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NODAL SOLUTIONS FOR SIXTH-ORDER M-POINT BOUNDARY-VALUE PROBLEMS USING BIFURCATION METHODS

YUDE JI, YANPING GUO, YUKUN YAO, YINGJIE FENG

Abstract. We consider the sixth-order m-point boundary-value problem

$$u^{(6)}(t)=f\big(u(t),u''(t),u^{(4)}(t)\big),\quad t\in(0,1),$$
$$u(0)=0,\quad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i),$$
$$u''(0)=0,\quad u''(1)=\sum_{i=1}^{m-2}a_iu''(\eta_i),$$
$$u^{(4)}(0)=0,\quad u^{(4)}(1)=\sum_{i=1}^{m-2}a_iu^{(4)}(\eta_i),$$

where $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a sign-changing continuous function, $m\geq 3$, $\eta_i\in(0,1)$, and $a_i>0$ for $i=1,2,\dots,m-2$ with $\sum_{i=1}^{m-2}a_i<1$. We first show that the spectral properties of the linearisation of this problem are similar to the well-known properties of the standard Sturm-Liouville problem with separated boundary conditions. These spectral properties are then used to prove a Rabinowitz-type global bifurcation theorem for a bifurcation problem related to the above problem. Finally, we obtain the existence of nodal solutions for the problem, under various conditions on the asymptotic behaviour of nonlinearity f by using the global bifurcation theorem.

1. Introduction

Multi-point boundary value problems for ordinary differential equations arise in different areas of applied mathematics and physics. The existence of solutions for second order and high order multi-point boundary value problems has been studied by many authors and the methods used are the nonlinear alternative of Leray-Schauder, coincidence degree theory, fixed point theorems in cones and global bifurcation techniques; see [2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15] and the references therein.

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We consider the sixth order m-point boundary value problem (BVP, for short)

$$\begin{gathered} u^{(6)}(t)=f\big(u(t),u''(t),u^{(4)}(t)\big),\quad t\in(0,1),\\ u(0)=0,\quad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i),\\ u''(0)=0,\quad u''(1)=\sum_{i=1}^{m-2}a_iu''(\eta_i),\\ u^{(4)}(0)=0,\quad u^{(4)}(1)=\sum_{i=1}^{m-2}a_iu^{(4)}(\eta_i), \end{gathered} \tag{1.1}$$

where $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a sign-changing continuous function, $m\geq 3$, $\eta_i\in(0,1)$, and $a_i>0$ for $i=1,2,\dots,m-2$ with

$$\sum_{i=1}^{m-2}a_i<1. \tag{1.2}$$

Ma and O'Regan [11] investigated the existence of nodal solutions of m-point boundary value problem

$$\begin{gathered} u''(t)+f(u)=0,\quad t\in(0,1),\\ u(0)=0,\quad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i). \end{gathered} \tag{1.3}$$

where $\eta_i\in\mathbb{Q}(i=1,2,\dots,m-2)$ with $0<\eta_1<\eta_2<\dots<\eta_{m-2}<1$, and $\alpha_i\in\mathbb{R}$ $(i=1,2,\dots,m-2)$ with $a_i>0$ and $\sum_{i=1}^{m-2}a_i\leq 1$. They obtained some results on the spectrum of the linear operator corresponding to (1.3) and gave conditions on the ratio $f(s)/s$ at infinity and zero that guarantee the existence of nodal solutions. The proofs of main results are based on bifurcation techniques.

Recently, An and Ma [1] extended this result, they considered the nonlinear eigenvalue problems

$$\begin{gathered} u''(t)+rf(u)=0,\quad 0<t<1,\\ u(0)=0,\quad u(1)=\sum_{i=1}^{m-2}a_iu(\eta_i). \end{gathered} \tag{1.4}$$

under the following conditions:

- (A0) $a_i>0$ for $i=1,2,\dots,m-2$ with $0<\sum_{i=1}^{m-2}a_i<1$, $r\in\mathbb{R}$;
- (A1) $f\in C^1(\mathbb{R},\mathbb{R})$ and there exist two constants $s_2<0<s_1$ such that $f(s_1)=f(s_2)=f(0)=0$;
- (A2) There exist $f_0,f_\infty\in(0,\infty)$ such that

$$f_0:=\lim_{|u|\to 0}\frac{f(u)}{u},\quad f_\infty:=\lim_{|u|\to\infty}\frac{f(u)}{u}.$$

Using Rabinowitz global bifurcation theorem, An and Ma established the following theorem.

Theorem 1.1. *Let* (A0), (A1), (A2) *hold. Assume that for some* $k\in\mathbb{N}$,

$$\frac{\lambda_k}{f_\infty}<\frac{\lambda_k}{f_0}\quad\Big(\textit{resp.},\ \frac{\lambda_k}{f_0}<\frac{\lambda_k}{f_\infty}\Big)$$

Then

(i) *if* $r \in (\frac{\lambda_k}{f_\infty}, \frac{\lambda_k}{f_0}]$ *(resp.,* $r \in (\frac{\lambda_k}{f_0}, \frac{\lambda_k}{f_\infty}]$*) then* (1.4) *has at least two solutions* $u^{\pm}_{k,\infty}$ *(resp.,*$u^{\pm}_{k,0}$*), such that* $u^{+}_{k,\infty} \in T^{+}_{k}$ *and* $u^{-}_{k,\infty} \in T^{-}_{k}$ *(resp.,* $u^{+}_{k,0} \in T^{+}_{k}$ *and* $u^{-}_{k,0} \in T^{-}_{k}$*),*

(ii) *if* $r \in (\frac{\lambda_k}{f_0}, \infty)$ *(resp.,* $r \in (\frac{\lambda_k}{f_\infty}, \infty)$*) then* (1.4) *has at least four solutions* $u^{\pm}_{k,\infty}$ *and* $u^{\pm}_{k,0}$*, such that* $u^{+}_{k,\infty}, u^{+}_{k,0} \in T^{+}_{k}$*, and* $u^{-}_{k,\infty}, u^{-}_{k,0} \in T^{-}_{k}$*.*

Where λ_k *is the* k*th eigenvalue of*

$$u''(t) + \lambda u(t) = 0, \quad 0 < t < 1, \quad u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i).$$

Remark 1.2. Comparing results in [11] and the above theorem, we see that as f has two zeros $s_1, s_2 : s_2 < 0 < s_1$, the bifurcation structure of the nodal solutions of (1.4) becomes more complicated: the component of the solutions of (1.4) from the trivial solution at $(\frac{\lambda_k}{f_0}, 0)$ and the component of the solutions of (1.4) from infinity at $(\frac{\lambda_k}{f_\infty}, \infty)$ are disjoint; two new nodal solutions are born when $r > \max\{\frac{\lambda_k}{f_0}, \frac{\lambda_k}{f_\infty}\}$.

Liu and O'Regan [8] studied the existence and multiplicity of nodal solutions for fourth order m-point BVPs:

$$\begin{gathered} u^{(4)}(t) = f(u(t), u''(t)), \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \\ u'''(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} a_i u''(\eta_i), \end{gathered}$$

where $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given sign-changing continuous function, $m \geq 3$, $\eta_i \in (0,1)$, and $a_i > 0$ for $i = 1, 2, \dots, m-2$ satisfies (1.2). The main tool be used is global results on bifurcation from infinity, while in [16] is results on bifurcation coming from the trivial solutions.

