multiple solutions

We have obtained some existence results of at least one or two distinct positive solutions to fractional differential (1.2) in the preceding section. In this section, we will further discuss the existence of at least 3, n or $2n-1$ positive solutions to fractional differential (1.2) by using different fixed point theorems.

For the notational convenience, we define

$$M = \int_{1/2}^{1} G(1,s) ds, \quad N = \gamma \int_{1/2}^{1} G(1,s)\,ds, \quad L = (\alpha - 1)\int_{0}^{1} G(1,s)\,ds.$$

6.1. **Existence of Three Solutions.** In this subsection, we investigate the existence of at least three distinct positive solutions of (1.2).

Theorem 6.1. *Let a, b and c be constants such that $0 < a < b < d \le c$ and $bL < cN$. In addition, if all assumptions of Lemma 4.1 hold and $f(t, u_1, u_2)$ satisfies the following conditions:*

- (i) $f(t, u_1, u_2) < \frac{a}{L}$ *for* $(t, u_1, u_2) \in [0, 1] \times [0, a] \times [-a, a]$;
- (ii) $f(t, u_1, u_2) > \frac{b}{N}$ *for* $(t, u_1, u_2) \in [\frac{1}{2}, 1] \times [b, d] \times [-c, c]$;
- (iii) $f(t, u_1, u_2) \le \frac{c}{L}$ *for* $(t, u_1, u_2) \in [0, 1] \times [0, c] \times [-c, c]$.

Then (1.2) *has at least three positive solutions $u_1, u_2, u_3 \in P_1$ such that*

$$0 < \|u_1\| < a, \quad b < \inf_{t\in[1/2,1]} u_2, \quad a < u_3 \quad \text{with} \quad \inf_{t\in[1/2,1]} u_3 < b. \tag{6.1}$$

Proof. By the definition of the completely continuous operator A and by Lemma 2.5, we consider all conditions of Lemma 2.5 with respect to A. Let

$$q(u) = \inf_{t\in[1/2,1]} u(t) \quad \text{for } u \in \overline{P}_1,$$

then $q(u)$ is a nonnegative continuous concave function and satisfies

$$q(u) \le \|u\| \text{ for } u \in P_{1_c} = \{u \in P_1 : \|u\| \le c\}.$$

Since

$$u \in [0, c] \text{ and } u' \in [-c, c] \text{ for } u \in P_{1_c},$$

according to condition (iii) and (5.1), we have

$$\begin{aligned} \|Au\| &\le (\alpha - 1) \int_0^1 G(1, s) f(s, u(s), u'(s))\, ds \\ &\le (\alpha - 1) \int_0^1 G(1, s) \frac{c}{L}\, ds \le c, \end{aligned} \tag{6.2}$$

which implies $A : \overline{P}_{1_c} \to \overline{P}_{1_c}$. When $u \in P_{1_a} = \{u \in P_1 : \|u\| \le a\}$, it implies that

$$u \in [0, a] \text{ and } u' \in [-a, a].$$

We observe that the conditions (ii) of Lemma 2.5 is true.

Let d be a fixed constant such that $b < d \le c$, then we have $q(d) = d > b$ and $\|d\| = d$. This means that

$$d \in P_1(q, b, d) = \{u \in P_1 : b \le q(u), \|u\| \le d\}.$$

For any $u \in P_1(q, b, d)$, we obtain

$$\|u\| \le d \text{ and } q(u) = \inf_{t\in[1/2,1]} u \ge b,$$

which implies

$$u \in [b, d] \text{ and } u' \in [-d, d] \quad \text{for } t \in [1/2, 1].$$

Hence,

$$\begin{aligned} q(Au) &= \inf_{t\in[1/2,1]} Au \\ &= \int_{1/2}^1 \inf_{t\in[1/2,1]} G(t, s) f(s, u(s), u'(s))\, ds \\ &> \int_{1/2}^1 \gamma G(1, s) \frac{b}{N}\, ds > b, \end{aligned}$$

which means that condition (i) of Lemma 2.5 holds.

For any $u \in P_1(q,b,c)$ with $\|Au\| > d$, it gives $\|u\| \le c$ and $\inf_{t\in[1/2,1]} u \ge b$. By using the same argument as the above, we see that $q(Au) > b$. This implies that the condition (iii) of Lemma 2.5 is fulfilled.

Consequently, all conditions of Lemma 2.5 are verified. That is, problem(1.2) has at least three distinct solutions distributed as (6.1). □

Corollary 6.2. *Assume that all assumptions of Lemma 4.1 hold. If the condition (iii) in Theorem 6.1 is replaced by*