Motivated by [1, 8, 16], in this paper we consider the existence of nodal solutions (that is, sign-changing solutions with a specified number of zeros) of BVP (1.1). To the best of our best knowledge, only [8, 17] seems to have considered the existence of nontrivial or positive solutions of the nonlinear multi-point BVPs for fourth order differential equations. Of course an interesting question is, as for sixth order m-point BVPs, whether we can obtain some new results which are similar to [1, 8]. Using the global bifurcation techniques, we study the global behavior of the components of nodal solutions of BVP (1.1) and give a positive answer to the above question. However, when m-point boundary value condition of (1.1) is concerned, the discussion is more difficult since the problem is nonsymmetric and the corresponding operator is disconjugate. Although the paper [3] has also obtained sign-changing solutions of (1.1), but no information is obtained regarding the number of zeros of the solution, and the method of proof is entirely different (relying on degree theory in cones).

The article is organized as follows. Section 2 gives some preliminaries. The results we obtain are similar to the standard spectral theory of the linear, separated Sturm-Liouville problem, with a slight difference in the nodal counting method used, to deal with the multi-point boundary conditions. We also show that the standard counting method is inadequate in this situation, and that the condition (1.2) is optimal for the results in Section 2 to hold. This section deals entirely with the linear eigenvalue problem. In Section 3 we first consider a bifurcation problem related to (1.1), and prove a Rabinowitz-type global bifurcation theorem for this problem. The proof uses the spectral properties of the linearisation obtained in Section 2. Finally, we use the global bifurcation theorem to obtain nodal solutions of (1.1), under various hypotheses on the asymptotic behaviour of f. Specifically, we consider the cases where f is asymptotically linear.

To conclude this section we give some notation and state four lemmas, which will be used in Section 3. Following the notation of Rabinowitz, let $F : \mathbb{R} \times E \to E$ where E is real Banach spaces and F is continuous. Suppose the equation $F(U) = 0$ possesses a simple curve of solutions $\mathscr{C}$ given by $\{U(t) | t \in [a, b]\}$. If for some $\tau \in (a, b)$, F possesses zeroes not lying on $\mathscr{C}$ in every neighborhood of $U(\tau)$, then $U(\tau)$ is said to be a bifurcation point for F with respect to the curve $\mathscr{C}$.

A special family of such equations has the form

$$u = G(\lambda, u). \tag{1.5}$$

where $\lambda \in \mathbb{R}, u \in E$, a real Banach space with norm $\|\cdot\|$ and $G : \mathbb{R} \times E \to E$ is compact and continuous. Equations of the form (1.5) are usually called nonlinear eigenvalue problems and arise in many contexts in mathematical physics. It is therefore of interest to investigate the structure of the set of their solutions.

Bifurcation phenomena occur in many parts of physics and have been intensively studied. It is often the case in applications that $F(\lambda, u) = u - (\lambda L u + H(\lambda, u))$ where $L : E \to E$ is a compact linear operator and $H : \mathbb{R} \times E \to E$ is compact(i.e., continuous and maps bounded sets into relatively compact sets) with $H(\lambda, u) = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals. The zeros $\mathscr{R} = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$ of F are then called the line of trivial solutions of

$$u = \lambda L u + H(\lambda, u). \tag{1.6}$$

If there exists $\mu \in \mathbb{R}$ and $0 \neq v \in E$ such that $v = \mu L v$, μ is said to be a real characteristic value of L. The set of real characteristic values of L will be denoted by $r(L)$. The multiplicity of $\mu \in r(L)$ is

$$\dim \cup_{j=1}^{\infty} N((I - \mu L)^j),$$

where I is the identity map on E and $N(A)$ denotes the null space of A. It is well known that if $\mu \in \mathbb{R}$, a necessary condition for $(\mu, 0)$ to be a bifurcation point of (1.6) with respect to $\mathscr{R}$ is that $\mu \in r(L)$. If μ is a simple characteristic value of L, let v denote the eigenvector of L corresponding to μ normalized so $\|v\| = 1$. By $\mathscr{S}$ we denote the closure of the set of nontrivial solutions of (1.6). A component of $\mathscr{S}$ is a maximal closed connected subset. The following are global results for (1.6) on bifurcation from the trivial solutions (see, Rabinowitz [18, Theorems 1.3, 1.25, 1.27]).

Lemma 1.3. *If $\mu \in r(L)$ is of odd multiplicity, then $\mathscr{S}$ contains a component $\mathscr{C}_\mu$ which can be decomposed into two subcontinua $\mathscr{C}_\mu^+, \mathscr{C}_\mu^-$ such that each of $\mathscr{C}_\mu^+, \mathscr{C}_\mu^-$ either*

(i) *meets infinity in $\mathscr{S}$, or*
(ii) *meets $(\widehat{\mu}, 0)$ where $\mu \neq \widehat{\mu} \in \sigma(L)$, or*
(iii) *contains a pair of points $(\lambda, u), (\lambda, -u), u \neq 0$.*

Lemma 1.4. *If $\mu \in r(L)$ is simple, then $\mathscr{S}$ contains a component $\mathscr{C}_\mu$ which can be decomposed into two subcontinua $\mathscr{C}_\mu^+, \mathscr{C}_\mu^-$ such that for some neighborhood $\mathscr{N}$ of $(\mu, 0)$,*

$$(\lambda, u) \in \mathscr{C}_\mu^+ (\mathscr{C}_\mu^-) \cap \mathscr{N}, \quad (\lambda, u) \neq (\mu, 0)$$

implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1), \|w\| = o(|\alpha|)$ at $\alpha = 0$.

Remark 1.5. We say a continuum $\mathscr{C}$ of $\mathscr{S}$ meets infinity if $\mathscr{C}$ is not bounded.

Remark 1.6. Lemmas 1.3 is the first important result on the existence of a subcontinuum of solutions for nonlinear equations by degree theoretic method. When using Lemmas 1.3 to study multiplicity of nodal solutions one needs to first study the spectrum structure of the linear operator L corresponding to the nonlinear eigenvalue problem. Fortunately, the spectrum structure of linear operators corresponding to most known nonlinear boundary value problems have been studied systematically. However, for the case of multi-point boundary value problems, a complete study of the spectrum structure is not available yet.

Rabinowitz showed in [19, Theorems 1.6, Corollary 1.8] that analogues of Lemmas 1.3 and 1.4 when one is dealing with ∞ rather than 0. Let L be as above and $K : \mathbb{R} \times E \to E$ be continuous with $K(\lambda, u) = o(\|u\|)$ at $u = \infty$ uniformly on bounded λ intervals. Consider the equation

$$u = \lambda L u + K(\lambda, u) \tag{1.7}$$

Let $\mathscr{T}$ denote the closure of the set of nontrivial solutions of (1.7). The following are global results for (1.7) on bifurcation from infinity.

Lemma 1.7. *Suppose L is compact and linear, $K(\lambda, u)$ is continuous on $\mathbb{R} \times E$, $K(\lambda, u) = o(\|u\|)$ at $u = \infty$ uniformly on bounded λ intervals, and $\|u\|^2 K(\lambda, \frac{u}{\|u\|^2})$ is compact. If $\mu \in r(L)$ is of odd multiplicity, then $\mathscr{T}$ possesses an unbounded component $\mathscr{D}_\mu$ which meets (μ, ∞). Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap r(L) = \{\mu\}$ and $\mathscr{M}$ is a neighborhood of (μ, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0, then either*

(i) *$\mathscr{D}_\mu \backslash \mathscr{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathscr{D}_\mu \backslash \mathscr{M}$ meets $\mathscr{R} = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ or*

(ii) *$\mathscr{D}_\mu \backslash \mathscr{M}$ is unbounded.*

If (ii) occurs and $\mathscr{D}_\mu \backslash \mathscr{M}$ has a bounded projection on $\mathbb{R}$, then $\mathscr{D}_\mu \backslash \mathscr{M}$ meets $(\widehat{\mu}, \infty)$ where $\mu \neq \widehat{\mu} \in \sigma(L)$.

Remark 1.8. A continuum $\mathscr{D}_\mu \subset \mathscr{T}$ of solution of (1.7) meets (λ_k, ∞) which means that there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathscr{D}_\mu$ such that $\|u_n\| \to \infty$ and $\lambda_n \to \lambda_k$.

Lemma 1.9. *Suppose the assumptions of Lemma 1.7 hold. If $\mu \in r(L)$ is simple, then $\mathscr{D}_\mu$ can be decomposed into two subcontinua $\mathscr{D}_\mu^+, \mathscr{D}_\mu^-$ and there exists a neighborhood $\mathscr{O} \subset \mathscr{M}$ of (μ, ∞) such that $(\lambda, u) \in \mathscr{D}_\mu^+ (\mathscr{D}_\mu^-) \cap \mathscr{O}$ and $(\lambda, u) \neq (\mu, \infty)$ implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $|\lambda - \mu| = o(1), \|w\| = o(|\alpha|)$ at $\alpha = \infty$.*

Remark 1.10. We say (μ, ∞) is a bifurcation point for (1.7) if every neighborhood of (μ, ∞) contains solutions of (1.7); i.e., there exists a sequence (λ_n, u_n) of solutions of (1.7) such that $\lambda_n \to \mu$ and $\|u_n\| \to \infty$.

2. Preliminaries

Let $X = C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let

$$Y = \{u \in C^1[0,1] | u(0) = 0, u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i)\},$$

$$Z = \{u \in C^2[0,1] | u(0) = 0, u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i)\}$$

with the norm

$$\|u\|_1 = \max\{\|u\|, \|u'\|\}, \quad \|u\|_2 = \max\{\|u\|, \|u'\|, \|u''\|\},$$

respectively. Then X, Y and Z are Banach spaces.

For any C^1 function u, if $u(t_0) = 0$, then t_0 is a simple zero of u if $u'(t_0) \neq 0$. For any integer $k \in \mathbb{N}$ and any $\nu \in \{\pm\}$, as in [20], define sets S_k^ν, T_k^ν subsets of $C^2[0,1]$ consisting of of functions $u \in C^2[0,1]$ satisfying the following conditions:

S_k^ν: (i) $u(0) = 0, \nu u'(0) > 0$; (ii) u has only simple zeros in $[0,1]$ and has exactly $k-1$ zeros in $(0,1)$;

T_k^ν: (i) $u(0) = 0, \nu u'(0) > 0$, and $u'(1) \neq 0$; (ii) u' has only simple zeros in $(0,1)$ and has exactly k zeros in $(0,1)$; (iii) u has a zero strictly between each two consecutive zeros of u'.

Remark 2.1. (i) If $u \in T_k^\nu$, then u has exactly one zero between each two consecutive zeros of u', and all zeros of u are simple. Thus, u has at least $k-1$ zeros in $(0,1)$, and at most k zeros in $(0,1]$; i.e., $u \in S_k^\nu$ or $u \in S_{k+1}^\nu$.

(ii) The sets T_k^ν are open in Z and disjoint.

(iii) Note $T_k^- = -T_k^+$ and let $T_k = T_k^- \cup T_k^+$. It is easy to see that the sets T_k^+ and T_k^- are disjoint and open in Z.

Remark 2.2. One could regard the sets S_k^ν as counting zeros of u, while the sets T_k^ν count 'bumps'. The nodal properties of solutions of nonlinear Sturm-Liouville problems with separated boundary conditions are usually described in terms of sets similar to S_k^ν (with an additional condition at $x = 1$ to incorporate the boundary condition there); see [1, 3, 5, 9, 14]. However, Rynne [20] stated that T_k^ν are in fact more appropriate than S_k^ν when the multi-point boundary condition (1.1) is considered.

Let $\mathbb{E} = \mathbb{R} \times Y$ under the product topology. As in [19], we add the points $\{(\lambda, \infty) | \lambda \in \mathbb{R}\}$ to the space $\mathbb{E}$. Let $\Phi_k^+ = \mathbb{R} \times T_k^+, \Phi_k^- = \mathbb{R} \times T_k^-$, and $\Phi_k = \mathbb{R} \times T_k$.

We first convert (1.1) into another form. Notice that

$$u''(t) + v(t) = 0, \quad t \in (0,1),$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i).$$

Thus $u(t)$ can be written as

$$u(t) = Lv(t), \tag{2.1}$$

where the operator L is defined by

$$Lv(t) = \int_0^1 G(t,s)v(s)\,\mathrm{d}s, \forall v \in Y, \tag{2.2}$$

where

$$G(t,s)=g(t,s)+\frac{t}{1-\sum_{i=1}^{m-2}a_i\eta_i}\sum_{i=1}^{m-2}a_ig(\eta_i,s), \tag{2.3}$$

$$g(t,s)=\begin{cases}s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases} \tag{2.4}$$

Let $v(t)=-u^{(4)}(t)$. Then we obtain the following equivalent form of (1.1)

$$\begin{gathered} v''(t)+f((-L^2v)(t),(Lv)(t),-v(t))=0,\quad t\in(0,1),\\ v(0)=0,\quad v(1)=\sum_{i=1}^{m-2}a_iv(\eta_i). \end{gathered} \tag{2.5}$$

For the rest of this paper we assume that the initial value problem

$$\begin{gathered} v''(t)+f((-L^2v)(t),(Lv)(t),-v(t))=0,\quad t\in(0,1),\\ v(t_0)=0,\quad v'(t_0)=0, \end{gathered} \tag{2.6}$$

has the unique trivial solution $v\equiv 0$ on $[0,1]$ for any $t_0\in[0,1]$; in fact some suitable conditions such as a Lipschitz assumption or $f\in C^1$ guarantee this.

Define two operators on Y by

$$(Av)(t):=(LFv)(t), \tag{2.7}$$

$$(Fv)(t):=f((-L^2v)(t),(Lv)(t),-v(t)),\quad t\in[0,1],\quad v\in Y. \tag{2.8}$$

Then it is easy to see the following lemma holds.

Lemma 2.3. *The linear operator L and operator A are both completely continuous from Y to Y and*

$$\|Lv\|_1\le M\|v\|\le M\|v\|_1,\quad \forall v\in Y,$$

where

$$M=\max\Big\{1,\ \frac{1}{2}+\frac{\sum_{i=1}^{m-2}a_i}{6(1-\sum_{i=1}^{m-2}a_i\eta_i)}\Big\}.$$

Moreover, $u\in C^6[0,1]$ is a solution of (1.1) if and only if $v=-u^{(4)}$ is a solution of the operator equation $v=Av$. In fact, if u is a solution of (1.1), then $v=-u^{(4)}$ is a solution of the operator equation $v=Av$. Conversely, if v is a solution of the operator equation $v=Av$, then $u=-L^2v$ is a solution of (1.1)

Let the function $\Gamma(s)$ be defined by

$$\Gamma(s)=\sin s-\sum_{i=1}^{m-2}a_i\sin\eta_is,\ \ s\in\mathbb{R}^+.$$

The following lemma can be found in [20].