(iii') $f^\infty = \lim_{(u_1,u_2)\to(\infty,\infty)} \sup_{t\in[0,1]} \frac{f(t,u_1,u_2)}{|u_2|} \le \frac{1}{L}$ *for* $u_1, u_2 \in P_1$,

then (6.1) *in Theorem 6.1 also holds.*

Proof. We only need to prove that the condition (iii') implies the condition (iii) in Theorem 6.1. That is, assume that (iii') holds, then there exists a number $c^* \ge d^*$ such that
$$f(t,u,u') \le \frac{c^*}{L} \text{ for } (t,u,u') \in [0,1]\times[0,c^*]\times[-c^*,c^*].$$
Conversely, we suppose that for any $c^* \ge d^*$, there exists
$$(u_c,u_c') \in [0,c^*]\times[-c^*,c^*]$$
such that
$$f(t,u_c,u_c') > \frac{c^*}{L} \quad \text{for } t\in[0,1].$$
Take
$$c_n^* > d^* \quad (n=1,2,\dots) \text{ with } c_n^* \to \infty.$$
Then, there exists
$$(u_n,u_n') \in [0,c_n^*]\times[-c_n^*,c_n^*]$$
such that
$$f(t,u_n,u_n') > \frac{c_n^*}{L} \quad \text{for } t\in[0,1], \tag{6.3}$$
$$\lim_{n\to\infty} f(t,u_n,u_n') = \infty \quad \text{for } t\in[0,1]. \tag{6.4}$$

Since condition (iii') holds, there is a $\tau > 0$ such that
$$f(t,u,u') \le \frac{|u'|}{L} \quad \text{for } (t,u,u') \in [0,1]\times[\tau,\infty)\times(-\infty,\tau]\cup[\tau,\infty). \tag{6.5}$$
Thus,
$$|u_n'(t)| \le \tau \quad \text{for } t\in[0,1],$$
which implies $u_n(t) \le \tau$ for $t\in[0,1]$. Otherwise, if
$$|u_n'(t)| > \tau \text{ and } u_n(t) > \tau \quad \text{for } t\in[0,1],$$
it follows from (6.5) that
$$f(t,u_n,u_n') \le \frac{|u_n'|}{L} \le \frac{c_n^*}{L} \quad \text{for } t\in[0,1],$$
which contradicts inequality (6.3).

Let
$$W = \max_{(t,u,u')\in[0,1]\times[0,\tau]\times[-\tau,\tau]} f(t,u,u'),$$
so we have
$$f(t,u_n,u_n') \le W \ (n=1,2,\dots).$$

This yields a contradiction with formula (6.4). The proof is complete. □

6.2. **Existence of n solutions.** In this subsection, the existence criteria for at least three or an arbitrary number n positive solutions to fractional differential (1.2) are obtained by using the generalized Avery-Henderson fixed point theorem.

We define the nonnegative, increasing, continuous functionals γ_1, β_1 and α_1 by

$$\gamma_1(u) = \beta_1(u) = \max\{\inf_{t\in[1/2,1]} u,\ \inf_{t\in[1/2,1]} |u'|\} \quad \text{for } u \in P_1,$$
$$\alpha_1(u) = \max\{\sup_{t\in[0,1]} u,\ \sup_{t\in[0,1]} |u'|\} \quad \text{for } u \in P_1,$$

so we have

$$\gamma_1(u) = \beta_1(u) \le \alpha_1(u) \quad \text{for each } u \in P_1.$$

Since $u(t) = \int_0^1 G(t,s)y(s)\,ds$, in view of Lemma 3.2, we deduce

$$\begin{aligned}
\inf_{t\in[1/2,1]} u(t) = \min_{t\in[1/2,1]} u(t) &= \int_0^1 \min_{t\in[1/2,1]} G(t,s)y(s)\,ds \\
&\ge \gamma \max_{t\in[0,1]} \int_0^1 G(t,s)y(s)\,ds \\
&= \gamma\|u\|_1.
\end{aligned}$$

According to Lemma 3.4,

$$\begin{aligned}
\inf_{t\in[1/2,1]} |u'(t)| &= \int_0^1 \min_{t\in[1/2,1]} \frac{\partial G(t,s)}{\partial t} y(s)\,ds \\
&\ge \gamma^* \max_{t\in[0,1]} \int_0^1 \frac{\partial G(t,s)}{\partial t} y(s)\,ds \\
&= \gamma^*\|u\|_2.
\end{aligned}$$

Hence, we have

$$\|u\| \le \max\{\frac{1}{\gamma}, \frac{1}{\gamma^*}\}\gamma_1(u) = \frac{1}{\gamma}\gamma_1(u) \quad \text{for all } u \in P_1.$$

Theorem 6.3. *Assume that there exist real numbers a', b', c' with $a' < b' < c'$ such that $0 < Lb' < a'N$. In addition, if all assumptions of Lemma 4.1 hold and $f(t,u_1,u_2)$ satisfies the following conditions:*

(i) $f(t,u_1,u_2) < \frac{c'}{L}$ *for* $(t,u_1,u_2) \in [1/2,1] \times [0, \frac{1}{\gamma}c'] \times [-\frac{1}{\gamma}c', \frac{1}{\gamma}c']$;
(ii) $f(t,u_1,u_2) > \frac{b'}{N}$ *for* $(t,u_1,u_2) \in [1/2,1] \times [0, \frac{1}{\gamma}b']] \times [-\frac{1}{\gamma}b', \frac{1}{\gamma}b']$;
(iii) $f(t,u_1,u_2) < \frac{a'}{L}$ *for* $(t,u_1,u_2) \in [0,1] \times [0,a'] \times [-a',a']$.