Lemma 2.4. *(i) For each $k\ge 1$, $\Gamma(s)$ has exactly one zero $s_k\in I_k:=((k-\frac{1}{2})\pi,(k+\frac{1}{2})\pi)$, so*

$$s_1<s_2<\dots<s_k\to\infty(k\to+\infty);$$

(ii) the characteristic value of L is exactly given by $\mu_k=s_k^2, k=1,2,\dots,$ and the eigenfunction ϕ_k corresponding to μ_k is $\phi_k(t)=\sin s_kt$;

(iii) the algebraic multiplicity of each characteristic value μ_k of L is 1;

(iv) $\phi_k\in T_k^+$ for $k=1,2,3,\dots,$ and ϕ_1 is strictly positive on $(0,1)$.

Lemma 2.5. *For $d = (d_1, d_2, d_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{(0,0,0)\}$, define a linear operator*

$$L_d v(t) = (d_1 L^3 + d_2 L^2 + d_3 L) v(t), \quad \forall t \in [0,1], v \in Y, \tag{2.9}$$

where L is defined as in (2.2). *Then the generalized eigenvalues of L_d are simple and are given by*

$$0 < \lambda_1(L_d) < \lambda_2(L_d) < \cdots < \lambda_k(L_d) \to \infty \quad \text{as } k \to +\infty,$$

where

$$\lambda_k(L_d) = \frac{\mu_k^3}{d_1 + d_2\mu_k + d_3\mu_k^2}.$$

The generalized eigenfunction corresponding to $\lambda_k(L_d)$is

$$\phi_k(t) = \sin s_k t,$$

where μ_k, s_k, ϕ_k are as in Lemma 2.4.

Proof. Suppose there exist λ and $u \neq 0$ such that $u = \lambda L_d u$. By (2.1)-(2.9), we have

$$\begin{gathered}
u^{(6)}(t) = \lambda(-d_1 u(t) + d_2 u''(t) - d_3 u^{(4)}(t)), \quad t \in (0,1), \\
u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \\
u''(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} a_i u''(\eta_i), \\
u^{(4)}(0) = 0, \quad u^{(4)}(1) = \sum_{i=1}^{m-2} a_i u^{(4)}(\eta_i).
\end{gathered} \tag{2.10}$$

Denote $D = \frac{d}{dt}$, Then there exist three complex numbers r_1, r_2 and r_3 such that

$$(D^6 + \lambda d_1 - \lambda d_2 D^2 + \lambda d_3 D^4) u(t) = (D^2 + r_1)(D^2 + r_2)(D^2 + r_3) u(t)$$

By the properties of differential operators, if (2.10) has a nonzero solution, then there exists $r_i (1 \le i \le 3)$ such that $r_i = \mu_k = s_k^2, k \in \mathbb{N}$. In this case, $\sin s_k t$ is a nonzero solution of (2.10). On substituting this solution into (2.10), we have

$$-\mu_k^3 = \lambda(-d_1 - d_2\mu_k - d_3\mu_k^2).$$

Hence, $\{\lambda_k = \frac{\mu_k^3}{d_1 + d_2\mu_k + d_3\mu_k^2}, k = 1, 2, \dots\}$ is the sequence of all eigenvalues of the operator L_d. Then λ is one of the values $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \dots$, and the eigenfunction corresponding to the eigenvalue λ_n is

$$u_n(t) = C \sin s_n t,$$

where C is a nonzero constant. By the ordinary method, we can show that any two eigenfunctions corresponding to the same eigenvalue λ_n are merely nonzero constant multiples of each other. Consequently,

$$\dim \ker(I - \lambda_n L_d) = 1.$$

Now we show that

$$\ker(I - \lambda_n L_d) = \ker(I - \lambda_n L_d)^2. \tag{2.11}$$

Obviously, we need to show only that

$$\ker(I - \lambda_n L_d)^2 \subset \ker(I - \lambda_n L_d). \tag{2.12}$$

For any $v \in \ker(I-\lambda_n L_d)^2$, $(I-\lambda_n L_d)v$ is an eigenfunction of the linear operator L_d corresponding to the eigenvalue λ_n if $(I-\lambda_n L_d)v \neq \theta$. Then there exists nonzero constant γ such that

$$(I-\lambda_n L_d)v = \gamma \sin s_n t, \quad t \in [0,1].$$

By direct computation, we have

$$\begin{gathered} v^{(6)}(t) = \lambda_n(-d_1 v(t) + d_2 v''(t) - d_3 v^{(4)}(t)) - \gamma \mu_n^3 \sin s_n t, \\ v(0) = 0, \quad v(1) = \sum_{i=1}^{m-2} a_i v(\eta_i), \\ v''(0) = 0, \quad v''(1) = \sum_{i=1}^{m-2} a_i v''(\eta_i), \\ v^{(4)}(0) = 0, \quad v^{(4)}(1) = \sum_{i=1}^{m-2} a_i v^{(4)}(\eta_i). \end{gathered} \tag{2.13}$$

The characteristic equation associated with (2.13) is

$$\lambda^6 - \frac{\mu_n^3}{d_1 + d_2\mu_n + d_3\mu_n^2}(-d_1 + d_2\lambda^2 - d_3\lambda^4) = 0.$$

Case (i): If there exists two real number $a > 0, b > 0$ and $a \neq b$ such that

$$\begin{aligned} &(\lambda^2 + \mu_n)(\lambda^2 - a)(\lambda^2 - b) \\ &= \lambda^6 - \frac{\mu_n^3}{d_1 + d_2\mu_n + d_3\mu_n^2}(-d_1 + d_2\lambda^2 - d_3\lambda^4) = 0. \end{aligned}$$

It is easy to see that the general solution of (2.13) is of the form

$$v(t) = C_1 e^{\sqrt{a}t} + C_2 e^{-\sqrt{a}t} + C_3 e^{\sqrt{b}t} + C_4 e^{-\sqrt{b}t} + C_5 \sin s_n t + C_6 \cos s_n t + Kt \cos s_n t,$$

for $t \in [0,1]$, where $C_1, C_2, C_3, C_4, C_5, C_6$ are six nonzero constants, and

$$K = \frac{\gamma s_n(d_1 + d_2\mu_n + d_3\mu_n^2)}{6d_1 + 4d_2\mu_n + 2d_3\mu_n^2}.$$

Applying the conditions $v(0) = 0, v''(0) = 0, v^{(4)}(0) = 0$, we obtain $C_1 + C_2 = 0, C_3 + C_4 = 0, C_6 = 0$, Then

$$\begin{aligned} v(t) &= C_1(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_3(e^{\sqrt{b}t} - e^{-\sqrt{b}t}) + C_5 \sin s_n t + Kt \cos s_n t, \\ v''(t) &= C_1 a(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_3 b(e^{\sqrt{b}t} - e^{-\sqrt{b}t}) - C_5 s_n^2 \sin s_n t \\ &\quad + K(-s_n^2 t \cos s_n t - 2s_n \sin s_n t), \\ v^{(4)}(t) &= C_1 a^2(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_3 b^2(e^{\sqrt{b}t} - e^{-\sqrt{b}t}) + C_5 s_n^4 \sin s_n t \\ &\quad + K(s_n^4 t \cos s_n t + 4s_n^3 \sin s_n t). \end{aligned}$$

Applying the conditions

$$\begin{gathered} v(1) = \sum_{i=1}^{m-2} a_i v(\eta_i), \quad v''(1) = \sum_{i=1}^{m-2} a_i v''(\eta_i), \\ v^{(4)}(1) = \sum_{i=1}^{m-2} a_i v^{(4)}(\eta_i), \quad \sin s_n = \sum_{i=1}^{m-2} a_i \sin \eta_i s_n, \end{gathered}$$

we have

$$\begin{gathered} C_1F + C_3G + KH = 0, \\ C_1aF + C_3bG - s_n^2KH = 0, \\ C_1a^2F + C_3b^2G + s_n^4KH = 0, \end{gathered} \tag{2.14}$$

where

$$\begin{gathered} F = e^{\sqrt{a}} - e^{-\sqrt{a}} - \sum_{i=1}^{m-2} a_i(e^{\sqrt{a}\eta_i} - e^{-\sqrt{a}\eta_i}), \\ G = e^{\sqrt{b}} - e^{-\sqrt{b}} - \sum_{i=1}^{m-2} a_i(e^{\sqrt{b}\eta_i} - e^{-\sqrt{b}\eta_i}), \\ H = \cos s_n - \sum_{i=1}^{m-2} a_i\eta_i \cos \eta_i s_n. \end{gathered}$$

If $H \neq 0$, then the solution of (2.14) is $C_1 = C_3 = K = 0$, which is a contradiction to $\gamma \neq 0$, and

$$v(t) = C_5 \sin s_n t \in \ker(I - \lambda_n L_d).$$

So, (2.12) holds. Hence, (2.11) holds. If $H = 0$, then

$$\cos s_n = \sum_{i=1}^{m-2} a_i\eta_i \cos \eta_i s_n.$$

By the Schwarz inequality, we obtain

$$\begin{aligned} 1 - \sin^2 s_n &= \Big(\sum_{i=1}^{m-2} a_i\eta_i \cos \eta_i s_n\Big)^2 \\ &\leq \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\big(\sum_{i=1}^{m-2} a_i^2 \cos^2 \eta_i s_n\Big) \\ &= \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2\Big) - \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2 \sin^2 \eta_i s_n\Big). \end{aligned}$$

Applying the condition $\sin s_n = \sum_{i=1}^{m-2} a_i \sin \eta_i s_n$, we obtain

$$\begin{aligned} 1 &\leq \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2\Big) + \Big(\sum_{i=1}^{m-2} a_i \sin \eta_i s_n\Big)^2 - \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2 \sin^2 \eta_i s_n\Big) \\ &= \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2\Big) + \Big(1 - \sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2 \sin^2 \eta_i s_n\Big) \\ &\quad + \sum_{i\neq j} a_i a_j \sin \eta_i s_n \sin \eta_j s_n \\ &\leq \Big(\sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2\Big) + \Big(1 - \sum_{i=1}^{m-2} \eta_i^2\Big)\Big(\sum_{i=1}^{m-2} a_i^2\Big) + \sum_{i\neq j} a_i a_j \\ &= \Big(\sum_{i=1}^{m-2} a_i\Big)^2. \end{aligned}$$

which is a contradiction to $\sum_{i=1}^{m-2} a_i < 1$. Thus, (2.11) holds. It follows from (2.11) and (2.12) that the algebraic multiplicity of the eigenvalue μ_n is 1.

Case (ii): There exists a real number $a > 0$ such that

$$(\lambda^2 + \mu_n)(\lambda^2 - a)^2 = \lambda^6 - \frac{\mu_n^3}{d_1 + d_2\mu_n + d_3\mu_n^2}(-d_1 + d_2\lambda^2 - d_3\lambda^4) = 0.$$

It is easy to see that the general solution of (2.13) is of the form

$$v(t) = (C_1 + C_2 t)e^{\sqrt{a}t} + (C_3 + C_4 t)e^{-\sqrt{a}t} + C_5 \sin s_n t + C_6 \cos s_n t + Kt\cos s_n t,$$

for $t \in [0,1]$, where $C_1, C_2, C_3, C_4, C_5, C_6$ are six nonzero constants, and $K = \frac{\gamma s_n(d_1+d_2\mu_n+d_3\mu_n^2)}{6d_1+4d_2\mu_n+2d_3\mu_n^2}$.

Applying the conditions $v(0) = 0$, $v''(0) = 0$, $v^{(4)}(0) = 0$, we obtain $C_1 + C_3 = 0$, $C_2 - C_4 = 0$, $C_6 = 0$, Then

$$\begin{aligned}
v(t) &= C_1(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_2 t(e^{\sqrt{a}t} + e^{-\sqrt{a}t}) + C_5 \sin s_n t + Kt\cos s_n t,\\
v''(t) &= C_1 a(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + 2C_2\sqrt{a}(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_2 at(e^{\sqrt{a}t} + e^{-\sqrt{a}t})\\
&\quad - C_5 s_n^2 \sin s_n t + K(-s_n^2 t\cos s_n t - 2s_n \sin s_n t),\\
v^{(4)}(t) &= C_1 a^2(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + 4C_2 a\sqrt{a}(e^{\sqrt{a}t} - e^{-\sqrt{a}t}) + C_2 a^2 t(e^{\sqrt{a}t} + e^{-\sqrt{a}t})\\
&\quad + C_5 s_n^4 \sin s_n t + K(s_n^4 t\cos s_n t + 4s_n^3 \sin s_n t).
\end{aligned}$$

Applying the conditions

$$\begin{gathered}
v(1) = \sum_{i=1}^{m-2} a_i v(\eta_i), \quad v''(1) = \sum_{i=1}^{m-2} a_i v''(\eta_i),\\
v^{(4)}(1) = \sum_{i=1}^{m-2} a_i v^{(4)}(\eta_i), \quad \sin s_n = \sum_{i=1}^{m-2} a_i \sin \eta_i s_n,
\end{gathered}$$

we have

$$\begin{gathered}
C_1 F + C_2 G + KH = 0,\\
C_1 aF + C_2(2\sqrt{a}F + aG) - s_n^2 KH = 0,\\
C_1 a^2 F + C_2(4a\sqrt{a}F + a^2 G) + s_n^4 KH = 0,
\end{gathered} \tag{2.15}$$

where

$$\begin{gathered}
F = e^{\sqrt{a}} - e^{-\sqrt{a}} - \sum_{i=1}^{m-2} a_i(e^{\sqrt{a}\eta_i} - e^{-\sqrt{a}\eta_i}),\\
G = e^{\sqrt{a}} + e^{-\sqrt{a}} - \sum_{i=1}^{m-2} a_i\eta_i(e^{\sqrt{a}\eta_i} + e^{-\sqrt{a}\eta_i}),\\
H = \cos s_n - \sum_{i=1}^{m-2} a_i \eta_i \cos \eta_i s_n.
\end{gathered}$$

If $H \neq 0$, then the solution of (2.15) is $C_1 = C_2 = K = 0$, which is a contradiction to $\gamma \neq 0$, and

$$v(t) = C_5 \sin s_n t \in \ker(I - \lambda_n L_d).$$

So, (2.12) holds. Hence, (2.11) holds.

If $H = 0$, in this case, the proof is similar to Case (i), we omit it.