Then (1.2) *has at least three distinct positive solutions $u_1, u_2, u_3 \in \overline{P_1}(\gamma_1, c')$ such that*

$$\begin{gathered}
0 < \|u_1\| < a' < \|u_2\|, \\
\max\{\inf_{t\in[1/2,1]} u_2,\ \inf_{t\in[1/2,1]} |u_2'|\} < b' < \max\{\inf_{t\in[1/2,1]} u_3,\ \inf_{t\in[1/2,1]} |u_3'|\} < c'.
\end{gathered} \tag{6.6}$$

Proof. We only need to check whether all conditions of Lemma 2.6 are fulfilled with respect to the operator A. By using a similar way as to the proof of inequality (6.2), we can see that

$$A : \overline{P}_1(\gamma_1, c) \to \overline{P}_1.$$

For arbitrary $u \in \partial P_1(\gamma_1, c')$, one has

$$\gamma_1(u) = \max\Big\{\inf_{t\in[1/2,1]} u,\ \inf_{t\in[1/2,1]} |u'|\Big\} = c',$$
$$\|u\| \le \frac{1}{\gamma}\gamma_1(u) = \frac{1}{\gamma}c'.$$

This implies that

$$0 \le u \le \frac{1}{\gamma}c', \quad t \in [1/2, 1],$$
$$-\frac{1}{\gamma}c' \le u' \le \frac{1}{\gamma}c', \quad t \in [1/2, 1].$$

According to the condition (i) and $\alpha > 3$, we have

$$\begin{aligned}
\gamma_1(Au) &= \max\big\{\inf_{t\in[1/2,1]} Au, \inf_{t\in[1/2,1]} |Au'|\big\}\\
&< (\alpha-1)\int_0^1 G(1,s)f(s,u(s),u'(s))ds\\
&\le (\alpha-1)\int_0^1 G(1,s)\frac{c'}{L}ds < c'.
\end{aligned}$$

We see that $\gamma_1(Au) < c'$ for $u \in \partial P_1(\gamma_1, c')$.

For any $u \in \partial P_1(\beta_1, b')$, we have

$$\beta_1(u) = \max\big\{\inf_{t\in[1/2,1]} u,\ \inf_{t\in[1/2,1]} |u'|\big\} = b',$$
$$\|u\| \le \frac{1}{\gamma}\beta_1(u) = \frac{1}{\gamma}\gamma_1(u) = \frac{1}{\gamma}b'.$$

This implies

$$0 \le u \le \frac{1}{\gamma}b', \quad t \in [1/2, 1],$$
$$-\frac{1}{\gamma}b' \le u' \le \frac{1}{\gamma}b', \quad t \in [1/2, 1].$$

Using condition (ii), we obtain

$$\begin{aligned}
\beta_1(Au) &= \max\big\{\inf_{t\in[1/2,1]} Au, \inf_{t\in[1/2,1]} |Au'|\big\}\\
&> \int_{1/2}^1 \min_{t\in[1/2,1]} G(t,s)f(s,u(s),u'(s))\,ds\\
&\ge \int_{1/2}^1 \gamma G(1,s)\frac{b'}{N}\,ds\\
&\ge \frac{b'}{N}\gamma\int_{1/2}^1 G(1,s)\,ds = b'.
\end{aligned}$$

So we have $\beta_1(Au) > b'$ for $u \in \partial P_1(\beta_1, b')$.

We now show that $P_1(\alpha_1, a') \neq \emptyset$ and $\alpha_1(Au) < a'$ for arbitrary $u \in \partial P_1(\alpha_1, a')$. Since $\frac{a'}{2} \in P_1(\alpha_1, a')$, for $u \in \partial P_1(\alpha_1, a')$ we have

$$\alpha_1(u) = \max\{\sup_{t\in[0,1]} u,\ \sup_{t\in[0,1]} |u'|\} = a',$$

which gives

$$0 \le u \le a' \text{ and } -a' \le u' \le a' \quad \text{for } t \in [0,1].$$

It follow from assumption (iii) and (5.1) that

$$\begin{aligned} \alpha_1(Au) &= \max\big\{\sup_{t\in[0,1]} Au, \sup_{t\in[0,1]} |A'u|\big\} \\ &= (\alpha-1)\int_0^1 G(1,s) f(s,u(s),u'(s))\,ds \\ &< (\alpha-1)\int_0^1 G(1,s)\frac{a'}{L}\,ds \\ &< (\alpha-1)\frac{a'}{L}\int_0^1 G(1,s)\,ds = a'. \end{aligned}$$

All conditions in Lemma 2.6 are satisfied. From (S1) and (S2), we know that solutions of (1.2) do not vanish identically on any closed subinterval of $[0,1]$. Consequently, (1.2) has at least three distinct positive solutions u_1, u_2 and u_3 belonging to $\overline{P}(\gamma_1, c')$ distributed as (6.6). □

The following result is regarded as a corollary of Theorem 6.3.

Corollary 6.4. *Assume that all assumptions of Lemma 4.1 hold and f satisfies the following conditions:*

(i) $f^0 = 0$ *and* $f^\infty = 0$*;*
(ii) *there exists a constant* $c_0 > 0$ *such that*

$$f(t,u_1,u_2) > \frac{\gamma c_0}{N} \quad \text{for } (t,u_1,u_2) \in [1/2,1]\times[0,c_0]\times[-c_0,c_0].$$

Then (1.2) *has at least three distinct positive solutions.*

Proof. Let $b' = \gamma c_0$. It follows from the condition (ii) that

$$f(t,u_1,u_2) > \frac{b'}{N} \quad \text{for } (t,u_1,u_2) \in [1/2,1]\times[0,\frac{b'}{\gamma}]\times[-\frac{1}{\gamma}b',\frac{1}{\gamma}b'],$$

which implies that the condition (ii) of Theorem 6.3 holds.