Case (iii): There exists a real number $a, b > 0$ such that

$$\begin{aligned}&(\lambda^2+\mu_n)[\lambda^2-(a^2-b^2+2abi)][\lambda^2-(a^2-b^2-2abi)]\\&=\lambda^6-\frac{\mu_n^3}{d_1+d_2\mu_n+d_3\mu_n^2}(-d_1+d_2\lambda^2-d_3\lambda^4)=0.\end{aligned}$$

It is easy to see that the general solution of (2.13) is of the form

$$\begin{aligned}v(t)&=(C_1\cos bt+C_2\sin bt)e^{at}+(C_3\cos bt+C_4\sin bt)e^{-at}\\&\quad+C_5\sin s_nt+C_6\cos s_nt+Kt\cos s_nt,\quad t\in[0,1].\end{aligned}$$

where $C_1, C_2, C_3, C_4, C_5, C_6$ are six nonzero constants, and

$$K=\frac{\gamma s_n(d_1+d_2\mu_n+d_3\mu_n^2)}{6d_1+4d_2\mu_n+2d_3\mu_n^2}.$$

Applying the conditions $v(0)=0$, $v''(0)=0$, $v^{(4)}(0)=0$, we obtain that $C_1+C_3=0$, $C_2-C_4=0$, $C_6=0$, Then

$$\begin{aligned}v(t)&=C_1\cos bt(e^{at}-e^{-at})+C_2\sin bt(e^{at}+e^{-at})+C_5\sin s_nt+Kt\cos s_nt,\\v''(t)&=C_1(a^2-b^2)\cos bt(e^{at}-e^{-at})-2C_1ab\sin bt(e^{at}+e^{-at})\\&\quad+2C_2ab\cos bt(e^{at}-e^{-at})+C_2(a^2-b^2)\sin bt(e^{at}+e^{-at})\\&\quad-C_5s_n^2\sin s_nt+K(-s_n^2t\cos s_nt-2s_n\sin s_nt),\\v^{(4)}(t)&=C_1(a^4+b^4-6a^2b^2)\cos bt(e^{at}-e^{-at})\\&\quad+4C_1(ab^3-a^3b)\sin bt(e^{at}+e^{-at})+4C_2(a^3b-ab^3)\cos bt(e^{at}-e^{-at})\\&\quad+C_2(a^4+b^4-6a^2b^2)\sin bt(e^{at}+e^{-at})+C_5s_n^4\sin s_nt\\&\quad+K(s_n^4t\cos s_nt+4s_n^3\sin s_nt).\end{aligned}$$

Applying the conditions

$$\begin{gathered}v(1)=\sum_{i=1}^{m-2}a_iv(\eta_i),\quad v''(1)=\sum_{i=1}^{m-2}a_iv''(\eta_i),\\v^{(4)}(1)=\sum_{i=1}^{m-2}a_iv^{(4)}(\eta_i),\quad \sin s_n=\sum_{i=1}^{m-2}a_i\sin\eta_is_n,\end{gathered}$$

we have

$$\begin{gathered}C_1F+C_2G+KH=0,\\C_1[(a^2-b^2)F-2abG]+C_2[2abF+(a^2-b^2)G]-s_n^2KH=0,\\C_1[(a^4+b^4-6a^2b^2)F+4(ab^3-a^3b)G]+C_2\big[4(a^3b-ab^3)F\\+(a^4+b^4-6a^2b^2)G\big]+s_n^4KH=0,\end{gathered}\tag{2.16}$$

where

$$\begin{aligned}F&=\cos b(e^a-e^{-a})-\sum_{i=1}^{m-2}a_i\cos b\eta_i(e^{a\eta_i}-e^{-a\eta_i}),\\G&=\sin b(e^a+e^{-a})-\sum_{i=1}^{m-2}a_i\sin b\eta_i(e^{a\eta_i}+e^{-a\eta_i}),\end{aligned}$$

$$H = \cos s_n - \sum_{i=1}^{m-2} a_i \eta_i \cos \eta_i s_n.$$

If $H \neq 0$, then the solution of (2.16) is $C_1 = C_2 = K = 0$, which is a contradiction to $\gamma \neq 0$, and

$$v(t) = C_5 \sin s_n t \in \ker(I - \lambda_n L_d).$$

So, (2.12) holds. Hence, (2.11) holds.

If $H = 0$, the proof is similar to Case (i), we omit it.

To sum up, the generalized eigenvalues of L_d are simple, and the proof of this lemma is complete. □

3. Main Results

We now list the following hypotheses for convenience.

(H1) There exists $a = (a_1, a_2, a_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{(0,0,0)\}$ such that
$$f(x,y,z) = -a_1 x + a_2 y - a_3 z + o(|(x,y,z)|), \quad \text{as } |(x,y,z)| \to 0,$$
where $(x,y,z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, and $|(x,y,z)| := \max\{|x|,|y|,|z|\}$.

(H2) There exists $b = (b_1, b_2, b_3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \backslash \{(0,0,0)\}$ such that
$$f(x,y,z) = -b_1 x + b_2 y - b_3 z + o(|(x,y,z)|), \text{ as } |(x,y,z)| \to \infty.$$

(H3) There exists $R > 0$ such that
$$|f(x,y,z)| < \frac{R}{M}, \quad \text{for } (x,y,z) \in \{(x,y,z) : |x| \leq M^2 R, |y| \leq MR, |z| \leq R\},$$
where M is defined as in Lemma 2.3.

(H4) There exist two constants $r_1 < 0 < r_2$ such that $f(x,y,-r_1) \geq 0$ and $f(x,y,-r_2) \leq 0$ for $(x,y) \in [-Mr^2, Mr^2] \times [-Mr, Mr]$, and $f(x,y,-z)$ satisfies a Lipschitz condition in z for $(x,y,z) \in [-Mr^2, Mr^2] \times [-Mr, Mr] \times [r_1, r_2]$, where $r = \max\{|r_1|, r_2\}$.

Now we are ready to give our main results. To set it up we first consider global results for the equation
$$v = \lambda A v, \tag{3.1}$$
on Y, where $\lambda \in \mathbb{R}$, and the operator A is defined as in (2.7). Under the condition (H1), Equation (3.1) can be rewritten as
$$v = \lambda L_a v + H_a(\lambda, v), \tag{3.2}$$
here $H_a(\lambda, v) = \lambda A v - \lambda L_a v$, L_a is defined as in (2.9) (replacing d with a). Obviously, by (H1) and Lemma 2.3-2.5, it can be seen that $H_a(\lambda, v)$ is $o(\|v\|_1)$ for v near 0 uniformly on bounded λ intervals and L_a is a compact linear map on Y. A solution of (3.1)) is a pair $(\lambda, v) \in E$. By (H1), the known curve of solutions $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set on nontrivial solutions of (3.1)) will be denoted by $\mathscr{S}$ as in Lemma 1.3.

If $H_a(\lambda, v) \equiv 0$, then (3.2) becomes a linear system
$$v = \lambda L_a v, \tag{3.3}$$
By Lemma 2.5, (3.3) possesses an increasing sequence of simple eigenvalues
$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \to \infty, \quad \text{as } k \to +\infty,$$

where

$$\lambda_k = \frac{\mu_k^3}{a_1 + a_2\mu_k + a_3\mu_k^2}. \tag{3.4}$$

Any eigenfunction $\phi_k(t) = \sin s_k t$ corresponding to λ_k is in T_k^+.

A similar analysis as in [8, Lemma 3.4, 3.5] and [20, Proposition 4.1] yield the following results.