We choose a sufficiently small $\varepsilon_5 > 0$ such that

$$\varepsilon_5 L = \varepsilon_5(\alpha-1)\Big(\int_0^1 G(1,s)\,ds\Big) < 1. \tag{6.7}$$

In view of $f^0 = 0$, there exists a sufficiently small $k_1 > 0$ such that

$$f(t,u_1,u_2) \le \varepsilon_5|u_2| \quad \text{for } (t,u_1,u_2) \in [0,1]\times[0,k_1]\times[-k_1,k_1]. \tag{6.8}$$

Without loss of generality, let $k_1 = a' < b'$. Because of $\max_{t\in[0,1]}|u_2| \le a'$, we have

$$\max_{t\in[0,1]} u_1 \le \max_{t\in[0,1]} |u_2| \le k_1.$$

It follows from (6.7) and (6.8) that

$$f(t,u_1,u_2) \le \varepsilon_5|u_2| \le \varepsilon_5 a' < \frac{a'}{L} \text{ for } (t,u_1,u_2) \in [0,1]\times[0,a']\times[-a',a'],$$

which implies that condition (iii) of Theorem 6.3 holds.

Choose ε_6 sufficiently small such that

$$\frac{\varepsilon_6}{\gamma}L = (\alpha-1)\frac{\varepsilon_6}{\gamma}\Big(\int_{1/2}^1 G(1,s)ds\Big) < 1.$$

By using the continuity of f, there exists a constant C^* such that

$$f(t,u_1,u_2) \le C^* \text{ for } (t,u_1,u_2) \in [0,1]\times[0,c'/\gamma]\times[-c',c']. \tag{6.9}$$

Since $f^\infty = 0$, there exists a sufficiently large $k_2 > LC^*$ such that

$$f(t,u_1,u_2) \le \varepsilon_6|u_2| \quad \text{for}(t,u_1,u_2) \in [0,1]\times[k_2,+\infty)\times[k_2,+\infty)\cup(-\infty,k_2].$$

Without loss of generality, let $k_2 > b'/\gamma$ and $c' = k_2$. We find

$$(t,u_1,u_2) \in [0,1]\times[0,c'/\gamma]\times[-c'/\gamma,-c']\cup[c',c'/\gamma],$$

and

$$f(t,u_1,u_2) \le \varepsilon_6|u_2| \le \varepsilon_6\frac{c'}{\gamma} < \frac{c'}{L}. \tag{6.10}$$

Moreover, in view of (6.9), one has

$$f(t,u_1,u_2) \le C^* < \frac{k_2}{L} = \frac{c'}{L} \quad \text{for } (t,u_1,u_2) \in [0,1]\times[0,\frac{1}{\gamma}c']\times[-c',c']. \tag{6.11}$$

From (6.10) and (6.11), we see that condition (i) of Theorem 6.3 is fulfilled. Hence, (1.2) has at least three distinct positive solutions according to Theorem 6.3. $\square$

According to Theorem 6.3, we can prove that the existence for multiple positive solutions to the (1.2) when conditions (i), (ii) and (iii) are modified appropriately on f.

Theorem 6.5. *If there exist constant numbers a_i', b_i' and c_i' such that $0 < a_1' < b_1' < c_1' < \dots < a_n' < b_n' < c_n'$ together with*

$$0 < Lb_1' < a_1'N < Lb_2' < a_2'N < \dots < Lb_n' < a_n'N, \quad n \in \mathbb{N}, \tag{6.12}$$

where $i = 1,2,\dots,n$. In addition, if all assumptions of Lemma 4.1 hold and the function f satisfies:

- (i) $f(t,u_1,u_2) < \frac{c_i'}{L}$ *for* $(t,u_1,u_2) \in [1/2,1]\times[0,\frac{1}{\gamma}c_i']\times[-\frac{1}{\gamma}c_i',\frac{1}{\gamma}c_i']$;
- (ii) $f(t,u_1,u_2) > \frac{b_i'}{N}$ *for* $(t,u_1,u_2) \in [1/2,1]\times[0,\frac{1}{\gamma}b_i']\times[-\frac{1}{\gamma}b_i',\frac{1}{\gamma}b_i']$;
- (iii) $f(t,u_1,u_2) < \frac{a_i'}{L}$ *for* $(t,u_1,u_2) \in [1/2,1]\times[0,a_i']\times[-a_i',a_i']$.

Then (1.2) *has at least n distinct positive solutions.*

Proof. (By Mathematical Induction) If $n = 1$, from the condition (iii), we have

$$A : \overline{P}_{a_1'} \to P_{a_1'} \subset \overline{P}_{a_1'}.$$

It follows from the Schauder fixed point theorem that A has at least one fixed point $u_{01} \in \overline{P}_{a_1'}$.

If $i = 2$, we let $a' = a_1'$, $b' = b_1'$ and $c' = c_1'$. By using Theorem 6.3, problem (1.2) has at least three distinct positive solutions u_{11}, u_{12} and u_{13} such that

$$0 < \|u_{11}\| < a_1' < \|u_{12}\|,$$
$$\max\Big\{\inf_{t\in[1/2,1]} u_{12},\ \inf_{t\in[1/2,1]} |u_{12}'|\Big\} < b_1' < \max\Big\{\inf_{t\in[1/2,1]} u_{13},\ \inf_{t\in[1/2,1]} |u_{13}'|\Big\} < c_1',$$

which implies that problem (1.2) has at least 2 distinct positive solutions.

Assume that problem (1.2) has at least $k-1$ distinct positive solutions when $n=k-1$. We denote by u_i again. It follows from the solution position and local properties that

$$0<\max\Big\{\inf_{t\in[1/2,1]} u_i,\ \inf_{t\in[1/2,1]}|u_i'|\Big\}<c_{k-1}',\quad i=1,2,\dots,k-1,\ \text{where } c_0'=a_1'. \tag{6.13}$$

When $n=k$, we let $a'=a_k'$, $b'=b_k'$ and $c'=c_k'$. According to Theorem 6.3, there exist at least three positive solutions u_{k1}, u_{k2} and u_{k3} such that

$$\begin{gathered}0<\|u_{k1}\|<a_k'<\|u_{k2}\|,\\ \max\Big\{\inf_{t\in[1/2,1]} u_{k2},\ \inf_{t\in[1/2,1]}|u_{k2}'|\Big\}<b_k'<\max\Big\{\inf_{t\in[1/2,1]} u_{k3},\ \inf_{t\in[1/2,1]}|u_{k3}'|\Big\}<c_k'.\end{gathered} \tag{6.14}$$

Combining (6.13) and (6.14) gives

$$\max\Big\{\inf_{t\in[1/2,1]} u_i,\ \inf_{t\in[1/2,1]}|u_i'|\Big\}<c_{k-1}'<b_k'<\max\Big\{\inf_{t\in[1/2,1]} u_{k3},\ \inf_{t\in[1/2,1]}|u_{k3}'|\Big\}.$$

This implies

$$u_i\neq u_{k3},\quad i=1,2,\dots,k-1.$$

Therefore, (1.2) has at least n distinct positive solutions. □

By Lemma 2.6 and Theorem 6.3, we can obtain the following results:

Theorem 6.6. *Assume that there exist positive numbers a',b',c' with $a'<b'<c'$ such that $c'L<b'N$. In addition, if all assumptions of Lemma 4.1 hold and $f(t,u_1,u_2)$ satisfies the following conditions:*

- (i) *$f(t,u_1,u_2)>\frac{c'}{N}$ for $(t,u_1,u_2)\in[1/2,1]\times[0,\frac{1}{\gamma}c']\times[-\frac{1}{\gamma}c',\frac{1}{\gamma}c']$;*
- (ii) *$f(t,u_1,u_2)<\frac{b'}{L}$ for $(t,u_1,u_2)\in[1/2,1]\times[0,\frac{1}{\gamma}b']\times[-\frac{1}{\gamma}b',\frac{1}{\gamma}b']$;*
- (iii) *$f(t,u_1,u_2)>\frac{a'}{N}$ for $(t,u_1,u_2)\in[0,1]\times[0,a']\times[-a',a']$.*

Then (1.2) has at least three distinct positive solutions $u_1,u_2,u_3\in\overline{P_1}(\gamma_1,c')$ such that

$$\begin{gathered}0\le\|u_1\|<a'<\|u_2\|,\\ \max\Big\{\inf_{t\in[1/2,1]} u_2,\ \inf_{t\in[1/2,1]}|u_2'|\Big\}<b'<\max\Big\{\inf_{t\in[1/2,1]} u_3,\ \inf_{t\in[1/2,1]}|u_3'|\Big\}<c'.\end{gathered}$$

Corollary 6.7. *Assume that all assumptions of Lemma 4.1 hold and f satisfies conditions*

- (i) *$f_0=\infty$ and $f_\infty=\infty$;*
- (ii) *there exists $c_0>0$ such that $f(t,u_1,u_2)<\frac{\gamma}{M}c_0$ for $(t,u_1,u_2)\in[1/2,1]\times[0,c_0]\times[-c_0,c_0]$.*

Then (1.2) has at least three distinct positive solutions.

Theorem 6.8. *Assume that all assumptions of Lemma 4.1 hold and there are positive numbers a_i',b_i',c_i' such that $a_1'<b_1'<c_1'<\dots<a_n'<b_n'<c_n'$ together with*

$$0<c_1'L<Nb_1'<c_2'L<Nb_2'<\dots<c_n'L<Nb_n',\quad n\in\mathbb{N},$$

where $i=1,2,\dots,n$. In addition, $f(t,u_1,u_2)$ satisfies the following conditions:

- (i) *$f(t,u_1,u_2)>\frac{c_i'}{N}$ for $(t,u_1,u_2)\in[1/2,1]\times[0,\frac{1}{\gamma}c_i']\times[-\frac{1}{\gamma}c_i',\frac{1}{\gamma}c_i']$;*

(ii) $f(t,u_1,u_2) < \frac{b_i'}{L}$ *for* $(t,u_1,u_2) \in [1/2,1] \times [0, \frac{1}{\gamma} b_i'] \times [-\frac{1}{\gamma} b_i', \frac{1}{\gamma} b_i']$;
(iii) $f(t,u_1,u_2) > \frac{a_i'}{N}$ *for* $(t,u_1,u_2) \in [0,1] \times [0, a_i'] \times [-a_i', a_i']$.