Lemma 3.1. *Suppose that (λ, v) is a solution of (3.1) and $v \neq 0$. Then $v \in \cup_{i=1}^{\infty} T_i$.*

Lemma 3.2. *Assume that (H1) holds and λ_k is defined by (3.4). Then for each integer $k > 0$ and each $\nu = +$, or $-$, there exists a continua $\mathscr{C}_k^\nu$ of solutions of (3.1)) in $\Phi_k^\nu \cup \{(\lambda_k, 0)\}$, which meets $\{(\lambda_k, 0)\}$ and ∞ in $\mathscr{S}$.*

Under condition (H2), (3.1) can be rewritten as

$$v = \lambda L_b v + K_b(\lambda, v), \tag{3.5}$$

here $K_b(\lambda, v) = \lambda Av - \lambda L_b v$, L_b is defined as in (2.9) (replacing d with b). Here $h(x, y, z) = f(x, y, z) + b_1 x - b_2 y + b_3 z$. Then from (H2) it follows that

$$\lim_{|(x,y,z)|\to\infty} \frac{h(x, y, z)}{|(x, y, z)|} = 0.$$

Define a function

$$\widehat{h}(r) := \max\{|h(x, y, z)| : |(x, y, z)| \leq r\}.$$

Then $\widehat{h}(r)$ is nondecreasing and

$$\lim_{r\to\infty} \frac{\widehat{h}(r)}{r} = 0. \tag{3.6}$$

Obviously, by (3.6) and Lemma 2.3, it can be seen that $K_b(\lambda, v)$ is $o(\|v\|_1)$ for v near ∞ uniformly on bounded λ intervals and L_b is a compact linear map on Y.

Similar to (3.3), by Lemma 2.5, L_b possesses an increasing sequence of simple eigenvalues

$$0 < \overline{\lambda}_1 < \overline{\lambda}_2 < \dots < \overline{\lambda}_k \to \infty, \quad \text{as } k \to +\infty,$$

where

$$\overline{\lambda}_k = \frac{\mu_k^3}{b_1 + b_2\mu_k + b_3\mu_k^2}. \tag{3.7}$$

Note $\phi_k(t) = \sin s_k t$ is an eigenfunction corresponding to $\overline{\lambda}_k$. Obviously, it is in T_k^+.

Lemma 3.3. *Assume that (H1)-(H2) hold. Then for each integer $k > 0$ and each $\nu = +, or-$, there exists a continua $\mathscr{D}_k^\nu$ of $\mathscr{T}$ in $\Phi_k^\nu \cup \{(\overline{\lambda}_k, \infty)\}$ coming from $\{(\overline{\lambda}_k, \infty)\}$, which meets $\{(\overline{\lambda}_k, 0)\}$ or has an unbounded projection on $\mathbb{R}$.*

Theorem 3.4. *Assume that (H1)–(H2) hold. Suppose there exists two integers $i_0 \geq 0$ and $k > 0$ such that either*

$$\frac{\mu_{i_0+k}^3}{a_1 + a_2\mu_{i_0+k} + a_3\mu_{i_0+k}^2} < 1 < \frac{\mu_{i_0+1}^3}{b_1 + b_2\mu_{i_0+1} + b_3\mu_{i_0+1}^2} \tag{3.8}$$

or

$$\frac{\mu_{i_0+k}^3}{b_1 + b_2\mu_{i_0+k} + b_3\mu_{i_0+k}^2} < 1 < \frac{\mu_{i_0+1}^3}{a_1 + a_2\mu_{i_0+1} + a_3\mu_{i_0+1}^2} \tag{3.9}$$

holds. Then (1.1) has at least $2k$ nontrivial solutions.

Proof. First suppose that (3.8) holds. Using the notation of (3.4) and (3.7), this means $\lambda_{i_0+k} < 1 < \overline{\lambda}_{i_0+1}$ and so from Lemma 2.5 we know that

$$\lambda_{i_0+1} < \lambda_{i_0+2} < \cdots < \lambda_{i_0+k} < 1 < \overline{\lambda}_{i_0+1} < \overline{\lambda}_{i_0+2} < \cdots < \overline{\lambda}_{i_0+k}.$$

Consider (3.2) as a bifurcation problem from the trivial solution. We need only show that

$$\mathscr{C}^{\nu}_{i_0+j} \cap (\{1\} \times Y) \neq \emptyset, \quad j = 1, 2, \dots, k;\ \nu = +, \text{or} -. \tag{3.10}$$

Suppose, on the contrary and without loss of generality, that

$$\mathscr{C}^{+}_{i_0+i} \cap (\{1\} \times Y) = \emptyset, \quad \text{for some } i,\ 1 \leq i \leq k. \tag{3.11}$$

By Lemma 3.2 we know that $\mathscr{C}^{+}_{i_0+i}$ joins $(\lambda_{i_0+i}, 0)$ to infinity in $\mathscr{S}$ and $(\lambda, v) = (0, 0)$ is the unique solution of (3.1) (in which $\lambda = 0$) in $\mathbb{E}$. This together with $\lambda_{i_0+i} < 1$ guarantee that there exists a sequence $\{(\xi_m, y_m)\} \subset \mathscr{C}^{+}_{i_0+i}$ such that $\xi_m \in (0, 1)$ and $\|y_m\|_1 \to \infty$ as $m \to \infty$. We may assume that $\xi_m \to \overline{\xi} \in [0, 1]$ as $m \to \infty$. Let $x_m := \frac{y_m}{\|y_m\|_1}, m \geq 1$. From the fact that

$$y_m = \xi_m L_b y_m + K_b(\xi_m, y_m),$$

it follows that

$$x_m = \xi_m L_b x_m + \frac{K_b(\xi_m, y_m)}{\|y_m\|_1}. \tag{3.12}$$

Notice that $L_b : Y \to Y$ is completely continuous. We may assume that there exists $\omega \in Y$ with $\|\omega\|_1 = 1$ such that $\|x_m - \omega\|_1 \to 0$ as $m \to \infty$.

Letting $m \to \infty$ in (3.12) and noticing $\frac{K_b(\xi_m, y_m)}{\|y_m\|_1} \to 0$ as $m \to \infty$ one obtains

$$\omega = \overline{\xi} L_b \omega.$$

Since $\omega \neq 0$, then $\overline{\xi} \neq 0$ is an eigenvalue of L_b; that is, $\overline{\xi} = \overline{\lambda}_{i_0+i}$, which contradicts $\overline{\lambda}_{i_0+i} > 1$. Thus (3.11) is not true, which means (3.10) holds.

Next suppose that (3.9) holds. This means

$$\overline{\lambda}_{i_0+1} < \overline{\lambda}_{i_0+2} < \cdots < \overline{\lambda}_{i_0+k} < 1 < \lambda_{i_0+1} < \lambda_{i_0+2} < \cdots < \lambda_{i_0+k}.$$

Consider (3.5) as a bifurcation problem from infinity. As above we need only to prove that

$$\mathscr{D}^{\nu}_{i_0+j} \cap (\{1\} \times Y) \neq \emptyset, \quad j = 1, 2, \dots, k;\ \nu = +, \text{or} -. \tag{3.13}$$

From Lemma 3.3, we know that $\mathscr{D}^{\nu}_{i_0+j}$ comes from $\{(\overline{\lambda}_{i_0+j}, \infty)\}$, meets $\{(\lambda_{i_0+j}, 0)\}$ or has an unbounded projection on $\mathbb{R}$. If it meets $\{(\lambda_{i_0+j}, 0)\}$, then the connectedness of $\mathscr{D}^{\nu}_{i_0+j}$ and $\lambda_{i_0+j} > 1$ guarantees that (3.13) is satisfied. On the other hand, if $\mathscr{D}^{\nu}_{i_0+j}$ has an unbounded projection on $\mathbb{R}$, notice that $(\lambda, v) = (0, 0)$ is the unique solution of (3.1) (in which $\lambda = 0$) in E, so (3.13) also holds. □