Then (1.2) *has at least n distinct positive solutions.*

6.3. **Existence of $2n-1$ solutions.** In this subsection, we are concerned with the existence of at least three or $2n-1$ positive solutions to (1.2).

Define the nonnegative continuous convex functionals ϕ and β, concave functional λ and functional φ on P_1 by

$$\phi(u) = \max\Big\{ \sup_{t\in[0,1]} u, \sup_{t\in[0,1]} |u'| \Big\},$$
$$\beta(u) = \varphi(u) = \sup_{t\in[1/2,1]} u, \quad \lambda(u) = \inf_{t\in[\frac{1}{2},1]} u.$$

Theorem 6.9. *Assume that all assumptions of Lemma 4.1 hold and there exist constants* a^*, b^*, d^* *such that* $0 < a^* < b^* < \frac{N}{L} d^*$. *In addition,* $f(t,u_1,u_2)$ *satisfies the following conditions:*

(i) $f(t,u_1,u_2) \le \frac{d^*}{L}$ *for all* $(t,u_1,u_2) \in [0,1] \times [0,d^*] \times [-d^*, d^*]$;
(ii) $f(t,u_1,u_2) > \frac{b^*}{N}$ *for all* $(t,u_1,u_2) \in [1/2,1] \times [b^*,d^*] \times [-d^*, d^*]$;
(iii) $f(t,u_1,u_2) < \frac{a^*}{M}$ *for all* $(t,u_1,u_2) \in [1/2,1] \times [0,a^*] \times [-d^*, d^*]$.

Then (1.2) *has at least three distinct positive solutions* u_1, u_2, u_3 *such that*

$$\|x_i\| \le d^* \text{ for } i=1,2,3, \quad b^* < \inf_{t\in[1/2,1]} u_1, \quad a^* < \sup_{t\in[\frac{1}{2},1]} u_2, \quad \inf_{t\in[\frac{1}{2},1]} u_2 < b^*$$

with $\sup_{t\in[\frac{1}{2},1]} u_3 < a^*$.

Proof. It suffices to show that all conditions of Lemma 2.7 hold with respect to the completely continuous operator A. For arbitrary $u \in P_1$, we have $\lambda(u) = \varphi(u)$ and $\|u\| = \phi(u)$. This implies that inequality (2.1) in Lemma 2.7 holds.

For any $u \in \overline{P_1(\phi, d^*)}$, from $\phi(u) = \|u\| \le d^*$ and the assumption (i), we have

$$\begin{aligned} \|Au\| &\le (\alpha-1)\int_0^1 G(1,s) f(s,u(s),u'(s))\,ds \\ &\le (\alpha-1)\frac{d^*}{L}\int_0^1 G(1,s)\,ds = d^*. \end{aligned}$$

This means that $A : \overline{P_1(\phi, d^*)} \to \overline{P_1(\phi, d^*)}$.

It remains to show that assumptions (i)-(iii) of Lemma 2.7 are fulfilled with respect to the operator A.

Let $u \equiv kb^*$, where $k = L/N$. It is obvious that $k > 1$, $u = kb^* > b^*$ and $\beta(u) = kb^*$. We see that $b^* < \frac{N}{L} d^*$ and $\phi(u) = kb^* < d^*$. So we have

$$\{u \in P_1(\phi,\beta,\lambda,b^*,kb^*,d^*) : \lambda(x) > b^*\} \ne \emptyset.$$

For any $u \in P_1(\phi,\beta,\lambda,b^*,kb^*,d^*)$, we obtain $b^* \le u \le d^*$ and $-d^* \le u' \le d^*$ for all $t \in [1/2,1]$. It follows from the assumption (ii) that

$$\begin{aligned} \lambda(Au) &= \int_{1/2}^1 \inf_{t\in[1/2,1]} G(t,s) f(s,u(s),u'(s))\,ds \\ &\ge \int_{1/2}^1 \gamma G(1,s)\frac{b^*}{N}\,ds = b^*, \end{aligned}$$

which implies that the assumption (i) of Lemma 2.7 is satisfied.

For any $u \in P_1(\phi, \lambda, b^*, d^*)$ with $\beta(Au) > kb^*$, we have $b^* \le u \le d^*$ and $-d^* \le u' \le d^*$ for $t \in [1/2, 1]$. So we have

$$\begin{aligned}\lambda(Au) &= \int_{1/2}^1 \inf_{t\in[1/2,1]} G(t,s) f(s, u(s), u'(s))\, ds \\ &\ge \int_{1/2}^1 \gamma G(1,s) \frac{b^*}{N}\, ds = b^*.\end{aligned}$$

This implies that assumption (ii) of Lemma 2.7 is fulfilled.

Since $\varphi(0) = 0 < a^*$, we have $0 \notin R(\phi, \varphi, a^*, d^*)$. If

$$u \in R(\phi, \varphi, a^*, d^*) \quad \text{with } \varphi(u) = \sup_{t\in[1/2,1]} u = a^*,$$

it reduces to

$$0 \le u \le a^* \text{ and } -d^* \le u' \le d^* \quad \text{for all } t \in [1/2, 1].$$

A straightforward calculation gives

$$\begin{aligned}\lambda(Au) &= \inf_{t\in[1/2,1]} \int_{1/2}^1 G(t,s) f(s, u(s), u'(s))\, ds \\ &\le \int_{1/2}^1 G(t,s) f(s, u(s), u'(s))\, ds \\ &< \int_{1/2}^1 G(1,s) \frac{a^*}{M} ds = a^*.\end{aligned}$$

It is easy to see the assumption (iii) of Lemma 2.7 is fulfilled too. The proof is complete. □

Corollary 6.10. *Assume that all assumptions of Lemma 4.1 hold and the condition* (i) *in Theorem 6.9 is replaced by* (i'), *then the conclusion of Theorem 6.9 also holds.*

Similar to the proof of Theorem 6.5 by mathematical induction, we have

Theorem 6.11. *Assume that all assumptions of Lemma 4.1 hold and there exist constants a_i^*, b_i^* and d_i^* such that*

$$0 < a_1^* < b_1^* < \frac{N}{L} d_1^* < a_2^* < b_2^* < \frac{N}{L} d_2^* < a_3^* < \dots < a_n^*, \quad n \in \mathbb{N},$$

where $i = 1, 2, \dots, n$. In addition, f satisfies the following conditions:

(i) $f(t, u_1, u_2) \le \frac{d_i^*}{L}$ *for all* $(t, u_1, u_2) \in [0,1] \times [0, d_i^*] \times [-d_i^*, d_i^*]$;
(ii) $f(t, u_1, u_2) > \frac{b_i^*}{N}$ *for all* $(t, u_1, u_2) \in [0,1] \times [b_i^*, d_i^*] \times [-d_i^*, d_i^*]$;
(iii) $f(t, u_1, u_2) < \frac{a_i^*}{M}$ *for all* $(t, u_1, u_2) \in [1/2,1] \times [0, a_i^*] \times [-d_i^*, d_i^*]$.

Then (1.2)*has at least* $2n - 1$ *positive solutions.*

7. Examples

In this section, we give two simple examples to illustrate our theoretical results. In Example 7.1, it shows the difference between two cases, which indicates our theorems presented in Sections 5 and 6 are complementary.

Example 7.1. Consider the equation

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t),u'(t))=0,\quad t\in(0,1),n-1<\alpha\le n,\\ u^{(i)}(0)=0,\quad i=0,1,2,\dots,n-2,\\ [D_{0+}^{\beta}u(t)]_{t=1}=0,\quad 1\le\beta\le n-2. \end{gathered}\tag{7.1}$$

Case 1: when $f(t,u,u')$ takes the form as

$$f(t,u,u')=\frac{t}{4(\alpha-1)G(1,t)}(u(t)+u'(t))\quad\text{for } (t,u,u')\in[0,1]\times[0,\infty)\times(-\infty,+\infty).$$

It is easy to see that $a_1(t)=0$ and $a_2(t)=\frac{t}{2(\alpha-1)G(1,t)}$. Moreover, we see that

$$(\alpha-1)\int_0^1 G(1,s)a_2(s)\,ds=\int_0^1\frac{s}{2}\,ds<1.$$

By means of Theorem 5.1, we find that the fractional differential equation (7.1) has at least one positive solution. However, it is difficult for us to obtain the existence of at least one positive solution to the fractional differential equation (7.1) by using theorems of the super-linearity and sub-linearity.

Case 2: when $f(t,u,u')$ takes the form as

$$f(t,u,u')=\begin{cases}-e^{u'+1} & \text{for } u'\in(-\infty,-1),\\ u'^3 & \text{for } u'\in[-1,1],\\ e^{u'-1} & \text{for } u'\in(1,+\infty).\end{cases}$$

Since f is continuous, we know that the operator A is completely continuous. It is easy to check that $f_0=0$ and $f_\infty=\infty$. By means of Theorem 5.4, we conclude that fractional differential equation (7.1) has at least one positive solution. But it is difficult for us to know the existence of positive solution to the fractional differential equation (7.1) if we use Theorem 5.1.

Example 7.2. Consider the equation

$$\begin{gathered} D_{0+}^{\frac{7}{2}}u(t)+f(t,u(t),u'(t))=0,\quad t\in(0,1),\\ u(0)=u^{(1)}(0)=0,\\ [D_{0+}^{2}u(t)]_{t=1}=0, \end{gathered}\tag{7.2}$$

where

$$f(t,u,u')=\begin{cases}2\Gamma(\frac{7}{2}) & \text{for } u\in[0,1],\\ \Gamma(\frac{7}{2})(57u-55) & \text{for } u\in(1,2),\\ 59\Gamma(\frac{7}{2}) & \text{for } u\in[2,+\infty).\end{cases}$$