Theorem 3.5. *Assume that* (H1), (H2) *hold and one of* (H3) *or* (H4) *hold. Suppose there exists two integers* i_0 *and* j_0 *such that*

$$\frac{\mu_{i_0}^3}{a_1 + a_2\mu_{i_0} + a_3\mu_{i_0}^2} < 1, \frac{\mu_{j_0}^3}{b_1 + b_2\mu_{j_0} + b_3\mu_{j_0}^2} < 1. \tag{3.14}$$

Then (1.1) *has at least* $2(i_0 + j_0)$ *nontrivial solutions.*

Proof. First suppose that (H3) holds. Then there exists $\varepsilon > 0$ such that

$$(1+\varepsilon)|f(x,y,z)| < \frac{R}{M}, (x,y,z) \in \{(x,y,z) : |x| \le M^2R, |y| \le MR, |z| \le R\}, \tag{3.15}$$

Let (λ, v) be a solution of (3.1) such that $0 \le \lambda < 1+\varepsilon$ and $\|v\|_1 \le R$. Then by (2.7), (2.8), (3.1), (3.15) and Lemma 2.3 it is easy to see

$$\begin{aligned} \|v\|_1 &= \lambda\|Av\|_1 = \lambda\|LFv\| \\ &\le \lambda M\|Fv\| = M \max_{t\in[0,1]} |\lambda f((-L^2v)(t), (Lv)(t), -v(t))| \\ &< M\frac{R}{M} = R, \end{aligned} \tag{3.16}$$

Therefore,

$$\mathscr{S} \cap ([0, 1+\varepsilon] \times \partial\overline{B}_R) = \emptyset. \tag{3.17}$$

This together with (3.16) and Lemma 3.2 and Lemma 3.3 implies that

$$\mathscr{C}_k^\nu \cap ([0, 1+\varepsilon] \times \overline{B}_R) \subset [0, 1+\varepsilon] \times B_R, \quad k = 1, 2, \dots, i_0; \tag{3.18}$$

$$\mathscr{D}_j^\nu \cap ([0, 1+\varepsilon] \times \partial\overline{B}_R) = \emptyset, \quad j = 1, 2, \dots, j_0; \tag{3.19}$$

where $B_R = \{v \in Y| \|v\|_1 < R\}$ and $\overline{B}_R = \{v \in Y| \|v\|_1 \le R\}$, $\mathscr{C}_k^\nu$ and $\mathscr{D}_j^\nu$ are obtained from Lemma 3.2 and Lemma 3.3, respectively.

Since $\mathscr{C}_k^\nu$ is a unbounded component of solutions of (3.1) joining $(\lambda_k, 0)$ in $\mathbb{E}$, it follows from (3.17) and (3.18) that $\mathscr{C}_k^\nu$ crosses the hyperplane $\{1\} \times Y$ with $(1, v^\nu)$ such that $\|v^\nu\|_1 < R, (\nu = +$ or $-$, $k = 1, 2, \dots, i_0)$. This implies that (2.5) has $2i_0$ nontrivial solutions $\{v_i^\nu\}_{i=1}^{i_0}$ in which $\{v_i^+\}$ and $\{v_i^-\}$ are positive and negative solutions, respectively.

On the other hand, by (3.17), (3.19), and Lemma 3.3 one can obtain

$$\mathscr{D}_j^\nu \cap (\{1\} \times (Y \backslash \overline{B}_R)) \ne \emptyset, \quad j = 1, 2, \dots, j_0.$$

This implies that (2.5) has $2j_0$ nontrivial solutions $\{\omega_i^\nu\}_{i=1}^{j_0}$ in which $\{\omega_i^+\}$ and $\{\omega_i^-\}$ are positive and negative solutions, respectively.

Now it remains to show this theorem holds when the condition (H4) is satisfied. From the above we need only to prove that (i) for $(\lambda, v) \in \mathscr{C}_k^\nu(\nu = +$ or $-$, $k = 1, 2, \dots, i_0)$,

$$r_1 < v(t) < r_2, \quad t \in [0,1].$$

(ii) for $(\lambda, v) \in \mathscr{D}_j^\nu(\nu = +$ or $-$, $j = 1, 2, \dots, j_0)$, we have that either

$$\max_{t\in[0,1]} v(t) > r_2, \quad t \in [0,1]$$

or

$$\min_{t\in[0,1]} v(t) < r_1, \quad t \in [0,1].$$

In fact, like in [1], suppose on the contrary that there exists $(\lambda, v) \in \mathscr{C}_k^\nu \cap \mathscr{D}_j^\nu$ such that either

$$\max\{v(t) : t \in [0,1]\} = r_2$$

or

$$\min\{v(t) : t \in [0,1]\} = r_1$$

for some i, j.

First consider the case $\max\{v(t) : t \in [0,1]\} = r_2$. Then there exists $\bar{t} \in [0,1]$ such that $v(\bar{t}) = r_2$. Let

$$\tau_0 := \inf\{t \in [0,\bar{t}] : v(s) \geq 0 \text{ for } s \in [t,\bar{t}]\},$$
$$\tau_1 := \sup\{t \in [\bar{t},1] : v(s) \geq 0 \text{ for } s \in [\bar{t},t]\}.$$

Then

$$\max\{v(t) : t \in [\tau_0,\tau_1]\} = r_2, \tag{3.20}$$
$$0 \leq v(t) \leq r_2, \quad t \in [\tau_0,\tau_1]. \tag{3.21}$$

Therefore, $v(t)$ is a solution of the following equation

$$-v''(t) = \lambda f((-L^2v)(t),(Lv)(t),-v(t)), \quad t \in (\tau_0,\tau_1)$$

with $v(\tau_0) = v(\tau_1) = 0$ if $0 \leq \tau_0 < \tau_1 < 1$.

By (H4), there exists $\overline{M} \geq 0$ such that $f(x,y,-z) + \overline{M}z$ is strictly increasing in z for $(x,y,z) \in [-Mr^2, Mr^2] \times [-Mr, Mr] \times [r_1, r_2]$, where $r = \max\{|r_1|, r_2\}$. Then

$$-v''(t) + \lambda\overline{M}v = \lambda(f((-L^2v)(t),(Lv)(t),-v(t)) + \overline{M}v), \quad t \in (\tau_0,\tau_1)$$

Using (H4) and Lemma 2.3 again, we can obtain

$$\begin{aligned}
&-(r_2 - v(t))'' + \lambda\overline{M}(r_2 - v(t))\\
&= -\lambda[f((-L^2v)(t),(Lv)(t),-v(t)) + \overline{M}v(t) - \overline{M}r_2]\\
&= -\lambda[f((-L^2v)(t),(Lv)(t),-v(t)) + \overline{M}v(t) - (f((-L^2v)(t),(Lv)(t),-r_2) + \overline{M}r_2)]\\
&\quad - \lambda f((-L^2v)(t),(Lv)(t),-r_2)\\
&\geq 0, \quad t \in (\tau_0,\tau_1).
\end{aligned} \tag{3.22}$$

and if $\tau_1 = 1$, by (1.2) we know $v(1) < r_2$. Therefore,

$$r_2 - v(\tau_0) > 0, \quad r_2 - v(\tau_1) > 0 \quad \text{if } 0 \leq \tau_0 < \tau_1 < 1;$$
$$r_2 - v(\tau_1) > 0 \quad \text{if } \tau_1 = 1.$$

This together with (3.22) and the maximum principle [12] imply that $r_2 - v(t) > 0$ in $[\tau_0,\tau_1]$, which contradicts (3.20).

The proof in the case $\min\{v(t) : t \in [0,1]\} = r_1$ is similar, so we omit it. $\square$

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