Since $\alpha=7/2$ and $\beta=2$, a straightforward calculation gives

$$\begin{aligned} N&=\gamma\int_{1/2}^1 G(1,s)\,ds\\ &\approx\frac{1}{\Gamma(\frac{7}{2})}\Big(0.1768\int_{1/2}^1(1-s)^{1/2}ds-0.1768\int_{1/2}^1(1-s)^{5/2}ds\Big)\\ &\approx\frac{1}{\Gamma(\frac{7}{2})}3.7207\times10^{-2}, \end{aligned}$$

and

$$L \approx \frac{1}{\Gamma(\frac{7}{2})} 0.3910.$$

Taking $a = 1$, $b = 2$, $d = 80$ and $c = 100$, we see that $0 < a < b < d < \frac{N}{L} c$ and

$$f(t, u, u') < \frac{a}{L} = \frac{\Gamma(\frac{7}{2})}{0.3910} \approx 2.625\Gamma(\frac{7}{2}) \quad \text{for } (t, u, u') \in [0, 1] \times [0, 1] \times [-1, 1],$$

$$f(t, u, u') > \frac{b}{N} = \frac{2\Gamma(\frac{7}{2})}{3.7207 \times 10^{-2}} \approx 53.753\Gamma(\frac{7}{2})$$
$$\text{for } (t, u, u') \in [0, 1] \times [2, 80] \times [-80, 80],$$

$$f(t, u_1, u_2) < \frac{c}{L} = \frac{100\Gamma(\frac{7}{2})}{0.17778} \approx 262.5\Gamma(\frac{7}{2})$$
$$\text{for } (t, u, u') \in [0, 1] \times [0, 100] \times [-100, 100].$$

It follows from Theorem 6.1 that the fractional differential equation (7.2) has at least three distinct positive solutions such that

$$0 < \|u_1\| < 1, \quad 2 < \inf_{t \in [1/2, 1]} u_2, \quad 1 < u_3 \quad \text{with} \quad \inf_{t \in [1/2, 1]} u_3 < 2.$$

Acknowledgments. This work is supported by Natural Science Foundation of China (No.11126091), and Jiangsu Overseas Research and Training Program for University Prominent Young Faculty.

References

[1] I. Podlubny; *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[2] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc, New York, 1993.
[3] J. Sabatier, O. P. Agrawal, J. A. T. Machado; *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.
[4] W. G. Glöckle, T. F. Nonnenmacher; *A fractional calculus approach to self similar protein dynamics*, Biophys. J. 68 (1995), 46–53.
[5] R. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmacher; *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. 103 (1995), 7180–7186
[6] K. Diethelm, N.J. Ford; *Analysis of fractional differential equations*, J. Math. Anal. Appl. 265 (2002), 229–248
[7] N. Laskin; *Fractional Schrödinger equation*, Physical Review E, 66 (2002), 056108-7
[8] S. P. Näsholm, S. Holm; *Linking multiple relaxation, power-law attenuation, and fractional wave equations*, J. Acoust. Soc. Amer. 130 (2011), 3038–3045.
[9] S. Abbas; *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electron. J. Differential Equations, 2011, No. 9, 1–11
[10] Z. B. Bai, H. Lü; *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. 311 (2005), 495–505.
[11] V. S. Erturk, S. Momani; *Solving systems of fractional differential equations using differential transform method*, J. Comput. Appl. Math. 215 (2008), 142–151.
[12] A. A. Kilbas, J. J. Trujillo; *Differential equations of fractional order: methods, results and problems-I*, Appl. Anal. 78 (2001), 153–192.
[13] V. Lakshmikantham, S. Leela, J. Vasundhara Devi; *Theory of Fractional Dynamic Systems, Cambridge Academic Publishers*, Cambridge, 2009.
[14] F. Jiao, Y. Zhou; *Existence of solutions for a class of fractional boundary value problems via critical point theory*, Comput. Math. Appl. 62 (2011), 1181–1199.
[15] Y. H. Su, Z. Feng; *Existence theory for an arbitrary order fractional differential equation with deviating argument*, Acta. Appl. Math. 118 (2012), 81–105.
[16] L. G. Lin, X. P. Liu, H. Q. Fang; *Method of upper and lower solutions for fractional differential equations*, Electron. J. Differential Equations, 2012, No. 100, 1–13

[17] X. Xu, D. Jiang, C. Yuan; *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal. 71 (2009), 4676-4688
[18] C. S. Goodrich; *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett. 23 (2010), 1050–1055.
[19] S. Zhang; *Positive solutions to singular boundary value problem for nonlinear fractional differential equation*, Comput. Math. Appl. 59 (2010), 1300–1309.
[20] M. Krasnosel'skii; *Positive solutions of operator equations*, Noordhoff, Groningen, 1964.
[21] R. P. Agarwal, M. Meehan, D. O'Regan; *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.
[22] R. W. Leggett, L. R. Williams; *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J. 28 (1979), 673–688.
[23] R. I. Avery, A. Peterson; *Three positive fixed points of nonlinear operators on ordered Banach spaces*, Comput. Math. Appl. 42 (2001), 313–422.

Cuiping Cheng
Department of Applied Mathematics, Hangzhou Dianzi University, Hangzhou 310018, China
E-mail address: chengcp0611@163.com

Zhaosheng Feng
Department of Mathematics, University of Texas–Pan American, Edinburg, TX 78539, USA
E-mail address: zsfeng@utpa.edu; fax: (956) 665-5091

Youhui Su
Department of Mathematics, Xuzhou Institute of Technology, Xuzhou 221116, China
Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mail address: youhsu@indiana.edu